LECTURE NOTE ON LOCAL LANGLANDS CONJECTURE

CONTENTS

1. LOCAL LANGLANDS CORRESPONDENCE FOR $\mathrm{GL}_n(F)$

The references for this section are [Tat79], [How77], [Moy86] and [Zel80].

1.1. Notation.

Let F be a nonarchimedean local field of characteristic zero, $\mathfrak{o}_F$ ring of integers, $\mathfrak{p}_F := \varpi_F \mathfrak{o}_F$ the maximal ideal, $\mathfrak{f}_F$ be the residue field of F, which is a finite field of q elements. Let p denote the characteristic of $\mathfrak{f}_F$. For simplicity, we shall assume that p is odd.

The chain of $\mathfrak{o}_F$-modules

$$\cdots \subset \mathfrak{p}_F^2 \subset \mathfrak{p}_F^1 \subset \mathfrak{o}_F \subset \mathfrak{p}_F^{-1} \subset \mathfrak{p}_F^{-2} \subset \cdots$$

forms a filtration of F. If x is a nonzero element of F, then there exists a unique $k \in \mathbb{Z}$ such that $x \in \mathfrak{p}_F^k \smallsetminus \mathfrak{p}_F^{k+1}$ and we define a map

$$\mathrm{ord}_F \colon F^\times \longrightarrow \mathbb{Z}$$

by $\mathrm{ord}_F(x) := k$. Define the *norm map*

$$|\cdot|_F \colon F^\times \longrightarrow \mathbb{C}^\times \quad \text{by } |x|_F := q^{-\mathrm{ord}_F(x)}.$$

We have a filtration of subgroups of $F^\times$:

$$\cdots \subset U_F^3 \subset U_F^2 \subset U_F^1 \subset U_F^0 \subset F^\times$$

where

$$U_F^i := \begin{cases} \{\, x \in F^\times \mid \mathrm{ord}_F(x) = 0 \,\} & \text{if } i = 0; \\ 1 + \mathfrak{p}_F^i := \{\, 1 + x \mid x \in \mathfrak{p}_F^i \,\} & \text{if } i \geq 1. \end{cases}$$

1.2. Weil Group and Weil-Deligne Group.

Let $\overline{F}$ be an algebraic closure of F. The *Weil group* W_F of F is the subgroup of Galois group $\Gamma := \mathrm{Gal}(\overline{F}/F)$ defined by the commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & I_F \arrow[r] \arrow[d, equal] & W_F \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & I_F \arrow[r] & \Gamma \arrow[r] & \mathrm{Gal}(F^{\mathrm{ur}}/F) \arrow[r] & 1
\end{tikzcd}$$

where $I_F := \mathrm{Gal}(\overline{F}/F^{\mathrm{ur}})$ is the *inertia group* of F and F^{ur} is the maximal unramified extension of F in $\overline{F}$. Hence we have an isomorphism $\mathrm{Gal}(F^{\mathrm{ur}}/F) \simeq \mathrm{Gal}(\overline{\mathfrak{f}}_F/\mathfrak{f}_F)$ where $\overline{\mathfrak{f}}_F$ is an algebraic closure of $\mathfrak{f}_F$. Fix an element ζ_F in W_F which maps to Fr of $\mathrm{Gal}(\overline{\mathfrak{f}}_F/\mathfrak{f}_F)$ where Fr is the Frobenius automorphism of $\overline{\mathfrak{f}}_F$. Such an element ζ_F is called a *geometric Frobenius element.* We have the disjoint union decomposition

$$W_F = \bigcup_{k \in \mathbb{Z}} \zeta_F^k I_F.$$

Theorem 1.1 (Local Class Field Theory). *Let W_F^{ab} denote the abelianization of W_F, i.e., W_F modulo the closure of its commutator subgroup. Then*

$$W_F^{\mathrm{ab}} \simeq F^\times.$$

Let $\Pi(\mathrm{GL}_1(F))$ denote the set of isomorphism classes of irreducible admissible representations of $\mathrm{GL}_1(F) \simeq F^\times$. Let $\Phi(\mathrm{GL}_1(F))$ denote the set of isomorphism classes of homomorphisms from W_F to $\mathrm{GL}_1(\mathbb{C}) \simeq \mathbb{C}^\times$ (the "*dual group*" of $\mathrm{GL}_1(F)$). Theorem 1.1 indicates a natural bijection between $\Pi(\mathrm{GL}_1(F))$ and $\Phi(\mathrm{GL}_1(F))$. This is the Local Langlands Correspondence for $\mathrm{GL}_1(F)$.

