Hecke Operators on Drinfeld Cusp Forms

joint work with
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Drinfeld Upper Half Plane

- $K = \mathbb{F}[T]$, where $|\mathbb{F}| = q$.
- $\infty = \text{place at infinity}$
- $A = \mathbb{F}[T] = \text{elements regular outside } \infty$
- $K_\infty = \text{completion of } K \text{ at } \infty = \mathbb{F}((1/T))$
- $\mathcal{O}_\infty = \text{ring of integers in } K_\infty$
- $\mathcal{P}_\infty = \text{the maximal ideal of } \mathcal{O}_\infty$
- $C = \widehat{K_\infty}$
- Drinfeld upper half plane $\Omega = C \setminus K_\infty$
  - $GL_2(K_\infty)$ acts by fractional linear transformations.
  - It is a rigid analytic space.
- For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(K_\infty)$, $m, k \in \mathbb{Z}$ and $f : \Omega \to C$, define
  
  $$(f \mid \gamma)(z) := f(\gamma z)(\det \gamma)^m(cz + d)^{-k}.$$
Drinfeld Cusp Forms

- $\Gamma = \text{congruence subgroup of } GL_2(A)$
- $\Gamma$ has finitely many cusps, represented by the orbits $\Gamma \backslash \mathbb{P}^1(K)$.
- A rigid analytic function $f : \Omega \to C$ is called a Drinfeld cusp form of weight $k$ and type $m$ for $\Gamma$ if it satisfies
  
  (i) $f \mid \gamma = f$ for all $\gamma \in \Gamma$;
  
  (ii) $f$ is holomorphic at all cusps;
  
  (iii) $f$ vanishes at all cusps.
- $S_{k,m}(\Gamma)$ = space of cusp forms.
- $S_{k,m}^2(\Gamma)$ = subspace of double cusp forms.
- Shall take $\Gamma = \Gamma_1(T)$ and $\Gamma(T)$. In this case $\det \Gamma = 1$ and the spaces $S_{k,m}(\Gamma)$ are independent of $m$.

$$\dim_C S_{k,m}(\Gamma) = (k - 1)(g_\Gamma + h_\Gamma - 1)$$
The Tree $\mathcal{T}$

- $\mathcal{Z}_\infty = \text{the center of } GL_2(K_{\infty})$
- $\mathcal{T} := GL_2(K_{\infty})/GL_2(O_{\infty})\mathcal{Z}_\infty$ is a $(q + 1)$-regular tree on which $GL_2(K_{\infty})$ acts by left translations.
- vertices of $\mathcal{T} \cong GL_2(K_{\infty})/GL_2(O_{\infty})\mathcal{Z}_\infty$
- directed edges $\cong GL_2(K_{\infty})/J_{\infty}\mathcal{Z}_\infty$, where
  $J_{\infty} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(O_{\infty}) : c \in \mathcal{P}_{\infty} \right\}$
  is the Iwahori subgroup of $GL_2(O_{\infty})$.
- $\langle g \rangle = \text{the edge coset represented by } g \in GL_2(K_{\infty})$.
- Two paths in $\mathcal{T}$ are equivalent if they differ at only finitely many edges.
- An end of $\mathcal{T}$ is an equivalence class of infinite paths $\{e_1, e_2, \ldots \}$.
- There is a canonical bijection from the set of ends to $\mathbb{P}^1(K_{\infty})$, the boundary of $\Omega$.
- The rational ends are $\mathbb{P}^1(K)$, corresponding to cusps.
Harmonic Cocycles

• $V(k, m)$ = vector space over $C$ with a basis $X^j Y^{k-2-j}$, $0 \leq j \leq k - 2$

• $GL_2(K_\infty)$ acts on $Hom(V(k, m), C)$.

• A harmonic cocycle of weight $k$ and type $m$ for $\Gamma$ is a function $c$ from the directed edges of $T$ to $Hom(V(k, m), C)$ satisfying

(a) For all vertices $v$ of $T$, 
$$\sum_{e \rightarrow v} c(e) = 0;$$

(b) For all edges $e$ of $T$, $c(\bar{e}) = -c(e)$, where $\bar{e} = e$ with reversed orientation;

(c) It is $\Gamma$-equivariant, i.e.,
$$c(\gamma e) = \gamma(c(e))$$
for all $\gamma \in \Gamma$.

