NON-ARCHIMEDEAN HYPERBOLICITY

JULIE TZU-YUEH WANG

Lecture Notes for the 2007 CTS Winter School

1. Introduction

A complex manifold X is said to be hyperbolic (in the sense of Brody) if every analytic map from the complex plane $\mathbb{C}$ to X is constant. From Picard's "little" theorem, an entire function missing more than two values must be constant. It is equivalent to say that $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ is hyperbolic. Picard's theorem also show that a Riemann surface of genus one omitting one point and Riemann surfaces of genus at least 2 are hyperbolic. The higher dimensional case is a much more difficult problem. It was proved by Siu and Yeung [20] that an abelian variety omitting a very ample divisor is hyperbolic. A deep conjecture is that complex manifolds of general type are hyperbolic. It was also conjectured by Kobayashi [16] and Zaidenberg [22] that the complements of "generic" hypersurfaces in $\mathbb{P}^n$ with degree at least $2n + 1$ are hyperbolic. There have been many results related to this conjecture. We will only mention a few here. This conjecture was verified by Green [14] in the case of $2n + 1$ hyperplanes in general position. More generally, Babets [4], Eremenko-Sodin [12] and Ru [18] independently showed that $\mathbb{P}^n \setminus \{\, 2n+1$ hypersurfaces in general position$\}$ is hyperbolic. When $n = 2$, the conjecture is correct for the case of four generic curves (cf.[10]). For the case of three generic curves C_1, C_2, C_3, Dethloff, Schmacher, and Wong ([10], [11]) showed that $\mathbb{P}^2 \setminus \cup_{i=1}^3 C_i$ is hyperbolic if $\deg C_i \geq 2$ for $i = 1, 2, 3$. When one of the C_i is a line they show that any holomorphic map $f : \mathbb{C} \to \mathbb{P}^2 \setminus \cup_{i=1}^3 C_i$ is algebraically degenerate if, up to enumeration, $d_1 = 1$, $d_2 \geq 3$ and $d_3 \geq 4$.

Similar questions can be asked in the case of non-archimedean ground fields. Let K be an algebraically closed field of arbitrary characteristic, complete with respect to a non-archimedean absolute value $|\ |$. A variety X over K is said to be K-hyperbolic if every analytic map from K to X is constant. In contrast to the situation over the complex numbers, it is much easier to study hyperbolic problems over non-archimedean ground fields. For example, an non-archimedean entire function with no zero is constant, i.e., $\mathbb{P}^1 \setminus \{0, \infty\}$ is hyperbolic. The non-archimedean Picard's theorem asserts that any curve of genus at least 1 is K-hyperbolic. It was proved by Cherry that abelian varieties over K are K-hyperbolic(cf. [6] and [7]). As an easy consequence of the second main theorem of Ru [19], $\mathbb{P}^n \setminus \{\, n+1$ hypersurfaces in general position$\}$ is K-hyperbolic. Similar to the conjecture of Kobayashi and Zaidenberg, we assert the following

Conjecture. Let D_1,...,D_q, $q \le n$, be q distinct generic hypersurfaces in $\mathbb{P}^n(K)$. $\mathbb{P}^n \setminus \cup_{i=1}^q D_i$ is K-hyperbolic if $\sum_{i=1}^q \deg D_i \ge 2n$.

The purpose of this note is to set up the basics to study non-archimedean analytic curves into projective varieties and introduce some recent results in this direction. The key ingredients include the non-archimedean Nevanlinna theory, some fundamental theorems from algebraic geometry, and the construction of differential forms (more generally, jet differential forms). In Section 2, we introduce the non-archimedean Nevanlinna theory and show that $\mathbb{P}^n \setminus \{\, n+1$ hypersurfaces in general position$\}$ is K-hyperbolic. In Section 3, we give a simple proof of the non-archimedean Picard's theorem for smooth plan curve of genus at least one. In Section 4, we will show the non-archimedean Kobayashi-Zaidenberg conjecture for the case of n-components. In Section 5, we will show a non-archimedean Schwarz Lemma for the case of symmetric product of 1-forms which will imply the non-archimedean Picard's theorem for curves and imply that an abelian variety over K is K-hyperbolic. In Section 6, we will introduce a most recent result in [2] and show how to construct a jet differential forms in $\mathbb{P}^2$ with logarithmic poles along a curve. Finally, we will conclude the non-archimedean Kobayashi-Zaidenberg conjecture for $\mathbb{P}^2$.

2. Non-archimedean Nevanllina Theory

Let K be an algebraically closed field of arbitrary characteristic, complete with respect to a non-archimedean absolute value $|\ |$. An typical example of such fields is $\mathbb{C}_p$ which is the p-adic completion of the algebraic closure of the field of p-adic numbers $\mathbb{Q}_p$.

Non-archimedean analytic function. An infinite sum converges in an non-archimedean absolute value if and only if its general term approaches zero. So an expression of the form

$$h(z) = \sum_{j=0}^{\infty} a_n z^n, \qquad a_n \in K$$

is well defined whenever

$$|a_n z^n| \to 0, \qquad \text{as } n \to \infty.$$

Such functions are called (non-archimedean) analytic functions (at a neighborhood of z). If h is analytic on K, then h is called a (non-archimedean) entire function.

If $h(z) = \sum_{j=0}^{\infty} a_n z^n$, $a_n \in K$, is an entire function then for each real number $0 < r$, we define

$$\begin{aligned} |h|_r &:= \sup_j |a_j| r^j \\ &= \sup\{|h(z)| : z \in K \text{ with } |z| = r\} \qquad \text{(triangle inequality of } |\,|) \\ &= \sup\{|h(z)| : z \in K \text{ with } |z| \leq r\}. \qquad \text{(maximum modulus principle)} \end{aligned}$$

Note that $|\,|_r$ has the following properties:

Proposition 1. *Let h_1 and h_2 be entire functions, then*

(1) $|h_1 + h_2|_r \leq |h_1|_r + |h_2|_r$,
(2) $|h_1 h_2|_r = |h_1|_r |h_2|_r$.

