Toward a Generalization of the Leland-Toft Optimal Capital Structure Model *

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The Central Issue of the Model

According to the works of

- Leland (*Journal of Finance, 1994*),

the optimal capital structure problem can be formulated as follows.

It is concerned with capital raise of a firm by issuing a debt within a given time interval. The debt will pay in exchange to the investor streams of payments paid continuously prior to and at default.

A portion of each debt payment made is applied towards reducing the debt principal and another portion of the payment is applied towards paying the interest (coupon) on the debt; similar to amortizing bond.

In case of default, part of the firm's asset will be liquidated to pay the default settlement and the remaining of which will go to the debt holder. Should default occur, the firm's manager tries to find an optimal default level in which the firm's equity value is maximized.
Literature Review and Contribution

As the source of randomness in the firm’s asset, Leland (Journal of Finance, 1994) and Leland and Toft (Journal of Finance, 1996) employed diffusion process. The model has been extended to those allowing jumps in the firm’s asset.

Extensions to the Jump-Diffusion process with jumps of exponential type:

- Hilberink and Rogers (Finance & Stochastics, 2002) - one-sided jumps.

Extensions to the Spectrally Negative Lévy process with general structure of jumps:

- Kyprianou and Surya (Finance & Stochastics, 2007)

Our contribution:

In Surya and Yamazaki (2011) we extend the above works by allowing bankruptcy costs, coupon rates and tax rebate to be dependent on the asset value. In the calculation, we use a few results from Egami and Yamazaki (2011).
The Leland-Toft Optimal Capital Structure Model

Let \( X = (X_t : t \geq 0) \) be source of randomness\(^1\) in the firm's underlying asset. We denote by \( \mathbb{P}_x \) the law of \( X \) under which the process \( X_t \) started at \( x \in \mathbb{R} \). For convenience we write \( \mathbb{P} = \mathbb{P}_0 \) and we shall write \( \mathbb{E}_x \) (resp., \( \mathbb{E} \)) the expectation operator associated with \( \mathbb{P}_x \) (resp., \( \mathbb{P} \)). The firm's asset value \( V_t \) evolves as \( V_t = e^{X_t} \).

We assume the existence of a default-free asset that pays a continuous interest rate \( r > 0 \). Furthermore, assume that under \( \mathbb{P} \), the discounted value \( e^{-(r-\delta)t} V_t \) of the firm's asset is \( \mathbb{P} \)-martingale, i.e.,

\[
\mathbb{E}(e^{-(r-\delta)t} V_t) = 1
\]

where \( \delta > 0 \) is the total payout rate to the firm's investors (bond and equity holders).

Default happens at the first time \( \tau_B^- \) the underlying falls to some level \( B \) or lower;

\[
\tau_B^- := \inf\{t \geq 0 : X_t < B\}, \quad B \in \mathbb{R}.
\]

The Leland-Toft Model: Continued

As the firm may declare default prior to debt maturity, the debt may not be paid back. Hence, the debt holder will charge a higher interest \( m \) on the debt than a default-free asset. Suppose that the up-front payment of the outstanding loan is \( P \) with principal \( p \) to be repaid in a periodical basis. Since the debt loan is an *amortizing loan*, the credit spread \( m \) can be determined as such that

\[
P = \int_0^\infty e^{-mt} p \, dt,
\]
i.e., \( m = \frac{p}{P} \).

- Following the aforementioned literature, the total value of debt can be written as

\[
D(x; B) := \mathbb{E}_x \left( \int_0^{\tau_B} e^{-(r+m)t} (P\rho + p) \, dt \right) + \mathbb{E}_x (e^{-(r+m)\tau_B} V_{\tau_B}^- (1 - \eta) 1_{\{\tau_B^- < \infty\}}),
\]

where \( \rho \) is the coupon rate and \( \eta \) is the fraction of firm’s asset value lost in default.

- The firm value is given by

\[
\mathcal{V}(x; B) := e^x + \mathbb{E}_x \left( \int_0^{\tau_B} e^{-rt} 1_{\{V_t \geq V_T\}} \tau \rho P \, dt \right) - \mathbb{E}_x (e^{-r\tau_B^-} \eta V_{\tau_B^-}).
\]

Here, it is assumed that there is a corporate tax rate \( \tau \) and its (full) rebate on coupon payments is gained if and only if \( V_t \geq V_T \) for some cut-off level \( V_T > 0 \).
The Leland-Toft Model: Continued

The optimal default level $B \in \mathbb{R}$ is found by solving the problem:

**Listing 1: Finding the optimal default boundary**

$$\max_{\{x \geq B\}} \mathcal{E}(x; B) := \mathcal{V}(x; B) - \mathcal{D}(x; B),$$

subject to the limited liability constraint $\mathcal{E}(x; B) \geq 0$. (1)

