Wave Propagation in Poroelastic Materials

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• Motivation

• Biot’s equation for poroelasticity

• Balance form and finite volume method

• Numerical results of a plane strain problem using CLAWPACK

• Application to cancellous bones in literature

• Numerical simulation for effective elastic moduli by boundary element method (BEM)

• Conclusion and Future Work
Images of cancellous bone and cortical bone

Main interest: Two classes of inverse problems for composite materials

- To infer statistical information from measurement of effective properties: Dehomogenization
- To infer effective properties from waves requires building an effective wave solver for the governing equation and a better inversion scheme [Buchanan-Gilbert-MYO-2011].
Effective permittivity with isotropic constituents

[Golden-Papanicolaou-83]

The complex permittivity of the medium is modeled by a (spatially) stationary random field \( \varepsilon(x, \eta) \), \( x \in \mathbb{R}^d \) and \( \eta \in \Omega \), where \( \Omega \) is the set of all realizations of the random medium,

\[
\varepsilon(x, \eta) = \varepsilon_1 \chi_1(x, \eta) + \varepsilon_2 \chi_2(x, \eta).
\]  

(1)

\( E(x, \eta) \): electric field, \( D(x, \eta) \): displacement field \( e_k \): unit vector in direction \( k \), \( < \cdot > \): ensemble average over \( \Omega \)

\[
\begin{align*}
\text{(PDE)} & \quad \nabla \cdot D = 0, \quad \nabla \times E = 0, \quad < E > = e_k. \\
\end{align*}
\]  

(2)

Effective complex permittivity \( \varepsilon^* \) \( \quad \varepsilon^* < E > \overset{\text{def}}{=} < D > \)
**Integral representation formula (IRF)**

**Theorem (Golden-Papanicolaou-83)**

Let $\epsilon^*$ be one of the diagonal component of the effective permittivity tensor and $s \overset{\text{def}}{=} 1/(1 - h), h \overset{\text{def}}{=} \epsilon_1/\epsilon_2$. Then $F(s) \overset{\text{def}}{=} 1 - \epsilon^*(h)/\epsilon_2$ is analytic off $[0, 1]$ in the $s$–plane with the integral representation

$$F(s) = 1 - \frac{\epsilon^*}{\epsilon_2} = \int_0^1 \frac{\mu \, dz}{s - z},$$

(3)

where the positive measure $\mu$ on $[0, 1]$ is the spectral measure of the self-adjoint operator $\Gamma \chi_1$, where $\Gamma = \nabla (-\Delta)^{-1} \nabla$. (Helmholtz projector).

Separation of contrast ($h$) and information of microstructure ($\mu$). This provides a characterization of microstructure (Cherkaev and Zhang). If the $\mu$ is known, then computation of the effective property amounts to only plug in the contrast $s$ into IRF.
Numerical results for measure $\mu$

Uniqueness of $\mu$ from data on an arc on the $\omega$-plane [E. Cherkaev 2001]

Figure: Reconstructions of the spectral function of Maxwell-Garnett composite using quadratic regularization (left figure) and nonnegativity constraint (right figure). [Cherkaev-MYO-08]
First few moments

Table: $\mu_n$ calculated for Maxwell-Garnett composite solving regularized problem for the spectral function $\mu$. Volume fractions: 11%, 25%, 43%. [Cherkaev-MYO-08]

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Note: Tik: Tikhonov; n-neg: Non-negativity
Relations between the moments of $\mu$ and the microstructure

[Golden-Papanicolaou-83]

$m(h) := \epsilon^*/\epsilon_2, \ h := \epsilon_1/\epsilon_2$, analytic outside $(-\infty, 0]$. Expanding $m$ at $h = 1$ and $F(s)$ at $s = \infty$

$$\frac{(-1)^{n-1}}{n!} m^{(n)}(1) = \int_0^1 z^{n-1} \mu(dz) =: \mu^{(n-1)}.$$

Differentiating $m$ at $h = 1$

$$m'(1) = \text{volume fraction of region occupied by } \epsilon_1.$$

$\mu^{(n-1)}$ is related to the integral of $n$-point correlation functions, $n = 1, 2, 3, \ldots$.