The *central index* $\nu(r, f)$ is defined by

$$\nu(r, h) := \max_{n \geq 0}\{\, n \,|\, |a_n| r^n = |h|_r\}.$$

Note that in the special case of $r = 0$ with $h(0) = 0$, we define

$$\nu(0, h) = \inf\{n : a_n \neq 0\}.$$

We have the following Poisson-Jensen formula. (cf. [9] or [15])

Poisson-Jensen Formula. *The central index $\nu(r, h)$ increases as r increases, and satisfies the formula*

$$\log |h|_r = \log |a_{\nu(0,h)}| + \int_0^r \frac{\nu(t, h) - \nu(0, h)}{t} dt + \nu(0, h) \log r.$$

We also recall the following theorem. (cf. [9] or [15])

Weierstrass Preparation Theorem. *There exist a unique monic polynomial P of degree $\nu(r, h)$ and a non-archimedean analytic function on $\mathbf{B}[r]$ $(= \{|z| \leq r\})$ such that $h = gP$, where g does not have any zero inside $\mathbf{B}[r]$, and P has exactly $\nu(r, h)$ zeros, counting multiplicity.*

Now, we consider (non-archimedean) analytic maps into projective spaces. Let $f : K \to \mathbb{P}^n(K)$ be a non-archimedean analytic curve. Let $\tilde{f} = (f_0, ..., f_n)$ be a reduced representative of f, where $f_0, ..., f_n$ are entire functions on K and have no common zeros. The *Nevanlinna Characteristic Function* $T_f(r)$ is defined by $T_f(r) = \log ||f||_r$, where

$$||f||_r = \max\{|f_0|_r, ..., |f_n|_r\}.$$

The above definition of $T_f(r)$ is independent, up to an additive constant, of the choice of the reduced representation of f. Let Q be a homogeneous polynomial (form) in $n+1$ variables with coefficients in K. We consider the entire function $Q \circ f = Q(f_0, ..., f_n)$ on K. Let $n_f(r, Q)$ be the number of zeros of $Q \circ f$ in the disk $\mathbf{B}[r]$, counting multiplicity. If $Q \circ f \not\equiv 0$, set

$$\begin{aligned} N_f(r,Q) &= \int_0^r \frac{n_f(t,Q) - n_f(0,Q)}{t} dt + n_f(0,Q) \log r; \\ m_f(r,Q) &= \log \frac{||f||_r^d}{|Q \circ f|_r}. \end{aligned}$$

The functions $N_f(r, D)$ and $m_f(r, D)$ are referred to as the counting and proximity functions, respectively. The Weierstrass Preparation Theorem shows that

$$n_f(r, Q) = \nu(r, Q \circ f),$$

and the Poisson-Jensen Formula implies that

$$N_f(r, Q) = \log |Q \circ f|_r - \log |a_{\nu(0, Q\circ f)}|.$$

Therefore, we have the following Nevanlinna first main theorem in non-archimedean fields.

First Main Theorem. *Let $f : K \to \mathbb{P}^n$ be an analytic map and let Q be a homogeneous form of degree d. If $Q \circ f \not\equiv 0$, then for every positive real number r,*

$$m_f(r, Q) + N_f(r, Q) = dT_f(r) + O(1).$$

Forms $Q_1, ..., Q_q$, $q > n$, are said to be in general position if no set of $n+1$ forms in this system has common zeros in $\mathbb{C}^{n+1} - \{0\}$.

The following Second Main Theorem is the simplest version since it contains no ramification term and follows easily from the first main theorem. This is very different from the complex case.

Second Main Theorem. *Let $f : K \to \mathbb{P}^n(K)$ be a non-archimedean analytic curve, and let $Q_1, ..., Q_q$, $q > n$, be forms in $n+1$ variables with coefficients in K and are in general position. If $Q_j \circ f$ are not identically 0 for $1 \le j \le q$, then, for any real number $r \ge 1$,*

$$\sum_{j=1}^{q} \frac{m_f(r, Q_j)}{\deg Q_j} \le nT_f(r) + O(1).$$

We now include a proof by Ru [19].

Proof. Let d be the least common multiple of $\deg Q_1, ..., \deg Q_q$. It is easy to see that we may reduce the theorem to the case when $\deg Q_1 = \cdots = \deg Q_q$ by showing the theorem for $Q_i^{d/\deg Q_i}$.

Suppose that $\deg Q_i = d$ for $1 \leq i \leq q$. Given a real number $0 < r < \infty$. By rearranging indices if necessary, we may assume that

$$|Q_1 \circ f|_r \leq |Q_2 \circ f|_r \leq \cdots \leq |Q_q \circ f|_r \tag{1}$$

Since $Q_1, ..., Q_q$ are in general position, by Hilbert's Nullstellensatz that for any integer k, $0 \leq k \leq n$, there is an integer $m_k \geq d$ such that

$$x_k^{m_k} = \sum_{i=1}^{n+1} b_{ik}(x_0, ..., x_n) Q_i(x_0, ..., x_n),$$

where b_{ik}, $1 \leq i \leq n+1$, are homogeneous polynomials with coefficients in K of degree $m_k - d$. Therefore,

$$|f_k|_r^{m_k} \leq C\|f\|_r^{m_k - d} \max\{|Q_1 \circ f|_r, ..., |Q_{n+1} \circ f|_r\},$$

where C is a positive constant depends only on the coefficients of b_{ik}, $1 \leq i \leq n+1$, $0 \leq k \leq n$, thus depends only on the coefficients of Q_i, $1 \leq i \leq n+1$. The above inequality then yields

$$\|f\|_r^d \leq C \max\{|Q_1 \circ f|_r, ..., |Q_{n+1} \circ f|_r\}. \tag{2}$$

By (1) and (2),

$$\prod_{j=1}^{q} \frac{\|f\|_r^d}{|Q_j \circ f|_r} \leq \frac{1}{C^{q-n}} \prod_{j=1}^{n} \frac{\|f\|_r^d}{|Q_j \circ f|_r}.$$

Therefore,

$$\sum_{j=1}^{q} m_f(r, Q_j) \leq \sum_{j=1}^{n} m_f(r, Q_j) + O(1).$$

Since the counting function is positive, it follows from the First Main Theorem that

$$\sum_{j=1}^{n} m_f(r, Q_j) \leq ndT_f(r) + O(1).$$

Therefore, we have

$$\sum_{j=1}^{q} m_f(r, Q_j) \leq ndT_f(r) + O(1)$$

that completes the proof. □

The Second Main Theorem implies the following.

Theorem 2. *Let $Q_1,\ldots,Q_q$ be a collection of homogeneous polynomials in $n+1$ variables over K. Assume that $q \geq n+1$ and $Q_1,\ldots,Q_q$ are in general position. Then any non-archimedean analytic map $f: K \to \mathbb{P}^n(K) \setminus \cup_{j=1}^q \{Q_j = 0\}$, must be constant. In other words, $\mathbb{P}^n(K) \setminus \cup_{j=1}^q \{Q_j = 0\}$ is K-hyperbolic.*

Proof of Theorem 2. For an analytic map $f: K \to \mathbb{P}^n(K) \setminus \cup_{j=1}^q \{Q_j = 0\}$, $N_f(r, Q_j) = 0$ for all $1 \leq j \leq q$. The First Main Theorem then gives that

$$m_f(r, Q_j) = (\deg Q_j) T_f(r) + O(1).$$

It then follows from the Second Main Theorem that

$$qT_f(r) \leq nT_f(r) + O(1).$$

Since $q \geq n+1$, this is implies that $T_f(r) \leq O(1)$ which implies that f is constant. □

Remark. A particular case of this theorem is that a nowhere vanishing non-archimedean entire function must be constant.

Exercise:

(1) Let h be an non-archimedean entire function. Prove that h is constant if $|h|_r \leq O(1)$ for all r.
(2) Let $f : K \to \mathbb{P}^n(K)$ be an analytic map. Prove that f is constant if $T_f(r) \leq O(1)$.