Let

$$W_F' := W_F \times \mathrm{SL}_2(\mathbb{C})$$

be the *Weil-Deligne group* of F. The product topology makes W_F' a locally compact group, and the inclusion $W_F \hookrightarrow \Gamma$ induces a natural homomorphism

$$W_F' \longrightarrow W_F \longrightarrow \Gamma$$

with dense image.

Let $\rho'\colon W_F' \to \mathrm{GL}_n(\mathbb{C})$ be an n-dimensional representation of W_F'. Define

$$\rho(w) := \rho'\left(w, \begin{bmatrix} |w|_F^{-1/2} & 0 \\ 0 & |w|_F^{1/2} \end{bmatrix}\right)$$

$$\exp(N) := \rho'\left(1, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\right).$$

Then $\rho\colon W_F \to \mathrm{GL}_n(\mathbb{C})$ is a representation of the Weil group W_F, N is a nilpotent endomorphism of $\mathbb{C}^n$ and

$$\rho(w)N\rho(w)^{-1} = |w|_F N$$

for all $w \in W_F$. Hence a representation of W_F' can be presented by a pair (ρ, N) satisfying above conditions. The representation ρ' of W_F' is irreducible if and only if the representation ρ of W_F is irreducible and $N = 0$.

Let $\Pi(\mathrm{GL}_n(F))$ denote the set of isomorphism classes of irreducible admissible representations of $\mathrm{GL}_n(F)$. Let $\Phi(\mathrm{GL}_n(F))$ denote the set of isomorphism classes continuous homomorphism $W_F' \to \mathrm{GL}_n(\mathbb{C})$ (the "*dual group*" of $\mathrm{GL}_n(F)$).

Theorem 1.2 (Local Langlands Correspondence for $\mathrm{GL}_n(F)$). *There exists a natural bijection*

$$\Pi(\mathrm{GL}_n(F)) \longleftrightarrow \Phi(\mathrm{GL}_n(F))$$

such that $\pi \in \Pi(\mathrm{GL}_n(F))$ is supercuspidal if and only if the corresponding $\rho' \in \Phi(\mathrm{GL}_n(F))$ is irreducible.

The homomorphism ρ' is called the *Langlands parameter* of the representation π.

1.3. Tame Local Langlands Correspondence of Supercuspidal Representations of $\mathrm{GL}_n(F)$.

Let $\Pi^0(\mathrm{GL}_n(F)) \subset \Pi(\mathrm{GL}_n(F))$ denote the set of isomorphism classes of irreducible supercuspidal representations of $\mathrm{GL}_n(F)$. Let $\Phi^0(\mathrm{GL}_n(F)) \subset \Phi(\mathrm{GL}_n(F))$ denote the set of isomorphism classes irreducible n-dimensional representations of W_F'. Note that the set $\Phi^0(\mathrm{GL}_n(F))$ is equal to the set of isomorphism classes of irreducible n-dimensional representations of W_F.

Definition 1.3 (Howe). A pair (E,ψ) is called an *admissible pair of F of degree n* if E is an extension field of F of degree n and $\psi\colon E^\times \to \mathbb{C}^\times$ is a character of $E^\times$ such that

- if there is extension field E' and a character ψ' of $E'^\times$ such that $F \le E' \le E$ and $\psi = \psi' \circ \mathrm{Nr}_{E/E'}$, then $E' = E$;
- if $\psi|_{U_E^1} = \psi' \circ \mathrm{Nr}_{E/E'}|_{U_E^1}$ where $F \le E' \le E$ and ψ' is a character of $U_{E'}^1$, then E/E' is unramified.

Two admissible pairs (E_1,ψ_1) and (E_2,ψ_2) are said to be *equivalent* if there is a field isomorphism $h\colon E_1 \to E_2$ over F such that $\psi_1 = \psi_2 \circ h|_{E_1^\times}$.

By Theorem 1.1, we have $W_E^{\mathrm{ab}} \simeq E^\times$, thus ψ is regarded as a character of W_E.

Proposition 1.4. *Assume that* $\gcd(n,p)=1$. *Let* (E,ψ) *be an admissible pair of* F *of degree* n.