If the permittivity $\epsilon_1$ or $\epsilon_2$ depends on parameters such as frequency or temperature, we may reconstruct the moments from information of $\epsilon_1$, $\epsilon_2$ and $\epsilon^*$. 
Reconstruction of Moments

\[ F(s) = \int_0^1 \frac{\mu \, dz}{s - z} = \frac{\mu_0}{s} + \frac{\mu_1}{s^2} + \frac{\mu_2}{s^3} + \ldots, \quad \mu_n = \int_0^1 z^n \mu \, dz \leq \mu_0 \]

- Reconstruction of coefficients of analytic functions in Hardy space \( H^2 \) in the unit circle \( |\xi| < 1, \xi = 1/s \).

- Polynomial interpolation of the kernel function \( \frac{1}{s_k - z} \) using \( n \) nodes \( z_i, i, k = 1, \ldots, n \) on \([0, 1]\) – this results in an inversion scheme for the weights of a finite sum of Dirac measures supported at \( z_i \) whose moments are close to those of \( \mu \).

- \([L/M]\) Padé approximation of the effective function \( F(s) \)– this results in an inversion scheme for the coefficients of Padé approximants whose first \( L + M + 1 \) Taylor coefficients agree with those of \( F(s) \) at \( s = \infty \). Analytic continuation of \( F(s) \) into \( |s| < 1 \) with simple poles on \([0, 1]\).
Biot’s equations for waves in poroelastic materials


Three pieces of physics

1. Strain-stress relation
2. Kinetic energy
3. Energy dissipation

\( u \): displacement vector for solid
\( U \): displacement vector for fluid

* Stress acting on the solid part of each face: \( \tau_{ij}, \ i, j = 1, 2, 3 \)
* Stress tensor acting on the fluid part of each face: \( s\delta_{ij}, \ s = -\beta p, \ \beta: \text{porosity} \).
* Strain tensor in the solid: \( e_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}), \ i, j = 1, 2, 3 \)
* Strain tensor in the fluid: \( \epsilon = \nabla \cdot U \)

Coupled stress-strain relation \( \tau_{ij} = 2N e_{ij} + \delta_{ij}(A\epsilon + Q\epsilon), \ s = Q\epsilon + Re. \) Here \( e \overset{\text{def}}{=} \nabla \cdot u \).
Kinetic energy \( T = \frac{1}{2} \rho_{11} |\dot{u}|^2 + \rho_{12} (\dot{u} \cdot \dot{U}) + \frac{1}{2} \rho_{22} |\dot{U}|^2 \)

Dissipation function
\( D = b |\dot{u} - \dot{U}|^2 = b[(\dot{u}_1 - \dot{U}_1)^2 + (\dot{u}_2 - \dot{U}_2)^2 + (\dot{u}_3 - \dot{U}_3)^2] \)

*\( \hat{b} \) is frequency-dependent for high frequency \( \omega > \omega_c \)

Balance of force in \( x_1 \)-direction
Solid \( \sum_{j=1}^{3} \frac{\partial \tau_{1j}}{\partial x_j} = \frac{\partial}{\partial t} \left( \frac{\partial T}{\partial \dot{u}_1} \right) + \frac{\partial D}{\partial \dot{u}_1} \)

Fluid \( \frac{\partial s}{\partial x_1} = \frac{\partial}{\partial t} \left( \frac{\partial T}{\partial \dot{U}_1} \right) + \frac{\partial D}{\partial \dot{U}_1} \)

