

Hecke Operators on Drinfeld Cusp Forms

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Drinfeld Upper Half Plane

- $K = \mathbb{F}[T]$, where $|\mathbb{F}| = q$.
- $\infty =$ place at infinity
- $A = \mathbb{F}[T] =$ elements regular outside ∞
- $K_\infty =$ completion of K at $\infty = \mathbb{F}((1/T))$
- $\mathcal{O}_\infty =$ ring of integers in K_∞
- $\mathcal{P}_\infty =$ the maximal ideal of \mathcal{O}_∞
- $C = \widehat{K_\infty}$
- Drinfeld upper half plane $\Omega = C \setminus K_\infty$
 - $GL_2(K_\infty)$ acts by fractional linear transformations.
 - It is a rigid analytic space.
- For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(K_\infty)$, $m, k \in \mathbb{Z}$ and $f : \Omega \rightarrow C$, define

$$(f \mid \gamma)_{k,m}(z) := f(\gamma z) (\det \gamma)^m (cz + d)^{-k}.$$

Drinfeld Cusp Forms

- $\Gamma =$ congruence subgroup of $GL_2(A)$
- Γ has finitely many cusps, represented by the orbits $\Gamma \backslash \mathbb{P}^1(K)$.
- A rigid analytic function $f : \Omega \rightarrow C$ is called a Drinfeld cusp form of weight k and type m for Γ if it satisfies
 - (i) $f|_{k,m} \gamma = f$ for all $\gamma \in \Gamma$;
 - (ii) f is holomorphic at all cusps;
 - (iii) f vanishes at all cusps.
- $S_{k,m}(\Gamma) =$ space of cusp forms.
- $S_{k,m}^2(\Gamma) =$ subspace of double cusp forms.
- Shall take $\Gamma = \Gamma_1(T)$ and $\Gamma(T)$. In this case $\det \Gamma = 1$ and the spaces $S_{k,m}(\Gamma)$ are independent of m .

$$\dim_C S_{k,m}(\Gamma) = (k - 1)(g_\Gamma + h_\Gamma - 1)$$

The Tree \mathcal{T}

- $\mathcal{Z}_\infty =$ the center of $GL_2(K_\infty)$
- $\mathcal{T} := GL_2(K_\infty)/GL_2(\mathcal{O}_\infty)\mathcal{Z}_\infty$ is a $(q + 1)$ -regular tree on which $GL_2(K_\infty)$ acts by left translations.
- vertices of $\mathcal{T} \approx GL_2(K_\infty)/GL_2(\mathcal{O}_\infty)\mathcal{Z}_\infty$
- directed edges $\approx GL_2(K_\infty)/\mathfrak{I}_\infty\mathcal{Z}_\infty$, where

$$\mathfrak{I}_\infty = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathcal{O}_\infty) : c \in \mathcal{P}_\infty \right\}$$
 is the Iwahori subgroup of $GL_2(\mathcal{O}_\infty)$.
- $\langle g \rangle =$ the edge coset represented by $g \in GL_2(K_\infty)$.
- Two paths in \mathcal{T} are equivalent if they differ at only finitely many edges.
- An end of \mathcal{T} is an equivalence class of infinite paths $\{e_1, e_2, \dots\}$.
- There is a canonical bijection from the set of ends to $\mathbb{P}^1(K_\infty)$, the boundary of Ω .
- The rational ends are $\mathbb{P}^1(K)$, corresponding to cusps.

Harmonic Cocycles

- $V(k, m)$ = vector space over C with a basis $X^j Y^{k-2-j}$, $0 \leq j \leq k-2$
- $GL_2(K_\infty)$ acts on $Hom(V(k, m), C)$.
- A harmonic cocycle of weight k and type m for Γ is a function \mathbf{c} from the directed edges of \mathcal{T} to $Hom(V(k, m), C)$ satisfying
 - (a) For all vertices v of \mathcal{T} ,

$$\sum_{e \mapsto v} \mathbf{c}(e) = 0;$$

- (b) For all edges e of \mathcal{T} , $\mathbf{c}(\bar{e}) = -\mathbf{c}(e)$, where $\bar{e} = e$ with reversed orientation;
- (c) It is Γ -equivariant, i.e.,

$$\mathbf{c}(\gamma e) = \gamma(\mathbf{c}(e))$$

for all $\gamma \in \Gamma$.