• $H_{k,m}(\Gamma) =$ space of harmonic cocycles
Cusp Forms and Harmonic Cocycles

• Building map: $\Omega \to \mathcal{T}$, commuting with the action of $GL_2(K_\infty)$.

• It maps the annulus

$$V = \{z \in \Omega : 1 < |z| < q\}$$

to a directed edge $e_0$, and $\gamma V$ to $\gamma e_0$ for $\gamma \in GL_2(K_\infty)$.

• $f(z)dz = \sum_{i \in \mathbb{Z}} a_i z^i dz$ holomorphic $C$-valued differential on $V$

• Let $\text{Res}_{e_0} f(z)dz = a_{-1}$, and

$$\text{Res}_{\gamma e_0} f(z)dz = \text{Res}_{e_0} g(u)du,$$

where $f(z)dz = f(\gamma u)d\gamma u = g(u)du$.

• Define $\text{Res} : S_{k,m}(\Gamma) \to H_{k,m}(\Gamma)$ by

$$\text{Res}(f)(e)(X^jY^{k-2-j}) = \text{Res}_e z^j f(z)dz$$

for $0 \leq j \leq k - 2$.

• Teitelbaum: $\text{Res}$ is an isomorphism.

Identify the two spaces.
This allows us to view a Drinfeld cusp form as a vector valued left $\Gamma$-equivariant function on $GL_2(K_\infty) / \mathcal{J}_\infty \mathcal{Z}_\infty$. When $k = 2$, it is a left $\Gamma$-invariant $C$-valued function. There is a suitable basis of $H_{2,m}(\Gamma)$ which take values in $\mathbb{Z}/p\mathbb{Z}$, and hence they can be lifted to $C$-valued automorphic forms on $GL_2(\mathbb{A}_K)$.

- $\Gamma_{[s]} = \Gamma$-stabilizer of an end $[s]$ representing a cusp of $\Gamma$.
- Böckle:
  (a) $V(k, m)^{\Gamma_{[s]}}$ is one-dimensional.
  (b) $\Gamma_{[s]}$ acts freely on $src([s])$ with finitely many orbits, represented by edges $e^{[s]}_1, \ldots, e^{[s]}_{l_\mathcal{Y}}$.
  (c) Let $f \in S_{k,m}(\Gamma)$ and $c = \text{Res}(f)$. Then $f \in S^2_{k,m}(\Gamma)$ iff for any cusp $[s]$,

$$\sum_{i=1}^{l_s} c(e^{[s]}_i)(g_s) = 0$$

for any $g_s \in V(k, m)^{\Gamma_{[s]}}$. 

\begin{itemize}
  \item $\dim_C S_{2,m}^2(\Gamma) = g_\Gamma$
  \item $\dim_C S_{k,m}^2(\Gamma)$
    
    \begin{equation*}
    = (k - 2)(g_\Gamma + h_\Gamma - 1) + g_\Gamma - 1
    \end{equation*}
    if $k \geq 3$.
\end{itemize}
Hecke Operators

- \( \Gamma = \Gamma_1(T) \) and \( \Gamma(T) \)
- \( \mathfrak{P} = (P) \neq (T) \) is a prime ideal of \( A \), \( P(0) = 1 \), \( \deg P = d \)
- The Hecke operator at \( \mathfrak{P} \) is
  \[
  T_{\mathfrak{P}} = \begin{array}{c}
P^{k-m-1} \left[( \begin{array}{cc} P & 0 \\ 0 & 1 \end{array} ) \\
  + \sum_{b \in A, \deg b < d} \left( \begin{array}{c} 1 \\ 0 \\ \frac{b(1-P)}{P} \end{array} \right) \right] 
  \end{array}
  \]
- Transport its action from \( S_{k,m}(\Gamma) \) to \( H_{k,m}(\Gamma) \)
  \[
  T_{\mathfrak{P}} \mathbf{c}(e) = P^{k-m-1} \left[( \begin{array}{cc} P & 0 \\ 0 & 1 \end{array} )^{-1} \mathbf{c} \left( \left( \begin{array}{cc} P & 0 \\ 0 & 1 \end{array} \right) e \right) \\
  + \sum_{b \in A, \deg b < d} \left( \begin{array}{c} 1 \\ 0 \\ \frac{b(1-P)}{P} \end{array} \right)^{-1} \mathbf{c} \left( \left( \begin{array}{cc} 1 \\ 0 \\ \frac{b(1-P)}{P} \end{array} \right) e \right) \right].
  \]