Dynamic equations
\( N \nabla^2 u + \nabla[(A + N)e + Q\varepsilon] = \frac{\partial^2}{\partial t^2}(\rho_{11}u + \rho_{12}U) + b \frac{\partial}{\partial t}(u - U) \)
\( \nabla[Qe + Re] = \frac{\partial^2}{\partial t^2}(\rho_{12}u + \rho_{22}U) - b \frac{\partial}{\partial t}(u - U) \)

Two longitudinal waves by taking \( \nabla \cdot \) on both sides to get wave equations
\( \nabla^2 (Pe + Q\varepsilon) = \frac{\partial^2}{\partial t^2}(\rho_{11}e + \rho_{12}\varepsilon) + b \frac{\partial}{\partial t}(e - \varepsilon) \)
\( \nabla^2 (Qe + Re) = \frac{\partial^2}{\partial t^2}(\rho_{12}e + \rho_{22}\varepsilon) - b \frac{\partial}{\partial t}(e - \varepsilon) \)

*\( P = A + 2N \)
Biot’s 1962 formulation in JAP

- Stress-strain relations $\mathbf{\tau} = C^u \cdot \mathbf{\epsilon}$

$$\mathbf{\tau} = [\tau_{11}, \tau_{22}, \tau_{33}, \tau_{13}, \tau_{12}, -p]^t$$

$$\mathbf{\epsilon} = [\epsilon_{11}, \epsilon_{22}, \epsilon_{33}, 2\epsilon_{23}, 2\epsilon_{13}, 2\epsilon_{12}, -\zeta]^t$$

$$\zeta = -div[\mathbf{\beta}(\mathbf{U} - \mathbf{u})] : \text{variation of Fluid content}$$

- Equations of motion

$$\partial_j (\tau_{ij}) = \rho \partial_t v_i + \rho_f \partial_t q_i + f_i, \ i = 1, 2, 3$$

$v$ solid velocity, $q = \mathbf{\beta}(\partial_t \mathbf{U} - \mathbf{v})$: fluid velocity rel. to solid, $\rho = (1 - \beta) \rho_s + \beta \rho_f$

- Dynamic Darcy’s Law

$$-\partial_i p = \rho_f \partial_t v_i + \psi_i(t) \star \partial_t q_i(t)$$

$\psi_i(t)$: viscodynamic operator
\[
\omega_c := \min_i \frac{\mu}{m_i \kappa_i}, \quad m_i := \frac{\rho_f T_i}{\beta}
\]

- For \( \omega < \omega_c \):
  \[
  \psi_i(t) = m_i \delta(t) + \frac{\mu}{\kappa_i} H(t)
  \]

  \[
  -\partial_i p = \rho_f \partial_t v_i + m_i \partial_t q_i + \frac{\mu}{\kappa_i} q_i
  \]

- For \( \omega \geq \omega_c \), approximate by the generalized Zener kernel
  \[
  \psi_i(t) = \psi_i^0 \left[ 1 - \frac{1}{L_i} \sum_{l=1}^{L_i} \left( 1 - \frac{\lambda_{il}}{\tau_{il}} \right) e^{-\frac{t}{\tau_{il}}} \right] H(t)
  \]
  \[
  \lambda_{il}, \tau_{il} : \text{relaxation times}, \lambda_{il} \geq \tau_{il}
  \]

  \[
  -\partial_i p = \rho_f \partial_t v_i + \psi_i^\infty q_i + \left( \psi_i^0 \sum_{l=1}^{L_i} \frac{\lambda_{il}}{\tau_{il}} \right) q_i + \psi_i^0 \sum_{l=1}^{L_i} e_{il}
  \]

  \[
  \partial_t e_{il} = \frac{-1}{\tau_{il}} e_{il} + \frac{1}{L_i \tau_{il}} \left( 1 - \frac{\lambda_{il}}{\tau_{il}} \right) q_i
  \]
A transversely isotropic case: plane strain problem