3. Non-archimedean Picard's theorem for algebraic curves

In this section we will discuss non-archimedean analogs of Picard's Theorems for non-archimedean analytic maps to algebraic curves. In the early 90's, Berkovich [5] developed a new theory of non-archimedean analytic spaces. Berkovich's theory is based on ideas of spectral theory. Using his theory and rather involved uniformization theory of algebraic curves over non-archimedean ground fields, Berkovich proved a non-archimedean analog of Picard's Theorem. Later, a more elementary proof based on the work of Green [13] was given in [8] where they used the tools from non-archimedean Nevanlinna theory. We will only present the proof for non-singular plane curves, so the readers can grasp the ideas easily. We will also assume that the characteristic of K is zero, even everything in this section also work for the case of positive characteristic.

The following is a non-archimedean analog of Nevanllina's Lemma on the Logarithmic Derivative, but it is much easier and more powerful.

Lemma 3 (Logarithmic Derivative Lemma)**.** *Let h be a non-archimedean analytic function on the open disc $\mathbf{B}(R)$. Then, for $0 < r \leq R$,*

$$|\frac{h^{(n)}}{h}|_r \leq \frac{1}{r^n}.$$

Here $h^{(n)}$ denotes the n-th derivative of h.

Proof. Write h as a power series $h(z) = \sum_{k=0}^{\infty} a_k z^k$. Then

$$h^{(n)}(z) = \sum_{k=n}^{\infty} a_k[k(k-1)\cdots(k-n+1)]z^{k-n}.$$

Because $[k(k-1)\cdots(k-n+1)]$ is an integer, it has absolute value ≤ 1, and so

$$|h^{(n)}|_r = \sup_k |a_k||[k(k-1)\cdots(k-n+1)]|r^{k-n} \leq \frac{\sup_k |a_k|r^k}{r^n} = \frac{|h|_r}{r^n}.$$

□

The main idea of Green is to compare the characteristic functions of two analytic maps. Let $f : K \to \mathbb{P}^n$ be an analytic map. We have defined its characteristic function in the previous section by taking a reduce representation. When we take a different projective representation (not necessarily a reduced representation) $(f_0, ..., f_n)$ of f, we will need to make some modification on the characteristic functions. Denote Z_f the set of common zeros of $f_0, ..., f_n$ counted with multiplicity. Then the characteristic function is defined by

$$T_f(r) := \log \|f\|_r - \sum_{z \in Z_f} \log \frac{r}{|z|}.$$

This definition extends the previous one and we have the following

Proposition 4. *Let $f = (f_0, ..., f_n)$ and $g = (g_0, ..., g_n)$ represent the same analytic map from K to $\mathbb{P}^n$. Then*

$$|T_g(r) - T_f(r)| \leq O(1).$$

The proof follows easily from the Poisson-Jensen's Formula. We leave it as an exercise. It also can be found in [8].

We now state the non-archimedean analog of the Picard's Theorem for algebraic curve.

Theorem 5 (Picard's Theorem)**.** *If f is a non-archimedean analytic map from K to a projective algebraic curve of genus $g \geq 1$, then f is constant.*

Proof. We will only prove the case when the curves are a smooth plane curves and the characteristic of K is zero. A complete proof of the general case can be found in [8].

Assume from now that C is a smooth irreducible plane curve. Then it is the zero set of a homogeneous polynomial $P(X_0, X_1, X_2)$ with coefficients in K. Let $f : K \to C$ be a non-constant non-archimedean analytic map. Then there is a reduced representation $f = (f_0, f_1, f_2)$ and $P(f) = P(f_0, f_1, f_2) = 0$. By the Euler formula, we have

$$f_0 P_0(f) + f_1 P_1(f) + f_2 P_2(f) = (\deg P)P(f) = 0,$$

where $P_i = \frac{\partial P}{\partial X_i}$. By taking derivative on $P(f) = 0$, we have

$$f_0' P_0(f) + f_1' P_1(f) + f_2' P_2(f) = (P(f))' = 0.$$

It then follows from the Cramer's rule that

$$\frac{\begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}}{P_0(f)} = \frac{\begin{vmatrix} f_2 & f_0 \\ f_2' & f_0' \end{vmatrix}}{P_1(f)} = \frac{\begin{vmatrix} f_0 & f_1 \\ f_0' & f_1' \end{vmatrix}}{P_2(f)}. \tag{3}$$

Denote by

$$W_f = \left(\begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix}, \begin{vmatrix} f_2 & f_0 \\ f_2' & f_0' \end{vmatrix}, \begin{vmatrix} f_0 & f_1 \\ f_0' & f_1' \end{vmatrix} \right),$$

and

$$\partial P \circ f = (P_0(f), P_1(f), P_2(f)).$$

Then (3) implies W_f and $\partial P \circ f$ define same analytic map from K into $C \subset \mathbb{P}^2$. By Proposition 4,

$$T_{W_f}(r) = T_{\partial P \circ f}(r) + O(1). \tag{4}$$

Since C is non-singular, P_0, P_1, P_2 has no common zero along C. Therefore,

$$T_{\partial P \circ f}(r) = (\deg P - 1)T_f(r) + O(1). \text{(Exercise)} \tag{5}$$

On the other hand, rewriting the determinant

$$\begin{vmatrix} f_i & f_j \\ f_i' & f_j' \end{vmatrix} = f_i f_j \left(\frac{f_j'}{f_j} - \frac{f_i'}{f_i} \right)$$

and applying Lemma 3, we get

$$T_{W_f}(r) \leq 2T_f(r) - \log r. \tag{6}$$

By (4), (5), and (6), we obtain

$$(\deg P - 3)T_f(r) \leq -\log r + O(1).$$

which is impossible since $\deg P \geq 3$. This shows that f must be constant.

In general, a projective curve is birational to an irreducible plane curve with at worst ordinary double points as singularities. To complete the proof, one needs to modify the computation of $T_{\partial P \circ f}(r)$ as P_0, P_1, and P_2 may have common zeros. We refer the proof to [8] and [13]. ∎

We also record the following non-compact Picard's theorem which is used all the time for the proof of non-archimedean hyperbolicity.

Lemma 6. *Let C be an irreducible projective curve. Then $C \setminus \{$two distinct points$\}$ is K-hyperbolic.*

Proof. It follows from the non-archimedean Picard's theorem that C is K-hyperbolic if the genus of C is at least one. It then suffices to consider when the genus of C is zero. Let ξ_0, ξ_1 be two distinct points in C. Then there exist a birational surjective morphism $\pi : \mathbb{P}^1 \to C$. For a given analytic map $f : K \to C \setminus \{\xi_0, \xi_1\}$, it induces an analytic map $\widetilde{f} : K \to \mathbb{P}^1 \setminus \{\pi^{-1}(\xi_0), \pi^{-1}(\xi_1)\}$ with $f = \pi \circ \widetilde{f}$. Without loss of generality, we may assume that $(0,1)$ and $(1,0)$ in $\pi^{-1}(\xi_0) \cup \pi^{-1}(\xi_1)$. Then $\widetilde{f}$ can be identified as an analytic function with no zero which can only be a constant function. Therefore, f is also constant. This concludes the proof. □

4. Toward the Non-archimedean Kobayashi-Zaidenberg conjecture: n-components

It follows from Theorem 2 that $\mathbb{P}^n \setminus \{n+1$ hypersurfaces in general position$\}$ is K-hyperbolic. We now consider when the number of the omitted hypersurfaces is fewer than $n+1$. W note that the results in this section is contained in [3].