Shall study the behavior of \( T_{\mathfrak{P}} \) for degree one prime ideal \( \mathfrak{P} \neq (T) \) on cusp forms and double cusp forms.
**Cusp Forms for $\Gamma_1(T)$**

- A fundamental domain of $\Gamma_1(T) \backslash \mathcal{T}$ is a path connecting two cusps, $[\infty] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $[0] = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, containing no stable vertices and one stable edge $\gamma_0 := \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle$.
- A harmonic cocycle is determined by its value (in $\text{Hom}(V(k, m), C)$) at $\gamma_0$.

- $$\dim_C S_{k,m}(\Gamma_1(T)) = k - 1,$$
  $$\dim_C S_{2,m}^2(\Gamma_1(T)) = 0,$$
  and
  $$\dim_C S_{k,m}^2(\Gamma_1(T)) = k - 3 \text{ for } k \geq 3.$$

- $$V(k, m)^{\langle \Gamma_1(T) \rangle}[\infty] = \langle Y^{k-2} \rangle$$
  and
  $$V(k, m)^{\langle \Gamma_1(T) \rangle}[0] = \langle X^{k-2} \rangle.$$

- $$S_{k,m}^2(\Gamma_1(T)) = \{ c \in S_{k,m}(\Gamma_1(T) : \ c(\gamma_0)(Y^{k-2}) = c(\gamma_0)(X^{k-2}) = 0 \}. $$
$S_{k,m}(\Gamma_1(T))$—Case I. $k$ Small

- Assume $q \geq k \geq 2$. For $0 \leq j \leq k - 2$ and $\wp = (P)$ with $P = 1 + \alpha T$,

$$T_{\wp} c(\gamma_0)(X^j Y^{k-2-j})$$

$$= \lambda_j(P) c(\gamma_0)(X^j Y^{k-2-j}),$$

where

$$\lambda_j(P) = \sum_{l=0}^{j} \binom{j}{l} \binom{k-2-j}{l} (1 - P)^l$$

$$= \sum_{l=0}^{\min\{j, k-2-j\}} \binom{j}{l} \binom{k-2-j}{l} (-\alpha T)^l.$$  

- $\deg \lambda_j(P) \leq \min\{j, k - 2 - j\}$. Note that $\lambda_0(P) = \lambda_{k-2}(P) = 1$ and $\lambda_j(P) = \lambda_{k-2-j}(P)$ for all $0 \leq j \leq k - 2$.

- For $0 \leq j \leq k - 2$ and any $k$, define a harmonic cocycle $c_j$ by specifying its value at $\gamma_0$ by:

$$c_j(\gamma_0)(X^j Y^{k-2-j}) = 1$$

and

$$c_j(\gamma_0)(X^l Y^{k-2-l}) = 0 \text{ for } l \neq j.$$
Theorem 1. Let $\mathfrak{p}$ be a prime ideal of $A$ generated by $P$ with $P(0) = 1$ and $\deg P = 1$. Suppose $q \geq k \geq 2$. Then

(1) Each $c_j$, $0 \leq j \leq k - 2$, is an eigenfunction of $T\mathfrak{p}$ with eigenvalue $\lambda_j(P)$; and

(2) The Hecke operators at the ideals of degree one are simultaneously diagonalized on $S_{k,m}(\Gamma_1(T))$ with respect to the basis $c_j$, $0 \leq j \leq k - 2$.