- Stress-strain relations

\[ \partial_t \tau_{xx} = c_{11}^u \partial_x u_x + c_{13}^u \partial_z u_z + \alpha_1 M (\partial_x q_x + \partial_z q_z) + \partial_t s_{11} \]  \hspace{1em} (4)
\[ \partial_t \tau_{zz} = c_{13}^u \partial_x u_x + c_{33}^u \partial_z u_z + \alpha_3 M (\partial_x q_x + \partial_z q_z) + \partial_t s_{33} \]  \hspace{1em} (5)
\[ \partial_t \tau_{xz} = c_{55}^u (\partial_z u_x + \partial_x u_z) + \partial_t s_{55} \]  \hspace{1em} (6)
\[ \partial_t p = -\alpha_1 M \partial_x u_x - \alpha_3 M \partial_z u_z - M (\partial_x q_x + \partial_z q_z) + \partial_t s_f \]  \hspace{1em} (7)

- Equations of motion

\[ \rho \partial_t u_x + \rho_f \partial_t q_x = \partial_x \tau_{xx} + \partial_z \tau_{xz} \]  \hspace{1em} (8)
\[ \rho \partial_t u_z + \rho_f \partial_t q_z = \partial_x \tau_{xz} + \partial_z \tau_{zz} \]  \hspace{1em} (9)

- Darcy's law for low frequency range

\[ -\partial_x p = \rho_f \partial_t u_x + m_1 \partial_t q_x + \left( \frac{\nu}{\kappa_1} \right) q_x \]  \hspace{1em} (10)
\[ -\partial_z p = \rho_f \partial_t u_z + m_3 \partial_t q_z + \left( \frac{\nu}{\kappa_3} \right) q_z \]  \hspace{1em} (11)
\[ \partial_t g + A \partial_x g + B \partial_z g = Dg + \partial_t s \]  

where

\[ \begin{align*}
g &= \begin{pmatrix} \tau_{xx} & \tau_{zz} & \tau_{xz} & v_x & v_z & p & q_x & q_z \end{pmatrix}^t \\
s &= \begin{pmatrix} \partial_t s_{11} & \partial_t s_{33} & \partial_t s_{55} & 0 & 0 & \partial_t s_f & 0 & 0 \end{pmatrix}^t 
\end{align*} \]

\[ A = - \begin{pmatrix}
0 & 0 & 0 & c_{11}^u & 0 & 0 & \alpha_1 M & 0 \\
0 & 0 & 0 & c_{13}^u & 0 & 0 & \alpha_3 M & 0 \\
0 & 0 & 0 & c_{55}^u & 0 & 0 & 0 & 0 \\
\frac{m_1}{\Delta_1} & 0 & 0 & 0 & 0 & \frac{\rho_f}{\Delta_1} & 0 & 0 \\
0 & \frac{m_3}{\Delta_3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\alpha_1 M & 0 & 0 & -M & 0 & 0 \\
\frac{-\rho_f}{\Delta_1} & 0 & 0 & 0 & 0 & \frac{-\rho}{\Delta_1} & 0 & 0 \\
0 & 0 & -\frac{\rho_f}{\Delta_3} & 0 & 0 & 0 & 0 & 0 
\end{pmatrix} \]
\[ \mathbf{B} = - \begin{pmatrix} 0 & 0 & 0 & 0 & c_{13}^u & 0 & 0 & \alpha_1 M \\ 0 & 0 & 0 & 0 & c_{33}^u & 0 & 0 & \alpha_3 M \\ 0 & 0 & 0 & m_1 / \Delta_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_3 / \Delta_3 & 0 & 0 & 0 & \rho_f / \Delta_3 & 0 \\ 0 & 0 & 0 & -\alpha_3 M & 0 & 0 & -M & 0 \\ 0 & 0 & -\rho_f / \Delta_3 & 0 & 0 & 0 & -\rho / \Delta_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \] (14)

\[ \mathbf{D} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \rho_f \nu / \Delta_{1\kappa_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho_f \nu / \Delta_{3\kappa_3} & 0 \\ 0 & 0 & 0 & 0 & 0 & -\rho \nu / \Delta_{1\kappa_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\rho \nu / \Delta_{3\kappa_3} & 0 \end{pmatrix} \] (15)
Finite Volume Method