Definition. Let $P_1, ..., P_q$, $q \leq n$, be non-constant homogeneous polynomials in $n+1$ variable over K.

1. $P_1, ..., P_q$ are in general position if the codimension of $\cap_{i=1}^q \{P_i = 0\}$ is q.
2. $(P_1, ..., P_q)$ is a regular sequence of $K[X_0, ..., X_n]$ if P_i is not a zero divisor of $K[X_0, ..., X_n]/(P_1, ..., P_{i-1})$ for each $1 \leq i \leq q$.

Clearly, the condition that $P_1, ..., P_q$ are in general position is equivalent to that $(P_1, ..., P_q)$ is a regular sequence of $K[X_0, ..., X_n]$.

Theorem 7. *Let $P_1, ..., P_q$, $2 \leq q \leq n$, be non-constant irreducible homogeneous polynomials in $n+1$ variables, and assume that P_1,...,P_q are in general position. Then the image of a non-constant analytic map $f : K \to \mathbb{P}^n \setminus \cup_{i=1}^q \{P_i = 0\}$ is contained in a subvariey of $\mathbb{P}^n$ of dimension $n - q + 1$.*

Proof of Theorem 7. Denote by $d_i := \deg P_i$ and let l be the least common multiple of $d_1, ..., d_i$. Replacing P_i by $P_i^{\frac{l}{d_i}}$, we may assume that $d_1 = \cdots = d_q = d$.

Let $(f_0, ..., f_n)$ be a reduced representation of f, i.e., $f_0, ..., f_n$ are K-analytic functions with no common zero and $f = [f_0 : ... : f_n]$. Then $f : K \to \mathbb{P}^n \setminus \cup_{i=1}^q \{P_i = 0\}$ implies that $P_i(f_0, ..., f_n)$ $(1 \leq i \leq q)$ is an analytic function with no zero. By the non-archimedean Picard's theorem, $P_i(f_0, ..., f_n)$ is a non-zero constant for each $1 \le i \le q$. Therefore, there exist non-zero constant $c_2, ..., c_q$ such that $P_i(f) - c_i P_1(f) \equiv 0$ for $2 \le i \le q$. In other words, the image of f is contained in $\cap_{i=2}^q \{P_i - c_i P_1 = 0\}$.

Next, we will show that $(P_2 - c_2 P_1, ..., P_q - c_q P_1)$ is a regular sequence of $K[X_0, ..., X_n]$ which implies immediately that the the dimension of $\cap_{i=2}^q \{P_i - c_i P_1 = 0\}$ is $n - q + 1$ and conclude the proof.

We now show that $[P_2 - c_2 P_1, ..., P_q - c_q P_1]$ is a regular sequence of $K[X_0, ..., X_n]$. It suffices to show that $[P_1, P_2 - c_2 P_1, ..., P_q - c_q P_1]$ is a regular sequence. Suppose it is not, then there exists an i $(2 \le i \le q)$ such that $P_i - c_i P_1$ is a zero divisor of $K[X_0, ..., X_n]/(P_1, P_2 - c_2 P_1, ..., P_{i-1} - c_{i-1} P_1)$. Then there exists $G \in K[X_0, ..., X_n]$ which is not in the ideal of $(P_1, P_2 - c_2 P_1, ..., P_{i-1} - c_{i-1} P_1)$ such that $(P_i - c_i P_1)G$ is an element in the ideal $(P_1, P_2 - c_2 P_1, ..., P_{i-1} - c_{i-1} P_1)$ of $K[X_0, ..., X_n]$. This implies that GP_i is in the ideal of $(P_1, ..., P_{i-1})$ of $K[X_0, ..., X_n]$. Since $(P_1, P_2 - c_2 P_1, ..., P_{i-1} - c_{i-1} P_1) = (P_1, ..., P_{i-1})$ as ideals, $G \notin (P_1, ..., P_{i-1})$. This shows that P_i is a zero divisor of $K[X_0, ..., X_n]/(P_1, ..., P_{i-1})$ which contradicts that fact that $(P_1, ..., P_q)$ is a regular sequence of $K[X_0, ..., X_n]$. $\square$

Next we will study the case of $\mathbb{P}^n$ omitting n hypersurfaces. Certainly, the assumption of in general position is not enough in this case. The following is an example:

Example. Let $H_i = \{X_i = 0\}$ for $i = 1, ..., n$. Then $f(z) = (z, 1, ..., 1)$ is a non-constant K-analytic map into $\mathbb{P}^n \setminus \cup_{i=1}^n H_i$. Therefore, $\mathbb{P}^n \setminus \cup_{i=1}^n H_i$ is not K-hyperbolic.

We will impose the following assumption in the case of n-components.

Definition. Nonsingular hypersurfaces $D_1, ..., D_n$ in $\mathbb{P}^n(K)$ *intersect transversally* if for every point $x \in \cap_{i=1}^n D_i$, $\cap_{i=1}^n \Theta_{D_i,x} = \{x\}$, where $\Theta_{D_i,x}$ is the tangent space to D_i at x.

Theorem 8. *Let $D_1, ..., D_n$ be nonsingular hypersurfaces in $\mathbb{P}^n(K)$ intersect transversally. Then $\mathbb{P}^n \setminus \cup_{i=1}^n D_i$ is K-hyperbolic if $\deg D_i \ge 2$ for each $1 \le i \le n$.*

Remark. The assumption on the degree of the hypersurface is sharp. For example, in the case of $\mathbb{P}^2$, we may choose $D_1 = \{X_0 = 0\}$ and $D_2 = \{X_0^2 + X_1^2 - X_2^2 = 0\}$ and let $f(z) = (1, z, z)$. Clearly, f is a non-constant analytic map into $\mathbb{P}^2 \setminus \{D_1 \cup D_2\}$. For general $\mathbb{P}^n$, we may choose $D_1 = \{X_0 = 0\}$, and $D_i = \{X_0^2 + a_{i1} X_1^2 + \cdots + a_{in} X_n^2 = 0\}$ with $a_{i1} + \cdots + a_{in} = 0$ for $2 \le i \le n$. These hypersurfaces intersect transversally provided every $n-1$ by $n-1$ submatrix of the matrix (a_{ij}), $2 \le i \le n, 1 \le j \le n$, has rank $n-1$. Clearly, the analytic map $f(z) = (1, z, z, ..., z)$ does not intersect any of the hypersurfaces D_i, $1 \le i \le n$.