Conjecture. Same assumptions as above except $\deg P = d \geq 1$. Let $\theta$ be a root of $P$. Then each $c_j$, $0 \leq j \leq k - 2$, is an eigenfunction of $T\mathfrak{p}$ with eigenvalue

$$\lambda_j(P) := \prod_{i=0}^{d-1} \lambda_j(1 + \theta^q T).$$

Consequently the Hecke operators are simultaneously diagonalized on $S_{k,m}(\Gamma_1(T))$. Conjecture verified for $d = 2$. 
Remark 2. If we factor the polynomial

$$\deg \lambda_j(1+T)$$

$$\lambda_j(1 + T) = \prod_{s=1} (1 + \delta_s T),$$

then

$$\deg \lambda_j(1+T)$$

$$\lambda_j(P) = \prod_{s=1} P(\delta_s T).$$

Observe that the degree of $\lambda_j(P)$ above is at most $d(k - 2)/2$. This may be regarded as the Ramanujan conjecture on Drinfeld cusp forms.

Theorem 3. Let $\mathfrak{P} = (P), P(0) = 1$, be a degree one prime ideal of $A$. If $q+2 \geq k \geq 4$, then $c_j, 1 \leq j \leq k - 3$, are eigenfunctions of $T_{\mathfrak{P}}$ on $S^2_{k,m}(\Gamma_1(T))$ with eigenvalue $\lambda_j(P)$. Consequently the Hecke operators for degree one prime ideals are simultaneously diagonalized on $S^2_{k,m}(\Gamma_1(T))$ with respect to the basis $c_j, 1 \leq j \leq k - 3$. 

13
$S_{k,m}(\Gamma_1(T))$— Case II. $k$ Large

- For $i = 0, 1, \ldots, q - 2$, let $S_{k,m}(\Gamma_1(T))_i$ be the subspace generated by

$$\{c_i, c_{i+(q-1)}, \ldots, c_{i+\left[\frac{k-2-i}{q-1}\right](q-1)}\}$$

so that $S_{k,m}(\Gamma_1(T)) = \bigoplus_{i=0}^{q-2} S_{k,m}(\Gamma_1(T))_i$.

- Each $S_{k,m}(\Gamma_1(T))_i$ is invariant under $T_\mathfrak{p}$ for $\mathfrak{p}$ of degree one. With resp. to the basis above, the action of $T_\mathfrak{p}$ can be represented by an explicit matrix.

- The degree one Hecke operators may or may not be diagonalized.
Example 4. $q = 2$ and $k = 5$. There is only one poly. $P = 1 + T$, and only one subspace. $T_{\mathfrak{g}}$ is represented by the matrix

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
T^2 & 1 & T & T^3 \\
T & T^2 & 1 & T^3 \\
0 & 0 & 0 & 1
\end{pmatrix}.
$$

Thus $T_{\mathfrak{g}}$ has eigenvalue 1 of multiplicity two with two linearly independent eigenfunctions $c_0$ and $c_3$, and eigenvalue $1 + T^{3/2}$ of multiplicity two but only one linearly independent eigenfunction $T^{1/2}c_1 + c_2$. Hence $T_{\mathfrak{g}}$ is not diagonalizable on $S_{5,m}(\Gamma_1(T))$. Further, since $c_1$ and $c_2$ span the space $S_{5,m}^2(\Gamma_1)$, this shows that $T_{\mathfrak{g}}$ is not diagonalizable on $S_{5,m}^2(\Gamma_1(T))$ either.
Cusp forms for $\Gamma(T)$

- A fundamental domain of $\Gamma(T) \backslash T$ contains $q + 1$ rays, corresponding to the cusps $[\infty] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $[r] = \begin{pmatrix} r \\ 1 \end{pmatrix}$, $r \in \mathbb{F}$, one stable vertex $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $q + 1$ stable edges $\gamma_r := \langle \begin{pmatrix} r & 1 \\ 1 & 0 \end{pmatrix} \rangle$, $r \in \mathbb{F}$, and $\gamma_\infty := \langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rangle$.