- Normal and transverse Riemann problem solver
- Godunov operator splitting with $\Delta t = O(10^{-5})$
- MC limiter

Preliminary results
1. Materials with different porosities. Left: 0.2, Right: 0.5
2. AMRCLAW $\tau_{zz}$ and $p$ excited at point source by Ricker pulse with peak frequency of 3135 Hz, sandstone and shale
Ultrasound in cancellous bone and relevant effective parameters

Hosokawa 2006, *Ultrasonics*

### Table 3

Biot’s parameters of bovine cancellous bone models with parallel and perpendicular trabecular orientations

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<th>Parameter</th>
<th>Parallel</th>
<th>Perpendicular</th>
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<td>22.0 GPa</td>
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<td>Poisson’s ratio of solid bone $v_s$</td>
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<td>Bulk modulus of bone marrow $K_f$</td>
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<td>Density of bone marrow $\rho_f$</td>
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<td>Poisson’s ratio of trabecular frame $v_b$</td>
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<td>Viscosity of bone marrow $\eta$</td>
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<td>Resistance coefficients $\gamma_{ii}$ (i = x, y)</td>
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<td>Resistance coefficient $\gamma_{xy}$</td>
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Fig. 2. Geometric drawings of bovine cancellous bone models in the viscoelastic FDTD method, with (a) parallel and (b) perpendicular trabecular orientations.
Hosakawa’s result

As the excitation condition, pulsed particle displacements in the \( x \)-direction were given at the corresponding points on the transmitting line. The time function is a single sinusoid multiplied by a Hamming window:

\[
u(t) = U_0 \sin(2\pi f_0 t) \cdot \left(1 + \frac{C_0}{4} \cos(2\pi f_0 t) \right) \cdot 54/C_0 + \frac{C_0}{138} \text{at } 0 \leq t \leq 1/f_0; (7)\]

where \( U_0 \) is a constant and \( f_0 \) is a frequency. In the simulations, \( f_0 = 0.75 \text{ MHz} \) was given by considering the transmitting and receiving characteristics in the experimental settings. The output was calculated as the summation of the sound pressures at the receiving points. Higdon’s second-order absorbing boundary conditions [13] were adopted at the boundaries surrounding the analysis region.

The spatial intervals \( D_x \) and \( D_y \) in the \( x \)-and \( y \)-directions were the same at 25 \( \mu \)m, and the time interval \( D_t \) was 2.5 ns.

Fig. 4 shows the viscoelastic FDTD (upper), Biot’s FDTD (middle) and experimental (lower) results of the pulsed waveforms propagating through bovine cancellous bone, together with the experimentally observed results (lower). Fig. 4 (a) shows the waveforms for the propagation in the direction parallel to the trabecular alignment, and Fig. 4 (b) shows the waveforms in the perpendicular direction. The experimental waveforms in the parallel and perpendicular directions were given in Refs. [3] and [5], respectively. Both the Biot’s fast and slow longitudinal waves can be clearly observed in the parallel direction. These waves in the perpendicular direction overlap completely because the propagation speed of the fast wave in the perpendicular direction is much slower than the speed in the parallel direction and because the slow wave is more highly attenuated. As a result, a single wave can be observed.