Proof of Theorem 8. Assume that there exits an analytic map $f : K \to \mathbb{P}^n \setminus \cup_{i=1}^n D_i$. By Theorem 7, we see that the image of f is contained in an irreducible curve C. Then f is an analytic map from K into $C \setminus \cup_{i=1}^n D_i$. If $C \cap \{\cup_{i=1}^n D_i\}$ consists at least two points, then it follows from Lemma 6 that f is a constant. It then remains to consider when $C \cap \{\cup_{i=1}^n D_i\}$ consists exactly only one point x. Since $\dim C + \dim D_i - n \ge 0$, we have $C \cap D_i \ne \emptyset$ for each i. This can only happen when $x \in \cap_{i=1}^n D_i$ and $C \cap D_i = \{x\}$ for each i. By Bezout's theorem, we have

$$(C, D_i)_x = \deg C. \deg D_i \ge \deg D_i \ge 2$$

for each i. Therefore, $\Theta_{C,x} \cap \Theta_{D_i,x} \supsetneq \{x\}$ for each i. Since $\cap_{i=1}^n \Theta_{D_i,x} = \{x\}$, this shows that C must have at least 2 different tangent lines at the point x. Let $\pi : \tilde{C} \to C$ be the normalization of C. Then $\#\pi^{-1}(x) \ge 2$. If $f : K \to C \setminus \{x\}$ then its lifting $\tilde{f} : K \to \tilde{C}$ misses $\pi^{-1}(x)$ containing at least 2 points thus $\tilde{f}$ is a constant and so $f = \pi \circ \tilde{f}$ is a constant. □

Definition. Let D be a curve of degree $d \ge 3$ in $\mathbb{P}^2$. A nonsingular point x of D is said to be a *maximal inflexion point* if there exits a line intersect D at x with multiplicity d.

Theorem 9. *Let D_1 and D_2 be nonsingular projective curves in $\mathbb{P}^2$. Assume that D_1 and D_2 intersect transversally and* $\deg D_1 \le \deg D_2$. *Then $\mathbb{P}^2 \setminus \{D_1 \cup D_2\}$ is K-hyperbolic if and only if either* $\deg D_1, \deg D_2 \ge 2$ *or* $\deg D_1 = 1$, $\deg D_2 \ge 3$ *and D_1 does not intersect D_2 at any maximal inflexion point.*

Remark. A generic curve D_2 of degree at least 4 have no maximal inflexion point. If the curve D_2 has degree 3 then it has at most 9 maximal inflexion points, then we can choose a generic curve D_1 of degree at least 1 such that D_1 does not cut D_2 at any maximal inflexion points. Therefore, the conditions of above theorem are satisfied for generic curves.

Corollary 10. *Let D_1 and D_2 be distinct generic curves in $\mathbb{P}^2$. If* $\deg D_1 + \deg D_2 \ge 4$ *then $\mathbb{P}^2 \setminus \{D_1 \cup D_2\}$ is K-hyperbolic.*

To show the theorem, we first give some cases that $\mathbb{P}^2 \setminus \{D_1 \cup D_2\}$ fails to be K-hyperbolic.

Lemma 11. *$\mathbb{P}^2 \setminus \{D_1 \cup D_2\}$ is not K-hyperbolic if*

(i) $\deg D_1 = 1$ *and* $\deg D_2 \leq 2$ *or*

(ii) $\deg D_1 = \deg D_2 = 2$ *and* D_1 *and* D_2 *intersect tangentially.*

Proof. To show the theorem, we will construct non-constant analytic map f from K into $\mathbb{P}^2 \setminus \{D_1 \cup D_2\}$ which implies that $\mathbb{P}^2 \setminus \{D_1 \cup D_2\}$ is not K-hyperbolic. For case (i), we first consider when $\deg D_1 = \deg D_2 = 1$. After linear change of coordinates, we may assume that $D_1 = \{X_0 = 0\}$ and $D_2 = \{X_1 = 0\}$. Let $f(z) = (1, z, z)$, then $D_1(f) = D_2(f) = 1$. In other words, f is a non-constant analytic map from K into $\mathbb{P}^2 \setminus \{D_1 \cup D_2\}$. Next, we consider when $\deg D_1 = 1$ and $\deg D_2 = 2$. Then the intersection of D_1 and D_2 contains at most two points. If it contains one point, then D_1 is tangent to D_2 at this point. After change of coordinates, we may assume that $D_1 = \{X_0 = 0\}$ and $D_1 \cap D_2 = (0, 0, 1)$. Both conditions force the defining condition of D_2 to be of the form $\{a_0X_0^2 + a_1X_1^2 + a_2X_0X_1 + a_3X_0X_2 = 0\}$. Since D_2 is non-singular, it has to be irreducible. Therefore, we may further assume that $a_1 \neq 0$ and $a_3 \neq 0$. Without loss of generality, we let $a_3 = 1$. Now let $f(z) = (1, z, 1 - a_0 - a_2z - a_1z2)$. Then $D_1(f) = D_2(f) = 1$, which shows that f is a non-constant analytic map from K into $\mathbb{P}^2 \setminus \{D_1 \cup D_2\}$. If D_1 and D_2 contains two points, then D_1 cannot be a tangent line of D_2. Again, we may assume that $(0, 0, 1)$ is one of the intersection points of D_1 and D_2, and $\{X_0 = 0\}$ is the tangent line to D_2 at $(0, 0, 1)$. The previous arguments show that the defining equation of D_2 is $a_0X_0^2 + a_1X_1^2 + a_2X_0X_1 + X_0X_2 = 0$ with $a_1 \neq 0$. Since $(0, 0, 1) \in D_1$, it is defined by $b_0X_0 + b_1X_1 = 0$. We may also assume that $b_1 = 1$ since D_1 is not the tangent line $\{X_0 = 0\}$ of D_2 which implies that $b_1 \neq 0$. Now let $f(z) = (0, 1, z)$, then $D_1(f) = 1$ and $D_2(f) = a_1 \neq 0$ that concludes this case.

For case (ii), we may assume that D_1 and D_2 intersect tangentially at $x = (0, 0, 1)$ and $L = \{X_0 = 0\}$ be the common tangent line to D_1, D_2 at the x. Then similarly, $D_1 = \{a_0X_0^2 + a_1X_1^2 + a_2X_0X_1 + X_0X_2 = 0\}$ and $D_2 = \{b_0X_0^2 + b_1X_1^2 + b_2X_0X_1 + X_0X_2 = 0\}$ with $a_1 \neq 0$, $b_1 \neq 0$.