- $H_{k,m}(\Gamma)$ = space of harmonic cocycles

Cusp Forms and Harmonic Cocycles

- Building map : $\Omega \rightarrow \mathcal{T}$, commuting with the action of $GL_2(K_\infty)$.
- It maps the annulus

$$V = \{z \in \Omega : 1 < |z| < q\}$$

to a directed edge e_0 , and γV to γe_0 for $\gamma \in GL_2(K_\infty)$.

- $f(z)dz = \sum_{i \in \mathbb{Z}} a_i z^i dz$ holomorphic C -valued differential on V
- Let $\text{Res}_{e_0} f(z)dz = a_{-1}$, and

$$\text{Res}_{\gamma e_0} f(z)dz = \text{Res}_{e_0} g(u)du,$$

where $f(z)dz = f(\gamma u)d\gamma u = g(u)du$.

- Define $\text{Res} : S_{k,m}(\Gamma) \rightarrow H_{k,m}(\Gamma)$ by $\text{Res}(f)(e)(X^j Y^{k-2-j}) = \text{Res}_e z^j f(z)dz$ for $0 \leq j \leq k-2$.
- Teitelbaum: Res is an isomorphism.

Identify the two spaces.

This allows us to view a Drinfeld cusp form as a vector valued left Γ -equivariant function on $GL_2(K_\infty)/\mathfrak{I}_\infty\mathcal{Z}_\infty$. When $k = 2$, it is a left Γ -invariant \mathbb{C} -valued function. There is a suitable basis of $H_{2,m}(\Gamma)$ which take values in $\mathbb{Z}/p\mathbb{Z}$, and hence they can be lifted to \mathbb{C} -valued automorphic forms on $GL_2(\mathbb{A}_K)$.

- $\Gamma_{[s]}$ = Γ -stabilizer of an end $[s]$ representing a cusp of Γ .
- Böckle:
 - (a) $V(k, m)^{\Gamma_{[s]}}$ is one-dimensional.
 - (b) $\Gamma_{[s]}$ acts freely on $src([s])$ with finitely many orbits, represented by edges $e_1^{[s]}, \dots, e_{l_s}^{[s]}$.
 - (c) Let $f \in S_{k,m}(\Gamma)$ and $\mathbf{c} = \text{Res}(f)$. Then $f \in S_{k,m}^2(\Gamma)$ iff for any cusp $[s]$,

$$\sum_{i=1}^{l_s} \mathbf{c}(e_i^{[s]})(g_s) = 0$$

for any $g_s \in V(k, m)^{\Gamma_{[s]}}$.

- $\dim_C S_{2,m}^2(\Gamma) = g_\Gamma$
- $\dim_C S_{k,m}^2(\Gamma)$
 $= (k - 2)(g_\Gamma + h_\Gamma - 1) + g_\Gamma - 1$
if $k \geq 3$.

Hecke Operators

- $\Gamma = \Gamma_1(T)$ and $\Gamma(T)$
- $\mathfrak{P} = (P) \neq (T)$ is a prime ideal of A ,
 $P(0) = 1$, $\deg P = d$
- The Hecke operator at \mathfrak{P} is

$$T_{\mathfrak{P}} = P^{k-m-1} \left[\begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} + \sum_{b \in A, \deg b < d} \begin{pmatrix} 1 & b(1-P) \\ 0 & P \end{pmatrix} \right]$$

- Transport its action from $S_{k,m}(\Gamma)$ to $H_{k,m}(\Gamma)$

$$T_{\mathfrak{P}}\mathbf{c}(e) = P^{k-m-1} \left[\begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}^{-1} \mathbf{c} \left(\begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} e \right) + \sum_{b \in A, \deg b < d} \begin{pmatrix} 1 & b(1-P) \\ 0 & P \end{pmatrix}^{-1} \mathbf{c} \left(\begin{pmatrix} 1 & b(1-P) \\ 0 & P \end{pmatrix} e \right) \right].$$

Shall study the behavior of $T_{\mathfrak{P}}$ for degree one prime ideal $\mathfrak{P} \neq (T)$ on cusp forms and double cusp forms.