(i) *The induced representation* $\mathrm{Ind}_{W_E}^{W_F}\psi$ *is an irreducible representation of* W_F *of degree* n.

(ii) *The map* $\rho'\colon (E,\psi) \mapsto \mathrm{Ind}_{W_E}^{W_F}\psi$ *induces a bijection between the set of equivalence classes of admissible pairs of F of degree n and the set of isomorphism classes of irreducible representations of W_F of degree n.*

Now we want to construct an irreducible supercuspidal representation $\pi(E,\psi)$ of $\mathrm{GL}_n(F)$ from an admissible pair (E,ψ) of F of degree n

Proposition 1.5 (Howe). *Let* (E,ψ) *be an admissible pair,* $E_0 := E$, *and, for* $i \ge 1$, *let* E_i/F *be the minimal subextension of* E/F *such that* $\psi|_{U_E^i}$ *factor through the norm* Nr_{E/E_i}. *Then, for each* $i \ge 0$, *there exists a character* ψ_i *of* $E_i^\times$ *satisfying the following conditions:*

(1) *for almost all* i, $\psi_i = 1$;

(2) $\psi = \prod_{i\ge 0}\psi_i \circ \mathrm{Nr}_{E/E_i}$.

The Howe factorization associates to an admissible pair (E,ψ) a chain of field extension

$$F < E_{i_r} < E_{i_{r-1}} < \cdots < E_{i_1} < E$$

where i_j are those i not in (1) of Proposition 1.5.

Regard E as an n-dimensional vector space over F. Then $\mathfrak{p}_E^k$ is a lattice in E for any k. Let

$$\begin{aligned} A &:= \mathrm{End}_F(E) = M_n(F),\\ \mathfrak{o}_A &:= \{\, x \in A \mid x(\mathfrak{p}_E^k) \subseteq \mathfrak{p}_E^k \text{ for each integer } k \,\},\\ \mathfrak{p}_A &:= \{\, x \in A \mid x(\mathfrak{p}_E^k) \subseteq \mathfrak{p}_E^{k+1} \text{ for each integer } k \,\}. \end{aligned}$$

Note that $A^\times = \mathrm{GL}_n(F)$. Multiplying by an element of E is clearly a vector space endomorphism of E over F. Hence, E is regarded as an F-subalgebra of A. It is clear to see that $\mathfrak{o}_A \cap E = \mathfrak{o}_E$ and $\mathfrak{p}_A \cap E = \mathfrak{p}_E$.

Define

$$\begin{aligned} A_i &:= \mathrm{End}_{E_i}(E) \subset A, \\ S &:= E^\times(\mathfrak{o}_A \cap A_1)^\times(1 + \sum_{i \geq 1} \mathfrak{p}_A^i \cap A_{2i+1}). \end{aligned}$$

Note that S is a compact modulo center open subgroup of $\mathrm{GL}_n(F)$.

Proposition 1.6 (Howe). *To an admissible pair (E, ψ), we can associate a representation ζ of S such that the induced representation*

$$\pi = \pi(E, \psi) := c\text{-}\mathrm{Ind}_S^{\mathrm{GL}_n(F)} \zeta.$$

is an irreducible supercuspidal representation of $\mathrm{GL}_n(F)$.

Proposition 1.7 (Moy). *Assume that* $\gcd(n, p) = 1$. *Every irreducible supercuspidal representation of $\mathrm{GL}_n(F)$ is of the form $\pi(E, \psi)$ for some admissible pair (E, ψ) of degree n.*

So now we have a bijection (under the assumption $\gcd(n, p) = 1$) between $\Pi^0(\mathrm{GL}_n(F))$ and $\Phi^0(\mathrm{GL}_n(F))$ given by

$$\pi(E, \psi) \longleftrightarrow \rho'(E, \psi)$$

where (E, ψ) is an admissible pair of F of degree n.