The simulated results obtained for the propagation parallel to the trabecular orientation were previously investigated [9]. As shown in Fig. 4 (a), both the fast and slow waves could be analyzed using Biot’s FDTD method, whereas it was difficult to exactly analyze the fast wave using the model with an insufficiently connected trabecular frame (namely the model with a low elastic frame) in the viscoelastic FDTD method because the fast wave corresponds to the motion of the frame. On the other hand, it is shown in Fig. 4 (b) that, using Biot’s FDTD method, the simulated waveform in the perpendicular direction is not in good agreement with the experimental waveform. The slower speed of the fast wave and the higher attenuation of the slow wave, which are the propagation characteristics in the perpendicular direction, could be analyzed by increasing value of the parameter \( n \) and by decreasing value of the permeability \( k \) because \( n \) and \( k \) are inversely proportional.
Fellah et. al. 2004, 2006, *JASA*

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<td>Thickness L (cm)</td>
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<td>1960</td>
<td>1960</td>
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<td>Porosity $\phi$</td>
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</tbody>
</table>

FIG. 15. Experimental setup for ultrasonic measurements.

In Fellah 2006, $\phi$, $\alpha_{\infty}$, $\Lambda$, $K_b$ and $N$ are sought after by solving the inverse problem.
Fellah et al’s results

Solid line: Experimental transmitted signal; Dashed line: Simulated transmitted signal
Translation between microstructure and effective parameters

Numerical simulation from images of cancellous bone to some effective properties. Ongoing numerical projects focusing on cancellous bones.

**Step 1** Image segmentation and 3D reconstruction from \( \mu\)-CT scans

(a) raw \( \mu\)-CT scan  (b) digitalized data  (c) 3-D structure

Figure: \( \mu\)-CT scans are provided by Prof. L. Cardoso from CUNY Med. Center
Numerical projects for effective parameters of cancellous bone, SCBLAP

with Dr. S. Nincheu at ORNL

Step 2 Meshing and solving (BVP) by BEM

Advantage Meshing only the interface and outer surface, not the entire domain.

(a) meshed trabeculae

(b) electric potential

Figure: Our preliminary results
Effective dielectric tensor $\epsilon^*$

\[
\text{(BVP)} \left\{ \begin{array}{lcl}
\text{Div} \left[ (\epsilon_1 \chi_1(x) + \epsilon_2 \chi_2(x)) \nabla \phi \right] &=& 0, \\
\phi |_{\partial \Omega} &=& \sum_{j=1}^{d} E_j^0 x_j
\end{array} \right.
\]

with the constant vector $E^0 = e^j$, the $j$-th unit vector in Cartesian coordinates ($(e^j)_i = \delta_{ij}$) followed by evaluating the integral

\[
\epsilon^*_{ik} \overset{\text{def}}{=} \frac{1}{|\Omega|} \int_{\Omega} \left[ \epsilon_1 \chi_1(x) + \epsilon_2 \chi_2(x) \right] \partial_i \phi \, dx
\]

\[
= \frac{1}{|\Omega|} \left[ \epsilon_1 \int_{\partial \Omega_1} \phi_1 n_1 \, dS + \epsilon_2 \int_{\partial \Omega_2} \phi_2 n_2 \, dS \right]
\]
Illustration of ideas

To solve Laplace equation: $\triangle u = 0$ in a simply connected bounded domain $\Omega$ with boundary $\Gamma$ and constant boundary elements.

Key: Green's identity

$$\int_{\Gamma} G(x, y) t(y) d\Gamma_y - \int_{\Gamma} H(x, y) \cdot n(y) u(y) d\Gamma_y = \begin{cases} u(x), & x \in \Omega \\ 0, & x \in \mathbb{R}^3 \setminus \overline{\Omega} \end{cases}$$

1. Triangulate $\Gamma$ into $N$ non-overlapping elements $\Gamma_j$, $j = 1, \ldots, N$.
2. Define $g_j(x) := \int_{\Gamma_j} G(x, y) d\Gamma_y$ and $h_j(x) := \int_{\Gamma_j} H(x, y) \cdot n(y) d\Gamma_y$.
3. Discretized system is $G\{t\} = H\{u\}$, with $u = (u^1, \ldots, u^N)^T$ and $u = (t^1, \ldots, t^N)^T$
   
   $G_{ij} = \lim_{x_{\epsilon} \to x^j} g_j(x_{\epsilon}), \quad H_{ij} = \lim_{x_{\epsilon} \to x^j} h_j(x_{\epsilon}), \quad x^j \in \Gamma_j, \ j = 1, \ldots, N$.
4. Substitute in boundary data and rearrange to obtain
   
   $$A\{z\} = \{b\}$$
   
   with $\{z\}$ containing those $u^i$ and $t^j$ which are unknown, and $\{b\}$ with the boundary data.