- A harmonic cocycle for $\Gamma(T)$ is determined by its values at $\gamma_r, r \in \mathbb{F}$, since

$$a(\gamma_\infty) + \sum_{r \in \mathbb{F}} a(\gamma_r) = 0.$$ 

- 

$$\dim_C S_{k,m}(\Gamma(T)) = (k - 1)q,$$

$$\dim_C S^2_{2,m}(\Gamma(T)) = 0$$

and

$$\dim_C S^2_{k,m}(\Gamma(T)) = (k-2)q-1 \text{ for } k \geq 3.$$ 

- 

$$V(k, m)^{\Gamma(T)}_{[\infty]} = \langle Y^{k-2} \rangle$$

$$V(k, m)^{\Gamma(T)}_{[r]} = \langle (X - rY)^{k-2} \rangle.$$ 

16
\[ S^2_{k,m}(\Gamma(T)) = \{ c \in S_{k,m}(\Gamma(T)) : \]
\[ c(\gamma_\infty)(Y^{k-2}) = 0 \quad \text{and} \quad c(\gamma_r)((X - rY)^{k-2}) = 0 \quad \text{for all } r \in \mathbb{F} \}. \]

- In view of this, use the basis
\[ (X - rY)^j Y^{k-2-j}, \quad 0 \leq j \leq k - 2, \]
for \( V(k, m) \) to describe the value of \( c(\gamma_r) \).
(different bases on different directed edges)

- Compute the action of \( T_\Psi \) with resp. to the new bases.

- Let \( c \in S_{k,m}(\Gamma(T)) \) be an eigenfunction of \( T_\Psi \), where \( \Psi \neq (T) \) has degree 1. If it is not a double cusp form, then the eigenvalue is 1.

- For \( q \geq k \geq 2 \), the functions \( c_j \) in \( S_{k,m}(\Gamma_1(T)) \) lift to eigenfunctions in \( S_{k,m}(\Gamma(T)) \) with eigenvalues \( \lambda_j(P) \).
Theorem 5. Let $\mathfrak{p} \neq (T)$ be a degree one prime ideal of $A$. For $q \geq k \geq 2$ the distinct eigenvalues for the Hecke operator $T_\mathfrak{p}$ on $S_{k,m}(\Gamma(T))$ are the distinct $\lambda_j(P)$, $0 \leq j \leq k - 2$.

This theorem is proved by using the following fact.

Let $c$ be an eigenfunction of $T_\mathfrak{p}$ with eigenvalue $\lambda \neq \lambda_n(P)$ for all $0 \leq n \leq k - 2$. Put

$$Z(r, j) = c(\gamma_r)(((X - rY)^jY^{k-2-j}).$$

Then

$$(\lambda - \lambda_n(P))Z(r, n) = \sum_{u=n+1}^{k-2} A_u^{(n)} \sum_{b \neq r} (b - r)^{n-u} Z(b, u),$$

where $A_u^{(n)} \in K$ depends only $u$ and $n$.

This expression also leads to a description of the $\lambda_n(P)$-eigenspace. More precisely, we have
Theorem 6. Suppose \( q \geq k \geq 2, \mathcal{P} = (P) \neq (T) \) degree one. Then \( \lambda_i(P), 0 \leq i \leq k - 2, \) are the eigenvalues of \( T_\mathcal{P} \) on \( S_{k,m}(\Gamma(T)) \). For \( 0 \leq n \leq k - 2, \) set
\[
A_n = \{i : 0 \leq i \leq k-2 \text{ and } \lambda_i(P) = \lambda_n(P)\}
\]
and denote the integers in \([0, k-2] \setminus A_n\) by \( l_0 < \cdots < l_t. \) Let \( c \) be an eigenfunction in \( H_{k,m}(\Gamma(T)) \) with eigenvalue \( \lambda_n(P); \) write
\[
Z(b, u) = c(\gamma_b)((X - bY)^u Y^{k-2-u}).
\]
Then \( c \) is determined by \( Z(b, u) \) with \( u \in A_n \) and \( b \in \mathbb{F} \) subject to the conditions
\[
0 = \sum_{0 \leq u \leq k-2} C_u(i, P) \sum_{b \neq r} (b-r)^{i-u} Z(b, u)
\]
for \( i \in A_n \) and \( r \in \mathbb{F}. \) The remaining \( Z(b, l) \)'s are determined by
\[
Z(b, l) = \sum_{0 \leq u < l} A_{u}^{(l,v)} \sum_{s \in \mathbb{F}} (s - b)^{l_v-u} Z(s, u)
\]
\[
+ \sum_{u = l_v + 1}^{k-2} A_{u}^{(l,v)} \sum_{s \neq b} (s - b)^{l_v-u} Z(s, u)
\]
from $v = t$ to $v = 0$. Here $C_u(i, P)$ and $A_u^{(1)}$ are explicitly given elements in $K$. 
Example. $S_{3,m}(\Gamma(T))$

