The Immersed Boundary Method with Porous Boundary

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Outline

- Motivation: Application Problems
- Introduction of the Immersed Boundary (IB) method
- IB method with Porous boundary
- Results
  - 2D Parachute
  - 2D Foam dynamics
  - 3D Foam dynamics
- Conclusions
**Motivation**

- Using the immersed boundary (IB) method,
- Investigate the interaction between a porous elastic material and fluid.
- Examples: Parachute, Foam, Lipid vesicles, and Cell membrane.
**Immersed Boundary Method**

- Two types of systems of equations:
  - Incompressible viscous flow (Eulerian)
  - Thin elastic material (Lagrangian)

- Interaction equations
  - Using the Dirac delta function
  - Elastic force in Lagrangian → Body force in Eulerian
  - Elastic boundary moves at a local fluid velocity (no slip condition)
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Equations of Motion

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\[ \rho\left(\frac{\partial u}{\partial t} + u \cdot \nabla u\right) = -\nabla p + \mu\nabla^2 u + f, \]

\[ \nabla \cdot u = 0, \]
\textbf{Equations of Motion}

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\[ \frac{\partial \mathbf{X}}{\partial t}(r, s, t) = \mathbf{u}(\mathbf{X}(r, s, t), t) \]

\[ = \int \mathbf{u}(\mathbf{x}, t) \delta(\mathbf{x} - \mathbf{X}(r, s, t))d\mathbf{x}. \]
Parachute with Porous Canopy

![Diagram of a parachute with porous canopy]

- $D_c$: Diameter of canopy
- $D_v$: Diameter of vent
- $h_s$: Height of suspension line

- Vent
- Suspension line
- Payload
1. Normal equilibrium of the boundary:

\[(p_1 - p_2) \left| \frac{\partial X}{\partial s} (s, t) \right| ds + \mathbf{F}(s, t) \cdot \mathbf{n} ds = 0.\]
1. Normal equilibrium of the boundary:

\[(p_1 - p_2) \left| \frac{\partial X}{\partial s} (s, t) \right| ds + F(s, t) \cdot nds = 0.\]

2. Darcy’s Law: Normal relative velocity is proportional to \((p_1 - p_2)\).
IB method with Porous Boundary
**IB method with Porous Boundary**

- Let \( \mathbf{U}(s, t) \) be fluid velocity at \( \mathbf{X}(s, t) \) and \( \frac{\partial \mathbf{X}}{\partial t}(s, t) \) be the boundary velocity.
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Normal flux: 
\[
(U(s, t) - \frac{\partial \mathbf{X}}{\partial t}(s, t)) \cdot \mathbf{n} = \beta \gamma (p_1 - p_2) ds,
\]
where \( \beta \) is the number density of pores and \( \gamma \) is the conductance.
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- Normal equilibrium of the boundary: \( (p_1 - p_2)\left| \frac{\partial X}{\partial s}(s,t) \right| ds + F(s,t) \cdot n ds = 0 \).

- \( (\frac{\partial X}{\partial t}(s,t) - U(s,t)) \cdot n = \frac{\beta\gamma}{\left| \frac{\partial X}{\partial s}(s,t) \right|} F(s,t) \cdot n \).

- \( \frac{\partial X}{\partial t}(s,t) = U(s,t) + \frac{\beta\gamma}{\left| \frac{\partial X}{\partial s}(s,t) \right|^2} (F(s,t) \cdot n) n \).
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- $\frac{\partial X}{\partial t}(s, t) = U(s, t) + \frac{\beta \gamma}{|\frac{\partial X(s, t)}{\partial s}|^2} (F(s, t) \cdot n)n$.