1.4. **Local Langlands Correspondence for $\mathrm{GL}_n(F)$.** Let $G := \mathrm{GL}_n(F)$. Suppose $n = mr$ tentatively. For the partition $n = m + m + \cdots + m$ (r times) and for an irreducible supercuspidal representation η of $\mathrm{GL}_m(F)$, we call a representation of $\mathrm{GL}_m(F) \times \cdots \times \mathrm{GL}_m(F)$ of the form

$$\begin{aligned} \Delta &:= \{\eta, \eta|\cdot|_F, \ldots, \eta|\cdot|_F^{r-1}\} \\ &= \eta \otimes \eta|\cdot|_F \otimes \eta|\cdot|_F^2 \otimes \cdots \otimes \eta|\cdot|_F^{r-1} \end{aligned}$$

a *segment* of length r. Let P be a parabolic subgroup of $\mathrm{GL}_n(F)$ with Levi factor

$$\mathrm{GL}_m(F) \times \cdots \times \mathrm{GL}_m(F).$$

The induced representation $\mathrm{Ind}_P^{\mathrm{GL}_n(F)}(\Delta)$ is reducible.

Proposition 1.8. *Let Δ be a segment for $\mathrm{GL}_n(F)$. The induced representation $\mathrm{Ind}_P^G(\Delta)$ has a unique irreducible quotient $Q(\Delta)$*

The unique irreducible quotient $Q(\Delta)$ is called the *Langlands quotient* of the induced representation $\mathrm{Ind}_P^G(\Delta)$.

Example 1.9. If $n = 1 + 1 + \cdots + 1$ is a partition of n and $\eta = |\cdot|_F^{\frac{1-n}{2}}$, then $P = B$, the Borel subgroup, and

$$\Delta = |\cdot|_F^{\frac{1-n}{2}} \otimes |\cdot|_F^{\frac{3-n}{2}} \otimes \cdots \otimes |\cdot|_F^{\frac{n-1}{2}}.$$

Then $Q(\Delta)$ is the *Steinberg representation* or the *special representation*. Here we have one segment of length n.

Two segments

$$\Delta_1 := \{\eta_1, \eta_1|\cdot|_F, \dots, \eta_1|\cdot|_F^{r_1-1}\}$$
$$\Delta_2 := \{\eta_2, \eta_2|\cdot|_F, \dots, \eta_2|\cdot|_F^{r_2-1}\}$$

are said to be *linked* if $\Delta_1 \not\subset \Delta_2$, $\Delta_2 \not\subset \Delta_1$ and $\Delta_1 \cup \Delta_2$ is a segment. We say that Δ_1 *precedes* Δ_2 if Δ_1 and Δ_2 are linked and if $\eta_2 = \eta_1|\cdot|_F^k$ for some positive integer k.

Example 1.10. $\{\eta, \eta|\cdot|_F, \eta|\cdot|_F^2\}$ and $\{\eta|\cdot|_F, \eta|\cdot|_F^2, \eta|\cdot|_F^3\}$ are linked and the first precedes the second.

Proposition 1.11 (Bernstein-Zelevinski). *Given segments $\Delta_1, \dots, \Delta_r$ assume that for $i < j$, Δ_i does not precede Δ_j.*

- (i) *The induced representation $\mathrm{Ind}_P^G(Q(\Delta_1) \otimes \cdots \otimes Q(\Delta_r))$ admits a unique irreducible quotient $Q(\Delta_1, \dots, \Delta_r)$.*
- (ii) *If $\Delta'_1, \dots, \Delta'_s$ is another collection of segments satisfying the "does not precede" condition, then $Q(\Delta_1, \dots, \Delta_r) \simeq Q(\Delta'_1, \dots, \Delta'_s)$ if and only if $r = s$ and $\Delta'_i = \Delta_{\tau(i)}$ for some permutation τ.*
- (iii) *Every irreducible admissible representation π of $\mathrm{GL}_n(F)$ is isomorphic to some $Q(\Delta_1, \dots, \Delta_r)$.*
- (iv) *The induced representation $\mathrm{Ind}_P^G(Q(\Delta_1) \otimes \cdots \otimes Q(\Delta_r))$ is irreducible if and only if no two of the segments Δ_i and Δ_j are linked.*

Example 1.12. The trivial representation is isomorphic to

$$Q(|\cdot|_F^{\frac{n-1}{2}}, |\cdot|_F^{\frac{n-3}{2}}, \dots, |\cdot|_F^{\frac{1-n}{2}}).$$

Here we have n segments each of which has length 1.

Denote by Φ^0 the set of isomorphism classes of irreducible finite-dimensional representations of W_F. Call a *segment* in Φ^0 any subset Δ' of the form

$$\Delta' = \{\sigma, \sigma|\cdot|_F, \dots, \sigma|\cdot|_F^{k-1}\}$$

for some $k \in \mathbb{N}$.