- $\lambda_0(P) = \lambda_1(P) = 1$,
  $$\dim_C S_{3,m}(\Gamma(T)) = 2q,$$
  $$\dim_C S_{3,m}^2(\Gamma(T)) = q - 1.$$  
- Only one relation
  $$\sum_{b \in \mathbb{F}} Z(b, 0) = \sum_{b \neq r} \frac{Z(b, 1)}{r - b} \text{ for all } r \in \mathbb{F}$$
- Solve the system
  $$M \begin{pmatrix} Z(0, 1) \\ Z(a, 1) \\ Z(a^2, 1) \\ \vdots \\ Z(a^{q-1}, 1) \end{pmatrix} = \begin{pmatrix} c \\ c \\ c \\ \vdots \\ c \end{pmatrix},$$
  where $M$ is the coefficient matrix
  $$M = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & \frac{1}{a} & 0 & \ldots & 0 \\ 0 & 0 & \frac{1}{a^2} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & \frac{1}{a^{q-1}} \end{pmatrix} C$$
with

\[
C = \begin{pmatrix}
0 & -\frac{1}{a} & -\frac{1}{a^2} & -\frac{1}{a^3} & \cdots & -\frac{1}{a^{q-1}} \\
1 & 0 & \frac{1}{1-a} & \frac{1}{1-a^2} & \cdots & \frac{1}{1-a^{q-2}} \\
1 & \frac{1}{1-a^{q-2}} & 0 & \frac{1}{1-a} & \cdots & \frac{1}{1-a^{q-3}} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \frac{1}{1-a} & \frac{1}{1-a^2} & \frac{1}{1-a^3} & \cdots & 0
\end{pmatrix}
\]

and \(c = \sum_{b \in \mathbb{F}} Z(b, 0)\).

- \(v_0 = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}\) and \(v_j = \begin{pmatrix} 0 \\ a^j \\ \vdots \\ a^{(q-2)j} \end{pmatrix}\), \(j = 1, \ldots, q-1\), are \(q\) linearly independent eigenvectors of \(C\) with the eigenvalues 0 and \(j\), resp.
- Since \(\mathbb{F}\) has characteristic \(p\), this shows that the nullity of \(M\) is \(q/p\).
- If \(c = \sum_{b \in \mathbb{F}} Z(b, 0) = 0\), then we obtain \((q-1) + q/p\) linearly independent eigenvectors for \(T_{\mathfrak{F}}\).
• When $c \neq 0$, $\mathbf{v} = \left( \begin{array}{c} 0 \\ ca \\ ca^2 \\ \vdots \\ ca^{q-1} \end{array} \right)$ is a solution of the system.

**Proposition 7.** Suppose $q \geq 3$ and characteristic $p$. For a degree one prime ideal $\mathfrak{p} \neq (T)$, 1 is the only eigenvalue of $T_{\mathfrak{p}}$ on $S_{3,m}(\Gamma(T))$. The eigenspace of $T_{\mathfrak{p}}$ has dimension $q + q/p$, hence $T_{\mathfrak{p}}$ is not diagonalizable on $S_{3,m}(\Gamma(T))$. But $T_{\mathfrak{p}}$ is diagonalizable on $S_{3,m}^{2}(\Gamma(T))$. 