- Assuming $\lambda = \frac{\beta \gamma}{|\frac{\partial X(s, t)}{\partial s}|^2}$,

$$\frac{\partial X}{\partial t}(s, t) = u(X(s, t), t) + \lambda (F(s, t) \cdot n(s, t))n(s, t).$$
2-D Parachute with Porous Canopy

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Motion of 2-D Parachute
2D Foam Dynamics: von Neumann relation
$2D$ Foam Dynamics: von Neumann relation

- $M$: permeability; $\gamma$: surface tension; $\kappa$: mean curvature.
- Assume that the gas diffuses through the wall at a rate $-M \gamma \kappa$ per unit length.
- $\frac{dA}{dt} = -M \gamma \int_\Gamma \kappa \, ds$. 
2D Foam Dynamics: von Neumann relation

- \( M \): permeability; \( \gamma \): surface tension; \( \kappa \): mean curvature.
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- \[
\frac{dA}{dt} = -M \gamma \int_{\Gamma} \kappa \ ds.
\]
- \[
\frac{dA}{dt} = -M \gamma (2\pi - \sum_{i=1}^{n} \alpha_i) = -2\pi M \gamma (1 - n/6),
\]
  where \( \alpha_i \): exterior angle; \( n \): number of walls.
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where \( \alpha_i \): exterior angle; \( n \): number of walls.
- The rate of change of the area of a given cell is independent of cell size and solely dependent on the number of walls (or edges) of the cell.
- The area is constant for 6-sided bubbles, bubbles with fewer than 6 sides tend to shrink, and bubbles with more than 6 sides tend to grow.
2D Foam Dynamics: Force and Normal slip

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2D Foam Dynamics: Force and Normal slip

\[ F(s, t) = \frac{\partial}{\partial s} (\gamma \tau) = \gamma \frac{\partial \tau}{\partial s}. \]

\[ \tau(s, t) = \frac{\partial X}{\partial s} / \left| \frac{\partial X}{\partial s} \right|. \]
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\[ \tau(s, t) = \frac{\partial X}{\partial s} / \left| \frac{\partial X}{\partial s} \right|. \]

\[ \frac{\partial X}{\partial t} (s, t) = \int u(x, t) \delta(x - X(s, t))dx + M \frac{\mathbf{F}}{\left| \frac{\partial X}{\partial s} \right|}. \]
Foam with 3 inner cells

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2D von Neumann relation

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Foam in an oscillatory flow

foam animation in a dynamic flow
Foam with multiple inner cells: Coarsening

The general foam animation
Coarsening with T1 and T2 processes

- **T1 process**: switch of side of foam boundaries.
- **The effect of T1 process**: reduce by one the number of edges of two cells and increase by one the number of edges of two other cells.
- **T2 process**: removal of a three-sided bubble.
- **Other topological changes are also possible through the combinations of these two processes.**
Foam with 200 cells

general foam animation with topological changes
3D Foam Dynamics


\[ \frac{dV}{dt} = -2\pi M\gamma (L(D) - \frac{1}{6} \sum_{i=1}^{6} e_i(D)), \]
where \( L(D) \) is a natural measure of the linear size of domain \( D \) and \( e_i \) is the length of triple line (edge) \( i \).

- Discretized version of the 3D von Neumann relation:

\[ \frac{dV}{dt} = -M\gamma \sum_{e \in E} L_e \theta_e, \]

where \( L_e \) is the length of edge \( e \) of the triangular facet, and \( \theta_e \) is the angle between the two facets with the same edge \( e \).
3D foam: Continuous force and normal slip

- Let $\mathbf{X}(r, s, t)$ be a foam boundary,

$$
\mathbf{F}(r, s, t) = -\frac{\partial E}{\partial \mathbf{X}},
$$

- $E[\mathbf{X}(\cdot, \cdot, t)] = \gamma \int \left| \frac{\partial \mathbf{X}}{\partial r} \times \frac{\partial \mathbf{X}}{\partial s} \right| \, dr \, ds$, where $\gamma$ is the surface tension.
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• Normal slip is

\[
\frac{\partial X}{\partial t}(r, s, t) = u(X(r, s, t), t) + M F/ \left| \frac{\partial X}{\partial r} \times \frac{\partial X}{\partial s} \right|,
\]