- The diffusion model admits analytical solutions (e.g., Leland and Toft, 1996).
- The spectrally negative model admits semi-analytical solutions in terms of the scale function (e.g., Kyprianou and Surya, 2007). Depending on the path regularity of the underlying Lévy process $X$, the optimal boundary is found by employing
  - Smooth-pasting condition when $X$ has paths of unbounded variation.
  - Continuous-pasting condition when $X$ has paths of bounded variation.
Towards Generalization of the Leland-Toft Model

- The original model of Leland-Toft assumes that
  - Default costs is a constant fraction $\eta$ of the asset value $V_t = e^{X_t}$.
  - The tax is a stepwise function of the asset value, as

$$
\text{the tax} = \begin{cases} 
\tau \rho P, & \text{when } X_t \geq \log V_T \\
0, & \text{otherwise}
\end{cases}
$$

or equivalently, $\text{the tax} = \tau \rho P 1_{\{X_t \geq \log V_T\}}$.
- Coupon is a constant fraction $\rho$ of up-front payment $P$ of total loan.

- In our work, we attempt to generalize the above assumptions in the following sense:
  - Default costs: $\eta e^{X_t} \rightarrow \eta(X_t)$.
  - The tax: $\tau \rho P 1_{\{X_t \geq \log V_T\}} \rightarrow f_2(X_t)$.
  - Coupon: from a constant $\rho \rightarrow \rho(X_t)$.

In the sequel below, we define a function

$$f_1(x) = P \rho(x) + p.$$
Towards Generalization of the Leland-Toft Model: Continued

By doing so,

• the total value of debt becomes

\[
\mathcal{D}(x; B) := \mathbb{E}_x \left( \int_0^{\bar{\tau}_B} e^{-(r+m)t} f_1(X_t) \, dt \right) + \mathbb{E}_x \left( e^{-(r+m)\bar{\tau}_B} \exp(X_{\bar{\tau}_B}) \left( 1 - \hat{\eta}(X_{\bar{\tau}_B}) \right) \mathbf{1}_{\{\bar{\tau}_B < \infty\}} \right),
\]

where \( \hat{\eta}(x) := e^{-x}\eta(x) \) is the ratio of default costs relative to the asset value.

• The firm value is given by

\[
\mathcal{V}(x; B) := e^x + \mathbb{E}_x \left( \int_0^{\bar{\tau}_B} e^{-rt} f_2(X_t) \, dt \right) - \mathbb{E}_x \left( e^{-r\bar{\tau}_B} \eta(X_{\bar{\tau}_B}) \right).
\]

The optimal default boundary \( B \in \mathbb{R} \) is found by solving the problem (1).
Spectrally Negative Lévy Processes

Because of the fact that the Lévy measure only charges the negative half-line, the characteristic exponent is well defined and analytic on ($\Im(\theta) \leq 0$). We refer among others to Kyprianou (2006).

Hence, it is therefore sensible to define a Laplace exponent

$$\kappa(\theta) = -\mu \theta + \frac{1}{2} \sigma^2 \theta^2 + \int_{(-\infty,0)} \left( e^{\theta y} - 1 - \theta y 1_{\{y > -1\}} \right) \Pi(dy),$$

and, hence, we see that the identity $\mathbb{E}(e^{\theta X_t}) = e^{t \kappa(\theta)}$ holds whenever $\Re(\theta) \geq 0$.

We denote by $\Phi : [0, \infty) \to [0, \infty)$ the right continuous inverse of $\kappa(\lambda)$, so that

$$\kappa(\Phi(\lambda)) = \lambda \quad \text{for all} \quad \lambda \geq 0$$

or, i.e., $\Phi(\alpha)$ is the largest positive root of $\Phi(\alpha) = \sup\{p > 0 : \kappa(p) = \alpha\}$.

The class of spectrally negative Lévy processes is very rich. Amongst other things it allows for processes which have paths of both unbounded and bounded variation.
Spectrally Negative Lévy Processes: Continued

It has bounded variation if and only if $\sigma = 0$ and $\int_{-\infty}^{0} |x| \Pi(dx) < \infty$.