To each segment Δ', define

$$\sigma(\Delta') = \sigma \oplus \sigma|\cdot|_F \oplus \cdots \oplus \sigma|\cdot|_F^{k-1}.$$

Let V_i be the space of the representation $\sigma|\cdot|_F^i$. Define $N(\Delta') \in \mathrm{End}_{\mathbb{C}}(\bigoplus_{i=0}^{k-1} V_i)$ via $N(\Delta')|_{V_0} = 0$ and $N(\Delta')\colon V_i \to V_{i-1}$ be the identity isomorphism for $i = 1, \dots, k-1$, i.e., N is of the block form

$$\begin{bmatrix} 0 & 0 & \cdots & 0 \\ I & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & I & 0 \end{bmatrix}$$

Finally define $\tau(\Delta') := (\sigma(\Delta'), N(\Delta'))$ be an object in Φ, i.e., a finite-dimensional representation of W'_F. We may denote $\tau(\Delta')$ by $\sigma \otimes \mathrm{sp}(k)$

Proposition 1.13. (a) *The representations $\tau(\Delta')$ of W'_F are indecomposable and mutually non-isomorphic, and each indecomposable representation of W'_F is of this form.*

(b) *Each representation of W'_F decomposes into the direct sum $\tau(\Delta'_1)\oplus\cdots\oplus\tau(\Delta'_r)$. This decomposition is unique up to permutation.*

Suppose now a bijection between $\Pi^0(\mathrm{GL}_m(F))$ and $\Phi^0(\mathrm{GL}_m(F))$ is given for any m. Then we can obtain a bijection between $\Pi(\mathrm{GL}_n(F))$ and $\Phi(\mathrm{GL}_n(F))$ as follows. Let π be an irreducible admissible representation of $\mathrm{GL}_n(F)$. By Proposition 1.11, $\pi \simeq Q(\Delta_1,\dots,\Delta_r)$ for some segments $\Delta_1,\dots,\Delta_r$. Suppose

$$\Delta = \{\eta, \eta|\cdot|_F, \dots, \eta|\cdot|_F^{k-1}\}$$

is a segment for some irreducible supercuspidal representation η of $\mathrm{GL}_m(F)$ for some m. Then there is an irreducible m-dimensional representation σ of the Weil group W_F corresponding to η. Then we have a segment $\Delta' = \{\sigma, \sigma|\cdot|_F, \dots, \sigma|\cdot|_F^{k-1}\}$ and an object $\tau(\Delta') = (\sigma(\Delta'), N(\Delta'))$. Then the direct sum

$$\tau(\Delta'_1)\oplus\tau(\Delta'_2)\oplus\cdots\oplus\tau(\Delta'_r)$$

is an n-dimensional representation of W'_F. The correspondence

$$Q(\Delta_1,\dots,\Delta_r) \longleftrightarrow \tau(\Delta_1)\oplus\cdots\oplus\tau(\Delta_r)$$

is the desired bijection between $\Pi(\mathrm{GL}_n(F))$ and $\Phi(\mathrm{GL}_n(F))$.

Example 1.14. Let χ be a character of $F^\times$. We also regard χ as a one-dimensional representation of W_F. The the induced representation

$$\mathrm{Ind}_B^G(\chi|\cdot|_F^3\otimes\chi|\cdot|_F^2\otimes\chi|\cdot|_F\otimes\chi)$$

of $\mathrm{GL}_4(F)$ has 8 irreducible constituents. The Langlands parameters and their corresponding representations are