Cusp Forms for $\Gamma_1(T)$

- A fundamental domain of $\Gamma_1(T) \backslash \mathcal{T}$ is a path connecting two cusps, $[\infty] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $[0] = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, containing no stable vertices and one stable edge $\gamma_0 := \langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle$.
- A harmonic cocycle is determined by its value (in $\text{Hom}(V(k, m), C)$) at γ_0 .

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$$\begin{aligned} \dim_C S_{k,m}(\Gamma_1(T)) &= k - 1, \\ \dim_C S_{2,m}^2(\Gamma_1(T)) &= 0, \end{aligned}$$

and

$$\dim_C S_{k,m}^2(\Gamma_1(T)) = k - 3 \quad \text{for } k \geq 3.$$

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$$V(k, m)^{(\Gamma_1(T))}_{[\infty]} = \langle Y^{k-2} \rangle$$

and

$$V(k, m)^{(\Gamma_1(T))}_{[0]} = \langle X^{k-2} \rangle.$$

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$$\begin{aligned} S_{k,m}^2(\Gamma_1(T)) &= \{ \mathbf{c} \in S_{k,m}(\Gamma_1(T)) : \\ \mathbf{c}(\gamma_0)(Y^{k-2}) &= \mathbf{c}(\gamma_0)(X^{k-2}) = 0 \}. \end{aligned}$$

$S_{k,m}(\Gamma_1(T))$ – **Case I. k Small**

- Assume $q \geq k \geq 2$. For $0 \leq j \leq k - 2$ and $\mathfrak{P} = (P)$ with $P = 1 + \alpha T$,

$$\begin{aligned} T_{\mathfrak{P}}\mathbf{c}(\gamma_0)(X^j Y^{k-2-j}) \\ = \lambda_j(P)\mathbf{c}(\gamma_0)(X^j Y^{k-2-j}), \end{aligned}$$

where

$$\begin{aligned} \lambda_j(P) &= \sum_{l=0}^j \binom{j}{l} \binom{k-2-j}{l} (1-P)^l \\ &= \sum_{l=0}^{\min\{j, k-2-j\}} \binom{j}{l} \binom{k-2-j}{l} (-\alpha T)^l. \end{aligned}$$

- $\deg \lambda_j(P) \leq \min\{j, k - 2 - j\}$.
Note that $\lambda_0(P) = \lambda_{k-2}(P) = 1$ and $\lambda_j(P) = \lambda_{k-2-j}(P)$ for all $0 \leq j \leq k-2$.
- For $0 \leq j \leq k - 2$ and any k , define a harmonic cocycle \mathbf{c}_j by specifying its value at γ_0 by:

$$\mathbf{c}_j(\gamma_0)(X^j Y^{k-2-j}) = 1$$

and

$$\mathbf{c}_j(\gamma_0)(X^l Y^{k-2-l}) = 0 \text{ for } l \neq j.$$

Theorem 1. *Let \mathfrak{P} be a prime ideal of A generated by P with $P(0) = 1$ and $\deg P = 1$. Suppose $q \geq k \geq 2$. Then*

- (1) *Each \mathbf{c}_j , $0 \leq j \leq k - 2$, is an eigenfunction of $T_{\mathfrak{P}}$ with eigenvalue $\lambda_j(P)$; and*
- (2) *The Hecke operators at the ideals of degree one are simultaneously diagonalized on $S_{k,m}(\Gamma_1(T))$ with respect to the basis \mathbf{c}_j , $0 \leq j \leq k - 2$.*

Conjecture. *Same assumptions as above except $\deg P = d \geq 1$. Let θ be a root of P . Then each \mathbf{c}_j , $0 \leq j \leq k - 2$, is an eigenfunction of $T_{\mathfrak{P}}$ with eigenvalue*

$$\lambda_j(P) := \prod_{i=0}^{d-1} \lambda_j(1 + \theta^{q^i} T).$$

Consequently the Hecke operators are simultaneously diagonalized on $S_{k,m}(\Gamma_1(T))$.