When $\deg D_1 = \deg D_2 = 2$ and D_1 and D_2 intersect tangentially at $x = (0, 0, 1)$, and assume $L = \{X_0 = 0\}$ be the common tangent line to D_1, D_2 at the x. Then $D_1 = \{a_0X_0^2 + a_1X_1^2 + a_2X_0X_1 + X_0X_2 = 0\}$ and $D_2 = \{b_0X_0^2 + b_1X_1^2 + b_2X_0X_1 + X_0X_2 = 0\}$ with $a_1 \neq 0$, $b_1 \neq 0$. Take $f(z) = (0, 1, z)$ then $D_1(f) = a_1 \neq 0$ and $D_2(f) = b_1 \neq 0$ which completes the proof. $\square$

Proof of Theorem 9. If $\deg D_1, \deg D_2 \geq 2$ then it follows from Theorem 8 that $\mathbb{P}^2 \setminus \{D_1 \cup D_2\}$ is K-hyperbolic. We now consider when $\deg D_1 = 1$, $\deg D_1 \geq 3$ and D_1 does not intersect D_2 at any maximal inflexion point of D_2. Let $f : K \to \mathbb{P}^2 \setminus \{D_1 \cup D_2\}$ be an analytic map. If it is non-constant, then by Theorem 7, the image of f is contained in an irreducible plane curve C. In other words, we have an analytic map $f : K \to C \setminus \{D_1 \cup D_2\}$. By Lemma 6, $C \cap \{\cup_{i=1}^n D_i\}$ can consist of only one point, otherwise f must be constant. Let $C \cap \{\cup_{i=1}^n D_i\}$ consists exactly only one point x. If $\deg C \geq 2$ has degree at least 2 then by Bezout's theorem, we have

$$(C, D_i)_x = C \cdot D_i = \deg C \cdot \deg D_i \geq \deg C \geq 2$$

for $i = 1, 2$. Then the arguments in the proof of Theorem 8 show that f must be a constant. Therefore, it remains to consider when $\deg C = 1$. In this case, $(C, D_2)_x = \deg D_2$ and $x \in D_1 \cap D_2$. This implies that x is a maximal inflexion point of D_2 and D_1 intersects D_2 in x, which is impossible by the hypothesis. In conclusion, f must be constant.

For the converse part, we have shown in Lemma 11 that $\mathbb{P}^2 \setminus \{D_1 \cup D_2\}$ is not K-hyperbolic if $\deg D_1 = 1$ and $\deg D_2 \leq 2$. It then remains to show for the case when $\deg D_1 = 1$, $\deg D_1 \geq 3$ and D_1 does intersect D_2 at an maximal inflexion point of D_2. Without loss of generality, we let $(0, 0, 1)$ is a maximal inflexion point of D_2 and $L = \{X_0 = 0\}$ is the tangent line of D_2 at $x = (0, 0, 1)$. Then the intersection multiplicity of L and D_2 at x equals $\deg D_2$, and therefore $L \cap D_2 = \{x\}$ by Bezout's theorem. On the other hand, as D_1 intersects D_2 transversally $D_1 \neq L$. The arguments in Lemma 11 degree imply that D_1 is defined by $bX_0 + X_1$. Let $f = (0, 1, z) : K \to \mathbb{P}^2$. Then $D_1(f) = 1$ and the image of f is contained in L. Since $L \cap D_2 = \{x\}$ and $x = (0, 0, 1) \notin f(K)$, $f(K) \cap D_2 = \emptyset$. This shows that in this case $\mathbb{P}^2 \setminus \{D_1, D_2\}$ is not K-hyperbolic. □

5. non-archimedean Schwartz lemma: the case of 1-forms.

Schwarz Lemma is one of the standard steps to show a variety is hyperbolic. It is a very powerful tool in the non-archimedean case. It will imply that curves with a regular 1-form and abelian varieties are non-archimedean hyperbolic. For starters, we will first present the proof for the simplest case of differential 1-forms. The readers who have more background in algebraic geometry should have no problem to generalize the results to symmetric product of 1-forms as well as jet differential forms.

Let X be a compact smooth projective curve and assume that there is a non-trivial global regular 1-form ω on X Locally, on a small enough affine neighborhood U of a smooth point x, $\omega = a(x)dx$ where $a(x)$ is a regular function on U.

Since X is projective, it is contained in some projective space $\mathbb{P}^n$. For $\mathbb{P}^n$, we may cover each coordinate neighborhood $U_i = \{Z_i \neq 0\} = \{(z_0, ..., z_{i-1}, 1, z_{i+1}, ..., z_n) \mid z_j \in$

K for $j \neq i\}$ by closed ball of the type $|z-a| \leq r$ where z and a are in U_i with the coordinate in Z_i equal 1. A closed ball is affinoid. It is also easy to show that there exists a finite subcovering $\mathcal{U}$ of this covering of $\mathbb{P}^n$. Moreover, $\mathcal{U}_X = \{U \cap X \mid U \in \mathcal{U}\}$ form an affinoid covering of X.

Theorem 12 (Schwarz Lemma)**.** *Let X be a non-singular projective curve defined over $(\mathbf{K}, |\ \ |)$. If X admits a non-trivial global regular 1-form ω then $\omega(f, f') \equiv 0$ for all analytic map $f : \mathbf{K} \to X$.*

Proof. Let ω be a regular differential one form on X. Choose a finite affine open covering $\mathcal{V}$ for X such that for each point $P \in X$ there is an open set $V \in \mathcal{V}$ such that $\omega|_V = \theta d\phi$ where θ, $\phi \in \mathcal{O}_X(V)$, i.e., rational functions of X regular on V. Choose an affinoid $U \in \mathcal{U}_X$ such that $U \cap V \neq \emptyset$. Then θ and ϕ is regular on $U \cap V$. For an analytic map $f : K \to X$, we have for $f(z) \in U \cap V$

$$\begin{aligned}\omega(f(z), f'(z)) &= \theta(f(z))d\phi(f(z)) = \theta(f(z))(\phi(f(z)))'dz \\ &= \theta(f(z))\phi(f(z))\frac{(\phi(f(z)))'}{\phi(f(z))}dz.\end{aligned}$$

Since θ and ϕ are regular on $U \cap V$, they are bounded on $U \cap V$ as U is affinoid. There exists a constant $c_{U\cap V}$ such that $|\theta(f(z))\phi(f(z))| \leq c_{U\cap V}$ foe all z with $f(z) \in U \cap V$. Since there are only finitely many U and V, there exists a constant c such that $c_{U\cap V} \leq c$ for all U and V. On the other hand, it follows from the Lemma of Logarithmic $|\frac{(\phi(f(z)))'}{\phi(f(z))}| \leq \frac{1}{|z|}$. Putting everything together, we have

$$|\theta(f(z))(\phi(f(z)))'| \leq c\frac{1}{|z|}.$$

Since f is an analytic map defined on K, we may choose $|z| \to \infty$ that conclude that $\theta(f(z))(\phi(f(z)))' \equiv 0$. Therefore, $\omega(f, f') \equiv 0$ and $f' \equiv 0$. Since the characteristic of K is zero, this also implies that f is constant. □

The previous arguments can be carried over to any compact non-singular projective variety admits a non-trivial global regular m-fold symmetric product of 1-form, i.e. $H^0(X, \bigodot^m T^*X) \neq 0$. We have the following similar result.

Theorem 13. *Let X be a non-singular projective variety defined over $(\mathbf{K}, |\ \ |)$. If X admits a non-trivial global regular m-fold symmetric product of 1-form ω then $\omega(f, f') \equiv 0$ for all holomorphic map $f : \mathbf{K} \to X$.*

A regular vector bundle $\mathcal{F}$ over a projective variety X of dimension n is said to be spanned if global regular sections span the fiber $\mathcal{F}_x$ over every point $x \in X$.