$\Phi(\mathrm{GL}_4(F))$	$\Pi(\mathrm{GL}_4(F))$
$\chi\vert\cdot\vert_F^3\oplus\chi\vert\cdot\vert_F^2\oplus\chi\vert\cdot\vert_F\oplus\chi$	$Q([\chi\vert\cdot\vert_F^3],[\chi\vert\cdot\vert_F^2],[\chi\vert\cdot\vert_F],[\chi])$
$(\chi\vert\cdot\vert_F^2\otimes\mathrm{sp}(2))\oplus\chi\vert\cdot\vert_F\oplus\chi$	$Q([\chi\vert\cdot\vert_F^2,\chi\vert\cdot\vert_F^3],[\chi\vert\cdot\vert_F],[\chi])$
$\chi\vert\cdot\vert_F^3\oplus(\chi\vert\cdot\vert_F\otimes\mathrm{sp}(2))\oplus\chi$	$Q([\chi\vert\cdot\vert_F^3],[\chi\vert\cdot\vert_F,\chi\vert\cdot\vert_F^2],[\chi])$
$\chi\vert\cdot\vert_F^3\oplus\chi\vert\cdot\vert_F^2\oplus(\chi\otimes\mathrm{sp}(2))$	$Q([\chi\vert\cdot\vert_F^3],[\chi\vert\cdot\vert_F^2],[\chi,\chi\vert\cdot\vert_F])$
$(\chi\vert\cdot\vert_F^2\otimes\mathrm{sp}(2))\oplus(\chi\otimes\mathrm{sp}(2))$	$Q([\chi\vert\cdot\vert_F^2,\chi\vert\cdot\vert_F^3],[\chi,\chi\vert\cdot\vert_F])$
$(\chi\vert\cdot\vert_F\otimes\mathrm{sp}(3))\oplus\chi$	$Q([\chi\vert\cdot\vert_F,\chi\vert\cdot\vert_F,\chi\vert\cdot\vert_F^3],[\chi])$
$\chi\vert\cdot\vert_F^3\oplus(\chi\otimes\mathrm{sp}(3))$	$Q([\chi\vert\cdot\vert_F^3],[\chi,\chi\vert\cdot\vert_F,\chi\vert\cdot\vert_F^2])$
$\chi\otimes\mathrm{sp}(4)$	$Q([\chi\vert\cdot\vert_F,\chi\vert\cdot\vert_F^2,\chi\vert\cdot\vert_F^3,\chi\vert\cdot\vert_F^4])$

2. LOCAL LANGLANDS CONJECTURE IN GENERAL

The references for this section are [Bor79], [Vog93], and [GR06].

2.1. Representations of rational forms. Suppose F is a nonarchimedean local field of characteristic zero, $\overline{F}$ is an algebraic closure of F, and

$$\Gamma := \mathrm{Gal}(\overline{F}/F)$$

is the Galois groups. Let G be a connected reductive algebraic group defined over $\overline{F}$. For simplicity we assume that G has trivial center. An *F-rational structure* or a *rational form* of G is a homomorphism

$$\sigma : \Gamma \longrightarrow \mathrm{Aut}(G(\overline{F}))$$

compatible with the action of Γ on $\overline{F}$. In this case the group of *F-rational points* is the common fixed points of all the automorphisms

$$G^{\sigma} := G(\overline{F})^{\sigma(\Gamma)}.$$

Notice that $G(\overline{F})$ acts by conjugation on the set of all F-rational forms of G; the orbits, of which there are finitely many, are called *equivalence classes of rational forms.* We have

$$G^{g.\sigma} = g G^{\sigma} g^{-1}.$$

for $g \in G(\overline{F})$.

We say that σ is *inner to* σ' if for each $\gamma \in \Gamma$ the automorphism $\sigma(\gamma)\sigma'(\gamma^{-1})$ is inner, i.e., it is given by conjugation by an element g_{γ} of $G(\overline{F})$. Equivalent rational forms are necessarily inner to each other. The relation of being inner therefore partitions the equivalence classes of rational forms into a finite number of pieces; each piece is called an *inner class* of rational forms. We assume that G is endowed with an inner class $\mathscr{C}$ of rational forms.

Example 2.1. The odd special orthogonal group SO_{2n+1} ($n \geq 1$) has two rational forms, which form an inner class:

SO(V)	dimension of V	Witt index of V	quasi-split
SO_{2n+1}	$2n+1$	n	Yes
SO^{-}_{2n+1}	$2n+1$	$n-1$	

Example 2.2. The even special orthogonal group SO_{2n} ($n \geq 2$) has the following rational forms and inner classes:

SO(V)	dimension of V	Witt index of V	quasi-split
SO_{2n}	$2n$	n	Yes
SO^{-}_{2n}	$2n$	$n-2$	
SO'_{2n}	$2n$	$n-1$	Yes
SO'_{2n}	$2n$	$n-1$	Yes

Note that in the second class, two groups are F-isomorphic even if their F-rational forms are not equivalent.