23
Example. $S_{4,m}(\Gamma(T))$

- $\lambda_0(P) = \lambda_2(P) = 1$ and $\lambda_1(P) = 2 - P$.
- $\dim_C S_{4,m}(\Gamma(T)) = 3q, \dim_C S_{4,m}^2(\Gamma(T)) = 2q - 1$.
- Two relations on 1-eigenspace

$$\sum_{b \in \mathbb{F}} Z(b, 0) = \sum_{b \neq r} \frac{Z(b, 2)}{(r - b)^2}$$

and

$$Z(r, 1) = \sum_{b \in \mathbb{F}} (b - r)Z(b, 0) + \sum_{b \neq r} \frac{Z(b, 2)}{(b - r)}$$

for all $r \in \mathbb{F}$.
- $(2 - P)$-eigenspace contains only double cusp forms.
- Relations are $Z(r, 2) = 0$ for all $r \in \mathbb{F}$ and

$$Z(r, 0) = -2 \sum_{b \neq r} \frac{Z(b, 1)}{b - r}$$

for all $r \in \mathbb{F}$.
- So $(2 - P)$-eigenspace is $q$-dim'1.
Proposition 8. Suppose $q \geq 4$ and characteristic $p$. For a degree one prime ideal $\frak{p} = (P) \neq (T)$, 1 and $2 - P$ are the two distinct eigenvalues of $T_{\frak{p}}$ on $S_{4,m}(\Gamma(T))$. The 1-eigenspace has dimension $q + 2q/p$ if $p > 2$ and dimension $q + q/p$ if $p = 2$. The $(2 - P)$-eigenspace has dimension $q$. Thus $T_{\frak{p}}$ is not diagonalizable on $S_{4,m}(\Gamma(T))$. But $T_{\frak{p}}$ on $S^2_{4,m}(\Gamma(T))$ is diagonalizable.
**Example.** $S_{5,m}(\Gamma(T))$

- $\lambda_0(P) = \lambda_3(P) = 1$ and
  $\lambda_1(P) = \lambda_2(P) = 3 - 2P$,
- $\dim \mathcal{S}_{5,m}(\Gamma(T)) = 4q$,
- $\dim \mathcal{S}_{5,m}^2(\Gamma(T)) = 3q - 1$.

First assume $p > 2$ so that $1 \neq 3 - 2P$.

- Three relations on 1-eigenspace

\[
Z(r, 1) = \frac{p+1}{2} \sum_{b \in \mathbb{F}} (b - r)Z(b, 0)
- \frac{p-3}{2} \sum_{b \neq r} \frac{Z(b,2)}{b-r}
- \frac{p-1}{2} \sum_{b \neq r} \frac{Z(b,3)}{(b-r)^2},
\]

\[
Z(r, 2) = \frac{1}{2} \sum_{b \in \mathbb{F}} (b-r)^2 Z(b, 0) + \frac{1}{2} \sum_{b \neq r} \frac{Z(b,3)}{b-r},
\]

and

\[
\sum_{b \in \mathbb{F}} Z(b, 0) = \sum_{b \neq r} \frac{Z(b,3)}{(r-b)^3}
\]

for all $r \in \mathbb{F}$.

- $(3 - 2P)$-eigenspace contains only double cusp forms.
• Relations are $Z(r, 3) = 0,$
\[
\sum_{b \in \mathbb{F}} Z(b, 1) = 2 \sum_{b \neq r} Z(b, 2) \frac{r}{r - b}
\]
and
\[
\sum_{b \in \mathbb{F}} Z(b, 2) = 0
\]
for all $r \in \mathbb{F}$.

**Proposition 9.** Suppose $q \geq 4$ and characteristic $p > 2$. For a degree one prime ideal $\mathfrak{p} = (P) \neq (T)$, $1$ and $3 - 2P$ are the two distinct eigenvalues of $T_{\mathfrak{p}}$ on $S_{5,m}(\Gamma(T))$. The $1$-eigenspace has dimension $q + 3q/p$ if $p > 3$ and dimension $q + q/p$ if $p = 3$. The $(3 - 2P)$-eigenspace has dimension $q + q/p$. Further, neither on $S_{5,m}(\Gamma(T))$ nor on $S_{5,m}^2(\Gamma(T))$ is $T_{\mathfrak{p}}$ diagonalizable.

**Proposition 10.** When $p = 2$, $1$ is the only eigenvalue of $T_{\mathfrak{p}}$ on $S_{5,m}(\Gamma(T))$ and neither on $S_{5,m}(\Gamma(T))$ nor on $S_{5,m}^2(\Gamma(T))$ is $T_{\mathfrak{p}}$ diagonalizable.