In that case one may rearrange the Laplace exponent into the form

$$\kappa(\lambda) = d\lambda - \int_{(-\infty,0)} (1 - e^{\lambda x}) \Pi(dx) \quad \text{for some } d > 0.$$ 

**Definition 1.** [Scale function] For a given SNLP $X$ with Laplace exponent $\kappa$ there exists for every $q \geq 0$ a increasing function $W^{(q)} : \mathbb{R} \to [0, \infty)$ such that $W^{(q)}(x) = 0$ for all $x < 0$ and otherwise is differentiable on $[0, \infty)$ satisfying,

$$\int_{0}^{\infty} e^{-\lambda x} W^{(q)}(x) dx = \frac{1}{\kappa(\lambda) - q}, \quad \text{for } \lambda > \Phi(q).$$

In the sequel, we will use the function $Z^{(q)}(x) := 1 + q \int_{0}^{x} W^{(q)}(y) dy$. The scale function $W^{(q)}(x)$ and $Z^{(q)}(x)$ appear in Laplace transform of exit times of SNLP.

In some cases, there are explicit expressions of the scale function $W^{(q)}(x)$. In general, one can apply numerical inversion of Laplace transform to compute $W^{(q)}(x)$. See for instance Surya (2008).
Our Main Results

Before we state our main results, let us define for all $B \in \mathbb{R}$ the following:

\[ G_j^{(q)}(B) := \int_0^\infty e^{-\Phi(q)y} f_j(y + B) dy, \quad j = 1, 2 \]

\[ H^{(q)}(B) := \int_0^\infty \Pi(du) \left\{ \int_0^u \! dz e^{-\Phi(q)z} [\eta(B) - \eta(B - (u - z))] \right\} \]

\[ J^{(r,m)}(B) := \left( \frac{r + m}{\Phi(r + m)} - \frac{r}{\Phi(r)} \right) \eta(B) - (H^{(r)}(B) - H^{(r+m)}(B)) \]

\[ K^{(r,m)}_1(B) := \frac{\kappa(1) - (r + m)}{1 - \Phi(r + m)} e^B - G^{(r+m)}_1(B) + G^{(r)}_2(B) - J^{(r,m)}(B) \]

\[ K^{(r)}_2(B) := G^{(r)}_2(B) + \frac{r}{\Phi(r)} \eta(B) + H^{(r)}(B) + \frac{\sigma^2}{2} \eta'(B). \]

We impose the following assumption throughout:

**Assumption 1.** $\int_0^\infty e^{-\Phi(q)x} f_j(x) dx < \infty, \quad j = 1, 2,$ for some $q > 0$.

**Assumption 2.** $\eta \in C^2(\mathbb{R})$ is bounded on $(-\infty, B)$ for a fixed $B \in \mathbb{R}$. 

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Remarks 1. The Assumption 2 is required just to show monotonicity of the function \( B \to E(x; B) \), but not necessary to later prove the optimality of default boundary. This makes the equity value \( E(x; B) \) is well defined for any \( x > B \).

Remarks 2. If \( \eta(B) \) is increasing in \( B \), then \( K_2^{(r)}(B) \) is uniformly positive.

Remarks 3. As \( \Phi(q) \) is increasing in \( q \) and \( \eta(.) > 0 \), \( J^{(r,m)}(B) \geq 0 \) \( \forall B \in \mathbb{R} \).

Also we define functions \( \mathcal{M}_j^{(q)}(x; B) \), \( j = 1, 2 \), \( \Lambda^{(q)}(x; B) \) and \( \Gamma^{(q)}(x) \) as follows

\[
\Lambda^{(q)}(x; B) = \eta(B) \left( Z^{(q)}(x - B) - \frac{q}{\Phi(q)} W^{(q)}(x - B) \right)
- W^{(q)}(x - B) H^{(q)}(B)
+ \int_0^\infty \Pi(du) \int_0^u dz W^{(q)}(x - z - B) \left[ \eta(B) - \eta(z + B - u) \right]
\]

\[
\mathcal{M}_j^{(q)}(x; B) = W^{(q)}(x - B) G_j^{(q)}(B) - \int_B^x W^{(q)}(x - y) f_j(y) dy
\]

\[
\Gamma^{(q)}(x) = \frac{\kappa(1) - q}{1 - \Phi(q)} W^{(q)}(x) + (\kappa(1) - q) e^x \int_0^x e^{-y} W^{(q)}(y) dy.
\]
Our Main Results: Continued

The corresponding debt and firm’s values are given by the following:

\[ D(x; B) = e^x - e^B \Gamma^{(r+m)}(x - B) + \mathcal{M}_1^{(r+m)}(x; B) - \Lambda^{(r+m)}(x; B) \]

\[ V(x; B) = e^x + \mathcal{M}_2^{(r)}(x; B) - \Lambda^{(r)}(x; B), \]