\[
= \int u(x, t)\delta(x - X(r, s, t))dx + M F/ \left| \frac{\partial X}{\partial r} \times \frac{\partial X}{\partial s} \right|.
\]
After triangulation of the foam boundary,

\[
E[X^n] = \gamma \sum_k |T^k| = \gamma \sum_k \frac{1}{2} |(X^k_2 - X^k_1) \times (X^k_3 - X^k_1)|,
\]

where \( T^k \) is a triangle with vertices \( \{X^k_1, X^k_2, X^k_3\} \) and \(|T^k|\) is the area of the triangle \( T^k \).
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Using the formula \( F^k_1 \Delta r \Delta s = -\nabla_{X^k_1} E \), where \( F^k_1 \) is the force density acting on \( X^k_1 \).

\[ F^k_1 = -\frac{\gamma}{\Delta r \Delta s} \sum_{k=1} \frac{1}{2} \frac{\partial}{\partial X^k_1} |(X^k_2 - X^k_1) \times (X^k_3 - X^k_1)|, \]
3D foam: Discrete force and slip using Triangulation

- After triangulation of the foam boundary,

\[ E[\mathbf{X}^n] = \gamma \sum_k |T^k| = \gamma \sum_k \frac{1}{2} |(\mathbf{X}_2^k - \mathbf{X}_1^k) \times (\mathbf{X}_3^k - \mathbf{X}_1^k)|, \]

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- Using the formula \( \mathbf{F}_1^k \Delta r \Delta s = -\nabla_{\mathbf{X}_1^k} E \), where \( \mathbf{F}_1^k \) is the force density acting on \( \mathbf{X}_1^k \).

\[ \mathbf{F}_1^k = -\frac{\gamma}{\Delta r \Delta s} \sum_{k=1} \frac{1}{2} \frac{\partial}{\partial \mathbf{X}_1^k} |(\mathbf{X}_2^k - \mathbf{X}_1^k) \times (\mathbf{X}_3^k - \mathbf{X}_1^k)|, \]

\[ \mathbf{F}_1^k = \frac{\gamma}{2\Delta r \Delta s} \sum_{k=1} (\mathbf{X}_3^k - \mathbf{X}_2^k) \times \mathbf{n}_1^k, \]

where \( \mathbf{n}_1^k = (\mathbf{X}_2^k - \mathbf{X}_1^k) \times (\mathbf{X}_3^k - \mathbf{X}_1^k)/|(\mathbf{X}_2^k - \mathbf{X}_1^k) \times (\mathbf{X}_3^k - \mathbf{X}_1^k)|. \]
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where $n^k_1 = (X^k_2 - X^k_1) \times (X^k_3 - X^k_1)/|(X^k_2 - X^k_1) \times (X^k_3 - X^k_1)|$.

• $X^{n+1}_1 - X^n_1 = \sum_x u^{n+1}(x) \delta_h(x - X^n_1) h^3 + \frac{M F^k_1 \Delta r \Delta s}{\sum_{j=1} |T^{k,j}|/3}$. 
3D Foam Dynamics with a single inner cell

- permeability=0
- permeability=0.01
- permeability=0.05

[Diagram showing pressure and vorticity with different permeability values]
3D Foam Dynamics: 3D von Neumann relation

![Graph showing volume and angle deviation over time for permeability=0 and permeability=0.01 with different values of n.]
3D General Foam with 40 Cells

permeability=0.01
3D General Foam with 40 Cells

The Immersed Boundary Method with Porous Boundary
Conclusions and Future Work:

• The results verify 2D and 3D von Neumann relations.
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• The IB method can be extended to handle wet foam dynamics with Plateau borders.
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- The 3D simulation method should include the topological changes.
- The IB method can be extended to handle wet foam dynamics with Plateau borders.
- The IB method with porous boundary can be applied to the lipid vesicle and cell boundary.