Theorem 14. *Let X be a non-singular projective variety and $f : \mathbf{K} \to X$ be an analytic map. If $\bigodot^m T^*X$ is spanned for some positive integer m then f is constant.*

Proof. Let x_1,...,x_n be a local coordinate system of a point $P \in X$. Since $\bigodot^m T^*X$ is spanned for some positive integer m, there exists a global regular m-fold symmetric product of 1-form ω_i, $1 \leq i \leq n$ such that $\omega_i|_V = dx_i^{\otimes s}$ where V is an affine open neighborhood of P. By Theorem 13, $\omega_i(f, f') \equiv 0$ that implies that $(x_i \circ f)'(z) = 0$ for $1 \leq i \leq n$ and z such that $f(z) = P$. As $x_1, ..., x_n$ are local coordinate system, this shows that $f'(z) = 0$ for all z. Therefore, f is constant. □

Corollary 15. *An abelian variety A defined over a non-archimedean field $\mathbf{K}$ is K-hyperbolic.*

6. NON-ARCHIMEDEAN SCHWARTZ LEMMA: THE CASE OF LOGARITHMIC JET DIFFERENTIAL FORMS.

Let X be a smooth variety of dimension n. We shall denote by J^kX the k-th jet bundle of X. These bundles are defined as follows. Let $\mathcal{H}_x, x \in X$, be the sheaf of germs of analytic curves:

$$\mathcal{H}_x = \{f : \mathbf{B}(r) \to X \text{ is holomorphic for some } r > 0 \text{ and } f(0) = x\}$$

where $\mathbf{B}(r) = \{|z|_p < r\}$ is the disc of radius r in $\mathbb{C}_p$. Define, for $k \in \mathbb{N}$, an equivalence relation by designating two elements $f, g \in \mathcal{H}_x$ as k-equivalent (written $f \sim_k g$) if

$$f_j^{(p)}(0) = g_j^{(p)}(0)$$

for all $1 \leq p \leq k$, where $f_j = z_j \circ f, z_1, ..., z_n$ are local analytic coordinates near x and $f_j^{(p)} = \partial^p f_j/\partial\zeta$ is the p-th order derivative relative to the variable $\zeta \in \mathbb{C}_p$. The sheaf of *parameterized k-jets* is defined by:

$$J^kX = \cup_{x\in X}\mathcal{H}_x/\sim_k .$$

Elements of J^kX will be denoted by $j^kf(0) = (f(0), f^{'}(0), ..., f^{(k)}(0))$.

We set $J^0X = \mathcal{O}_X$. It is clear that $J^1X = TX$ but, in general, for $k \geq 2, J^kX$ is not locally free. There is, however, a natural K^*-action on J^kX defined via parameterization. Namely, for $\lambda \in \mathbb{C}^*$ and $f \in \mathcal{H}_x$, a map $f_\lambda \in \mathcal{H}_x$ is defined by $f_\lambda(t) = f(\lambda t)$. Then $j^kf_\lambda(0) = (f_\lambda(0), f_\lambda^{'}(0), ..., f_\lambda^{(k)}(0)) = (f(0), \lambda f^{'}(0), ..., \lambda^k f^{(k)}(0))$. So the K^*-action is given by

$$\lambda \cdot j^kf(0) = (f(0), \lambda f^{'}(0), ..., \lambda^k f^{(k)}(0)).$$

Note that even though J^kX is not a vector bundle, the zero section (that is, $f^{(i)}(0) = 0$ for all $i = 1, ..., k$) still makes sense.

For the tangent bundle TX we have the dual $T^*X = \Omega^1_X$ which is the sheaf associated to the presheaf

$$\Omega^1_U = \{\omega : TX|_U \to \mathbb{C}_p \text{ analytic } \mid \omega(\lambda \cdot j^1 f) = \lambda\omega(j^1 f), \lambda \in \mathbb{C}_p\}.$$

Analogously, we define for positive integers m, k, *the sheaf of germs of k-jet differentials of weight m*, denoted $\mathcal{J}^m_k X$, to be the sheaf associated to the presheaf

$$\mathcal{J}^m_k U = \{\omega : J^kX|_U \to \mathbb{C}_p \text{ analytic } \mid \omega(\lambda j^k f) = \lambda^m\omega(j^k f), \lambda \in \mathbb{C}_p\}.$$

Note that $\mathcal{J}^1_1 X = T^*X = \Omega^1_X$. We also set $\mathcal{J}^m_0 X = \mathcal{O}_X$ for all m.

We now recall the definition of $\mathcal{J}^m_k X(\log D)$ where X is a nonsingular projective variety and D is an effective divisor. We use $\mathcal{J}^m_k X(lD)$ to denote the sheaf of k-jet differential of weight m with poles of order at most l along D. Then, $\mathcal{J}^m_k X(\log D)$ is the subsheaf of $\cup_{l=0}^{\infty}\mathcal{J}^m_k X(lD)$ such that locally in an open neighborhood U with ϕ a defining function for D on U, we have $\phi\omega$ and $\phi d\omega$ regular on U for all $\omega \in \mathcal{J}^m_k X(\log D)$. In application, we usual require that D to be simple normal crossing which means that if D splits into a sum of nonsingular irreducible components

$$D = D_1 + \cdots + D_q,$$

and at each point x in the support of D, the local defining function for the components of D meeting at X can be taken as part of a system of local coordinates at x. For example, if $X = \mathbb{P}^n$ and D is the sum of $q(\leq n)$ irreducible smooth hypersurfaces in general position, then D is a simple normal crossing divisor.

The non-archimedean Schwarz Lemma for $\mathcal{J}^m_k X(\log D)$ can be deduced similar to the case of symmetric product of 1-forms. We refer to [2] for a complete proof.

Lemma 16. (Non-archimedean Schwarz Lemma) *Let X be a projective variety and $f : K \to X$ be an analytic map. Then*

$$\omega(j^k f) \equiv 0$$

for all $\omega \in H^0(X, \mathcal{J}^m_k X)$ or $H^0(X, \mathcal{J}^m_k X(\log D))$, where D is an effective divisor with simple normal crossings.

A deeper result related to the study of K-hyperbolicity is the following theorem in [2].