A *representation of a rational form* of G is a pair (π, σ), with σ a rational form of G and π is an admissible representation of G^{σ}. The group $G(\overline{F})$ acts on representations of rational forms by

$$g.(\pi,\sigma) = (\pi \circ \mathrm{Ad}_{g^{-1}}, g.\sigma).$$

We write $\Pi(G/F)$ for the set of equivalence classes of irreducible representations of rational forms in the class $\mathscr{C}$. Choose a representative σ_i $(i = 1, \ldots, r)$ for each rational form in the inner class $\mathscr{C}$. Then $\Pi(G/F)$ may be identified with the disjoint union of the sets of irreducible admissible representations of each of the rational forms G^{σ_i}:

$$\Pi(G/F) \simeq \coprod_{i=1}^{r} \Pi(G^{\sigma_i}).$$

2.2. L-**groups.** Let G be a reductive group defined over F and let T be a maximal split torus in G. Define

$$X = X(T) := \mathrm{Hom}(T, \mathrm{GL}_1),$$
$$Y = Y(T) := \mathrm{Hom}(\mathrm{GL}_1, T).$$

Both X and Y are free abelian group of the same finite rank and there is nondegenerate pairing

$$\langle , \rangle : X \times Y \longrightarrow \mathbb{Z}.$$

Let $\Delta \subset X$ be the set of roots and $\Delta^{\vee} \subset Y$ be the set of coroots. The quadruple

$$\Psi := (X, \Delta, Y, \Delta^{\vee})$$

is called the *root datum* associated to the group G. Let ${}^{\vee}G$ be the complex points of the algebraic group whose root datum is

$$\Psi^{\vee} := (Y, \Delta^{\vee}, X, \Delta).$$

${}^{\vee}G$ is called the *dual group* of G.

Example 2.3. We have

G	${}^{\vee}G$
GL_n	$\mathrm{GL}_n(\mathbb{C})$
SL_n	$\mathrm{PGL}_n(\mathbb{C})$
SO_{2n+1}	$\mathrm{Sp}_{2n}(\mathbb{C})$
Sp_{2n}	$\mathrm{SO}_{2n+1}(\mathbb{C})$
SO_{2n}	$\mathrm{SO}_{2n}(\mathbb{C})$

The inner class $\mathscr{C}$ of rational forms of G determines an *L-group* for G. This is a pro-algebraic group ${}^LG$ endowed with a short exact sequence

$$1 \longrightarrow {}^{\vee}G \longrightarrow {}^LG \longrightarrow \Gamma \longrightarrow 1$$

and a ${}^{\vee}G$-conjugacy class $\mathscr{D}$ of *splittings*

$${}^{\vee}\delta : \Gamma \longrightarrow {}^LG$$

where ${}^{\vee}\delta \in \mathscr{D}$. If $\mathscr{C}$ includes the split form of G, then ${}^LG$ is just the direct product ${}^{\vee}G \times \Gamma$.

Write $W := W(\Delta, \Delta^{\vee})$ for the group of automorphism of X^* generated by the various reflection s_α, and $W^{\vee} := W(\Delta^{\vee}, \Delta)$ for the group of automorphism of X_* generated by the $s_{\alpha^{\vee}}$. Then

$$W \simeq W^{\vee} \qquad w \mapsto (w^{-1})^t.$$

This isomorphism carries s_α to $s_{\alpha^{\vee}}$.

2.3. **Langlands Parameters.** Recall that the Weil-Deligne group $W_F' := W_F \times \mathrm{SL}_2(\mathbb{C})$. A continuous homomorphism

$$\varphi \colon W_F \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^LG$$

is called *admissible* if the following conditions are satisfied:

- φ is a homomorphism over W_F, i.e., the following diagram commutes:

$$\begin{tikzcd} W_F \times \mathrm{SL}_2(\mathbb{C}) \arrow[r, "\varphi"] \arrow[d] & {}^LG \arrow[d] \\ W_F \arrow[r] & \Gamma \end{tikzcd}$$

- $\varphi(W_F)$ consists of semisimple elements of ${}^LG$
- $\varphi|_{\mathrm{SL}_2(\mathbb{C})}$ is an algebraic representation.

Two admissible homomorphisms φ and φ' are *equivalent* if there is a element $g \in {}^{\vee}G$ such that

$$\varphi'(t) = g\varphi(t)g^{-1}$$

for all $t \in W_F \times \mathrm{SL}_2(\mathbb{C})$.