**Proposition 1.** It can be shown after some algebra that for a fixed \( x \in \mathbb{R} \)

\[ \frac{\partial}{\partial B} \mathcal{E}(x; B) = - \left( \Theta^{(r+m)}(x - B) K_1^{(r,m)}(B) \right. \]

\[ + \left. \left[ \Theta^{(r)}(x - B) - \Theta^{(r+m)}(x - B) \right] K_2^{(r)}(B) \right) \quad \forall B < x. \]

where \( \Theta^{(q)}(x) := W^{(q)'}(x) - \Phi(q)W^{(q)}(x) \) is the resolvent measure of the ascending ladder height process of \( X \). For a fixed \( x > 0, \Theta^{(q)}(x) \searrow q. \)

**Listing 2: Optimality**

If there exists \( B^* \) such that \( K_1^{(r,m)}(B) \geq 0 \iff B \geq B^* \) and \( K_2^{(r)}(B) \geq 0 \) for every \( B \geq B^* \), then \( B^* \), if it exists, is the optimal default boundary.
Sufficient Condition and Examples

Listing 3: Example

Suppose that (1) $\eta$ is increasing, (2) $\hat{\eta}$ is decreasing, (3) $\rho$ is decreasing, (4) $f_2$ is increasing, (5) $0 \leq \hat{\eta}(.) \leq 1$. Then $B^*$ is the optimal default boundary.

In other words, the optimality holds when, monotonically in the asset value,

- the loss amount at default is increasing,
- its proportion relative to the asset value is decreasing,
- the coupon rate is decreasing,
- and the value of tax benefits is increasing.
Sufficient Condition and Examples: Continued

We consider the case
\[
\hat{\eta}(x) = \eta_0(e^{-a(x-b)} \wedge 1), \quad f_2(x) = \tau P_\rho(e^{x-c} \wedge 1).
\]
and a constant coupon rate \( \rho \).

Regarding the source of randomness, we consider \( B \) as a jump diffusion process with \( \sigma > 0 \) and jumps of exponential type with Lévy measure: \( \Pi(dx) = \lambda \beta e^{-\beta x} dx \).

See, e.g., [5] and [3] for an explicit expression of the Scale function \( W^{(q)} \).

The model parameters used in the computation are given by \( r = 7.5\%, \delta = 7\%, \tau = 35\%, \sigma = 0.2, \lambda = 0.5, \beta = 9 \). We set \( V_0 = 100 \).

We look at two cases:

- Case 1: \( \eta_0 = 0.9, a = 0.5, b = 0 \) and \( c = 5 \).
- Case 2: \( \eta_0 = 0.5, a = 0.01, b = 5 \) and \( c = 0 \).
Numerical Results

Figure 1: The plots of $K_1^{(r,m)}(B)$. Applying the bisection method to $K_1^{(r,m)}(B) = 0$, we obtain $B^* = 3.61$ and $B^* = 3.64$ for cases 1 and 2, respectively.
Numerical Results: Continued

Figure 2: The equity/debt/firm values as a function of $V_0$ for various values of $B$.
Numerical Results: Continued

Figure 3: The equity/debt/firm values as a function of $V_0$ for various values of $B$. 

debt value (case 1) 

debt value (case 2)
Numerical Results: Continued

Figure 4: The firm value as a function of $P$ for the two-stage problem. The optimal face values of debt are given by $P^* = 73.7$ and $P^* = 39$, respectively.
Toward a Generalization of the Leland-Toft Optimal Capital Structure Model

Some Ideas for Future Works

These are some ideas which can be pursued to extend the current work.

- Allowing the jumps of $X$ to be two-sided, having an explicit Wiener-Hopf factorization, such as jump-diffusion process with exponential jumps or phase-type.
- Another consideration would be the following finite-time model. In case of the firm is in financial distress, the firm’s manager may be looking for a stopping time $\tau$ in a finite time interval $[0, t]$, such that the equity value of the firm is maximized. That is to say that he/she is trying to solve the optimal stopping problem:

$$
\sup_{0 \leq \tau \leq t} \mathcal{E}_x(\tau) := \mathcal{V}_x(\tau) - \mathcal{D}_x(\tau),
$$

where $\mathcal{V}_x(\tau)$ is the firm’s value defined as

$$
\mathcal{V}_x(\tau) := e^x + \mathbb{E}_x \left( \int_0^\tau e^{-r s} f_2(X_s) ds \right) - \mathbb{E}_x \left( e^{-r \tau} \eta(X_\tau) \right),
$$

whereas $\mathcal{D}_x(\tau)$ is the total debt outstanding of the firm, given by

$$
\mathcal{D}_x(\tau) := \mathbb{E}_x \left( \int_0^\tau e^{-(r+m) s} f_1(X_s) ds \right) + \mathbb{E}_x \left( e^{-(r+m) \tau} \exp(X_\tau)(1 - \hat{\eta}(X_\tau)) \right).
$$
Main References

References