Corollary 17. *Let X be a projective variety and assume that $H^0(X, \mathcal{J}^m_k X) \neq \{0\}$ (resp. $H^0(X, \mathcal{J}^m_k X(\log D)) \neq \{0\}$) for some m and k. Then the image of every* *holomorphic map $f : K \to X$ (resp. $f : K \to X \setminus D$) is contained in a subvariety Y in X of strictly lower dimension.*

Corollary 18. *Let V be an irreducible hypersurface of $\mathbb{P}^n$ with degree at least $2n$. Then the image of every holomorphic map $f : K \to \mathbb{P}^n \setminus V$ degenerates properly.*

Proof. Let $(Z_0, ..., Z_n)$ be the projective coordinates on $\mathbb{P}^n$, and $x_i = Z_i/Z_0$, $1 \leq i \leq n$. Let $P(Z_0, ..., Z_n)$ be the defining equation of D. Let

$$\omega = \frac{dx_1 \cdots dx_n}{P(1, x_1, ..., x_n)}.$$

It is clear that ω has no pole on the open set $U_0 = \{Z_0 \neq 0\}$. Now if $y_i = Z_i/Z_j$, $j = 0, ..., \hat{j}, ..., n$ are coordinates on $U_j = \{Z_j \neq 0\}$, then

$$x_i = \frac{y_i}{y_0}, \qquad i \neq j; \qquad x_j = \frac{1}{y_0}$$

which gives

$$\frac{dx_i}{x_i} = \frac{dy_i}{y_i} - \frac{dy_0}{y_0}, \qquad i \neq j; \qquad \frac{dx_j}{x_j} = -\frac{dy_0}{y_0}.$$

Therefore, in term of $\{y_i\}$,

$$\begin{aligned}\omega &= \frac{x_1 \cdots x_n \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}}{P(1, x_1, ..., x_n)} \\ &= -\frac{y_0^{\deg P} \cdot \frac{y_1}{y_0} \cdots \frac{y_{j-1}}{y_0} \frac{1}{y_0} \frac{y_{j+1}}{y_0} \cdots \frac{y_n}{y_0} \frac{dy_0}{y_0} \prod_{i \neq j} (\frac{dy_i}{y_i} - \frac{dy_0}{y_0})}{P(y_0, .., y_{j-1}, 1, y_{j+1}, ..., y_n)}\end{aligned}$$

which clearly has no poles on U_j if $\deg P \geq 2n$. Therefore, we may conclude that $H^0(X, \mathcal{J}_1^n X(\log D))$ is not trivial, and the previous corollary implies that f degenerates properly. ☐

Corollary 19. *Let D be a generic plane curve of degree at least 4, then $\mathbb{P}^2 \setminus D$ is K-hyperbolic.*

Proof. Let D be a generic plane curve of degree d. Let f be a holomorphic map from K to $\mathbb{P}^2 \setminus D$. Then the image of f is contained in an irreducible curve C. It follows from Xu's result in [21] that $C \cap D$ contains at least $d - 2$ distinct points. This yields that the image of f is contained in C omitting at least two points which can only happen when f is constant. ☐

Remark. We have showed in Corollary 10 that $\mathbb{P}^2 \setminus D_1 \cup D_2$ is K-hyperbolic if D_1 and D_2 are generic and $\deg D_1 + \deg D_2 \geq 4$. It also follows from [1] that $\mathbb{P}^2 \setminus \cup_{i=1}^q D_i$ is K-hyperbolic if $q \geq 3$ and $D_1, ..., D_q$ are in general position. Together with this corollary, we have the following: $\mathbb{P}^2 \setminus \cup_{i=1}^q D_i$ is K-hyperbolic if D_1,...,D_q are generic and $\deg D_1 + \cdots + \deg D_q \geq 4$.

References

[1] T. T. H. An, *A defect relation for non-Archimedean analytic curves in arbitrary projective varieties,* to appear by Proceeding A.M.S.

[2] T. T. H. An, J. T.-Y. Wang and P.-M. Wong, *On p-adic hyperbolicity,* preprint.

[3] T. T. H. An, J. T.-Y. Wang and P.-M. Wong, *Non-archimedean analytic curves in the complements of divisors,* preprint.

[4] V. A. Babets, *Picard-type theorems for holomorphic mappings,* Siberian Math. J. 25 (1984), 195–200

[5] V. Berkovich *Spectral theory and analytic geometry over non-archimedean fields*, AMS survey and monographs 33, Amer. Math. Soc. 1990

[6] W. Cherry , *Non-archimedean analytic curves in abelian varieties*. Math. Ann. 300 (1994), 393–404.

[7] W. Cherry and M. Ru, *Rigid Analytic Picard Theorems,* Amer. J. Math. 126, 873-889, 2004

[8] W. Cherry and J. T.-Y. Wang, *Non-archimedean analytic maps to algebraic curves,* Value distribution theory and complex dynamics (W. Cherry and C.-C, Yang, eds.), Comtemp. Math. 303, Amer. Math. Soc., Providence, RI, 2002, 7–36

[9] W. Cherry and Z. Ye, *Non-Archimedean Nevanllina theory in several variables and non-Archimedean Nevanllina inverse problem,* Trans. Amer. Math. Soc. 349, 5037-5071, 1997

[10] G. Dethloff, G. Schumacher and P.-M. Wong, *Hyperbolicity of the complements of plane algebraic curves*. Amer. J. Math. 117 (1995), no. 3, 573–599.

[11] G. Dethloff, G. Schumacher and P.-M. Wong, *On the hyperbolicity of the complements of curves in algebraic surfaces: the three-component case*. Duke Math. J. 78 (1995), no. 1, 193–212.

[12] A. E. Eremenko and M. L. Sodin, *The value distribution of meromorphic functions and meromorphic curves from the point of view of potential theory*. St. Petersburg Math. J. 3 (1992), 109–136.

[13] M.L. Green, *Some Picard theorems for holomorphic maps to algebraic varieties*. Amer. J. Math 97 (1975) 43–75.

[14] M.L. Green, *The hyperbolicity of the complement of* $2n+1$ *hyperplanes in general position in* P_n *and related results*. Proc. Amer. Math. Soc. 66 (1977), no. 1, 109–113.

[15] P.-C. Hu and C.-C. Yang, *Meromorphic functions over non-Archimedean fields*. Mathematics and its applications 522, Kluwer Academic Publishers, 2000

[16] S. Kobayashi, *Hyperbolic manifolds and holomorphic mappings*. Pure and Applied Mathematics, 2 Marcel Dekker, Inc., New York 1970 ix+148 pp.

[17] J. Noguchi and J. Winkelmann, *Holomorphic curves and integral points off divisors*. Math. Z. 239 (2002), no. 3, 593–610.

[18] M. Ru, *Integral points and hyperbolicity of the complement of hypersurfaces*. J. Reine Angew. Math. 442 (1993), 163–176.

[19] M. Ru, *A note on p-adic Nevanlinna theory*. Proc. Amer. Math. Soc. 129 (2001), 1263–1269.

[20] Y.T. Siu and S.K. Yeung, *A generalized Bloch's theorem and the hyperbolicity of the complement of an ample divisor in an abelian variety*. Math. Ann. 306 (1996), 743–758.

[21] G. Xu, *On the complement of a generic curve in the projective plane*. Amer. J. Math. 118 (1996), no. 3, 611–620.

[22] M. Zaidenberg, *Stability of hyperbolic embeddedness and construction of examples*. Math. USSR-Sb. 63 (1989),351–361.

Institute of Mathematics, Academia Sinica, Nankang, Taipei 11529, Taiwan, R.O.C.
E-mail address: jwang@math.sinica.edu.tw