Let $\Phi(G/F)$ denote the set of equivalence classes of admissible homomorphisms

$$\varphi \colon W_F \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow {}^LG.$$

Conjecture 2.4 (Langlands). *Suppose G is a connected reductive algebraic group over $\overline{F}$ endow with an inner class of F-rational forms. Then to each equivalence class $\varphi \in \Phi(G/F)$ of Langlands parameters is associated a set of representations $\Pi_\varphi \in \Pi(G/F)$, called the L-*packet *of φ. This correspondence should have the following properties.*

(1) *The sets Π_φ partition $\Pi(G/F)$.*

(2) *If σ is a rational form of G, then the set*

$$\Pi_\varphi(\sigma) := \{\, \pi \in \Pi(G^\sigma) \mid (\pi, \sigma) \in \Pi_\varphi \}$$

is finite. If σ is quasi-split, it is nonempty.

(3) *The following three conditions on ϕ are equivalent:*
   (a) *some representation in Π_φ is square-integrable modulo center;*
   (b) *all representations in Π_φ is square-integrable modulo center;*
   (c) *the image of φ is not contained in any proper Levi subgroup of ${}^L G$.*

Example 2.5 (Unramified Representations). Suppose for simplicity that our fixed inner class of F-rational forms of G includes the split form. A Langlands parameter φ may therefore identified with a continuous homomorphism

$$\varphi_0 \colon W_F' \longrightarrow {}^\vee G$$

carrying W_F to semisimple elements and $\left[\begin{smallmatrix} 0 & * \\ 0 & 0 \end{smallmatrix}\right] \simeq \mathbb{C}$ algebraically to unipotent elements. The parameter is called *unramified* if ϕ_0 is trivial on I_F. An unramified Langlands parameter may be identified with a pair (y, N) such that

- $y := \phi_0(\mathrm{Fr})$ is a semisimple element of ${}^\vee G$;
- N is a nilpotent element of ${}^\vee\mathfrak{g} := \mathrm{Lie}({}^\vee G)$; and
- $\mathrm{Ad}_y(N) = q_F N$.

Fix $B \supset T$ a Borel subgroup and a maximal torus defined over F. Principal series representations correspond to continuous complex characters of the group T^σ. The characters of T^σ may be identified as

$$\widehat{T^\sigma} \simeq X^*(T) \otimes_{\mathbb{Z}} \mathrm{Hom}(F^\times, \mathbb{C}^\times).$$

Here the last Hom is the group of continuous complex characters of the locally compact group $F^\times$. A character of $F^\times$ is called *unramified* if it is trivial on the group $U_F = U_F^0$. These characters may be identified as

$$\widehat{T^\sigma_{\mathrm{unram}}} \simeq X^*(T) \otimes_{\mathbb{Z}} \mathrm{Hom}(F^\times / U_F, \mathbb{C}^\times).$$

Because $F^\times / U_F$ is naturally isomorphic to $\mathbb{Z}$, this last Hom may be identified with $\mathbb{C}^\times$:

$$\widehat{T^\sigma_{\mathrm{unram}}} \simeq X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}^\times.$$

On the other hand, fix ${}^\vee B \supset {}^\vee T$ a Borel subgroup and a maximal torus in ${}^\vee G$. Once these choices are made, the definition of the dual group provides a natural identification

$$X_*({}^\vee T) \simeq X^*(T).$$

We find that

$$\widehat{T^\sigma_{\mathrm{unram}}} \simeq X_*({}^\vee T) \otimes_{\mathbb{Z}} \mathbb{C}^\times \simeq {}^\vee T.$$

That is, given our fixed choices of Borel subgroups and maximal tori, there is a natural bijection:

$$\{\text{unramified principal series representations of } G^\sigma\} \longleftrightarrow \{\text{elements of } {}^\vee T\}$$

Definition 2.6. Suppose φ is a Langlands parameter. Define the *centralizer* $Z_{^\vee G}(\varphi)$ in $^\vee G$ of φ to be the subgroup of $^\vee G$ consisting of elements which commutes with the image of φ. Let A_φ denote the finite group $Z_{^\vee G}(\varphi)/Z^0_{^\vee G}(\varphi)$ of connected components of $Z_{^\vee G}(\varphi)$. A *complete Langlands parameter* is a pair (ϕ, τ), with ϕ a Langlands parameter and τ an irreducible representation of A_φ

Conjecture 2.7. *There is a natural bijection*

$$\Pi_\varphi \leftrightarrow \mathrm{Irr} A_\varphi.$$

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