5. Solve the linear system to obtain $u^j$ and $t^j$, $j = 1, \ldots, N$.
6. For any $x \in \Omega$, we have

$$u(x) = \sum_{j=1}^N t^j g_j(x) - \sum_{j=1}^N u^j h_j(x)$$
### 3. Real Trabecula and Marrow

A small piece of trabecula is used in this test case. The following are the results:

![Figure (viii) Trabecula and Marrow](image)

![Figure (ix) Cross-sectional View at z = 15 (Blue = Trabecula, Red = Marrow)](image)

| Number of Iteration | $|\Omega|\epsilon_{11}^*$ | $|\Omega|\epsilon_{12}^*$ | $|\Omega|\epsilon_{13}^*$ | Energy Norm |
|---------------------|--------------------------|--------------------------|--------------------------|-------------|
| 1                   | 1065.41269091            | -4.50359725              | -5.43014418              | 11.19163210 |
| 2                   | 1074.38794353            | -2.08527009              | -2.1662289               | 11.15119673 |
| 3                   | 1072.75162457            | -2.78742893              | -3.36231617              | 11.15116477 |
| 4                   | 1073.13390244            | -2.58523184              | -2.96664855              | 11.15084327 |
| 5                   | 1073.03101623            | -2.64195294              | -3.09887452              | 11.15086061 |
| 6                   | 1073.06116521            | -2.62658027              | -3.05355851              | 11.15085455 |
| 7                   | 1073.05182945            | -2.63055032              | -3.06947068              | 11.15085535 |
| 8                   | 1073.05483621            | -2.62960307              | -3.06377016              | 11.15085501 |
| 9                   | 1073.05383794            | -2.62979556              | -3.06584441              | 11.15085514 |
| 10                  | 1073.05417768            | -2.62977221              | -3.06508078              | 11.15085508 |
| 11                  | 1073.05405965            | -2.62976631              | -3.06536433              | 11.15085510 |
| 12                  | 1073.05410136            | -2.62977293              | -3.06525839              | 11.15085509 |
| 13                  | 1073.05408641            | -2.62976912              | -3.06529814              | 11.15085510 |
| 14                  | 1073.05409183            | -2.62977095              | -3.06528318              | 11.15085510 |
| 15                  | 1073.05408985            | -2.62977014              | -3.06528882              | 11.15085510 |
| 16                  | 1073.05409058            | -2.62977048              | -3.06528669              | 11.15085510 |
| 17                  | 1073.05409031            | -2.62977034              | -3.06528750              | 11.15085510 |
| 18                  | 1073.05409041            | -2.62977040              | -3.06528719              | 11.15085510 |
| 19                  | 1073.05409037            | -2.62977037              | -3.06528731              | 11.15085510 |
| 20                  | 1073.05409039            | -2.62977038              | -3.06528726              | 11.15085510 |
| 21                  | 1073.05409038            | -2.62977038              | -3.06528728              | 11.15085510 |
| 22                  | 1073.05409038            | -2.62977038              | -3.06528727              | 11.15085510 |
To generalize dehomogenization schemes for viscoelastic composites by using the 2-parameter IRF [MYO-2011]

To develop a better image processing scheme in order to handle bone sample of realistic size.

To derive integral representation formula for permeability tensor.

To compare the simulated results with lab measurement.

To build highly parallel numerical codes for effective parameters computation for bones.

Numerical analysis of the FVM and BEM codes.
Thank you.