Conjecture verified for $d = 2$.

Remark 2. If we factor the polynomial

$$\lambda_j(1+T) = \prod_{s=1}^{\deg \lambda_j(1+T)} (1 + \delta_s T),$$

then

$$\lambda_j(P) = \prod_{s=1}^{\deg \lambda_j(1+T)} P(\delta_s T).$$

Observe that the degree of $\lambda_j(P)$ above is at most $d(k-2)/2$. This may be regarded as the Ramanujan conjecture on Drinfeld cusp forms.

Theorem 3. *Let $\mathfrak{P} = (P)$, $P(0) = 1$, be a degree one prime ideal of A . If $q+2 \geq k \geq 4$, then \mathbf{c}_j , $1 \leq j \leq k-3$, are eigenfunctions of $T_{\mathfrak{P}}$ on $S_{k,m}^2(\Gamma_1(T))$ with eigenvalue $\lambda_j(P)$. Consequently the Hecke operators for degree one prime ideals are simultaneously diagonalized on $S_{k,m}^2(\Gamma_1(T))$ with respect to the basis \mathbf{c}_j , $1 \leq j \leq k-3$.*

$S_{k,m}(\Gamma_1(T))$ – **Case II. k Large**

- For $i = 0, 1, \dots, q - 2$, let $S_{k,m}(\Gamma_1(T))_i$ be the subspace generated by

$$\left\{ \mathbf{c}_i, \mathbf{c}_{i+(q-1)}, \dots, \mathbf{c}_{i+\lfloor \frac{k-2-i}{q-1} \rfloor (q-1)} \right\}$$

so that $S_{k,m}(\Gamma_1(T)) = \bigoplus_{i=0}^{q-2} S_{k,m}(\Gamma_1(T))_i$.

- Each $S_{k,m}(\Gamma_1(T))_i$ is invariant under $T_{\mathfrak{P}}$ for \mathfrak{P} of degree one. With resp. to the basis above, the action of $T_{\mathfrak{P}}$ can be represented by an explicit matrix.
- The degree one Hecke operators may or may not be diagonalized.

Example 4. $q = 2$ and $k = 5$. There is only one poly. $P = 1 + T$, and only one subspace. $T_{\mathfrak{P}}$ is represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ T^2 & 1 & T^2 & T^3 \\ T & T & 1 & T^3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus $T_{\mathfrak{P}}$ has eigenvalue 1 of multiplicity two with two linearly independent eigenfunctions \mathbf{c}_0 and \mathbf{c}_3 , and eigenvalue $1 + T^{3/2}$ of multiplicity two but only one linearly independent eigenfunction $T^{1/2}\mathbf{c}_1 + \mathbf{c}_2$. Hence $T_{\mathfrak{P}}$ is not diagonalizable on $S_{5,m}(\Gamma_1(T))$. Further, since \mathbf{c}_1 and \mathbf{c}_2 span the space $S_{5,m}^2(\Gamma_1)$, this shows that $T_{\mathfrak{P}}$ is not diagonalizable on $S_{5,m}^2(\Gamma_1(T))$ either.

Cusp forms for $\Gamma(T)$

- A fundamental domain of $\Gamma(T) \backslash \mathcal{T}$ contains $q+1$ rays, corresponding to the cusps $[\infty] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $[r] = \begin{pmatrix} r \\ 1 \end{pmatrix}$, $r \in \mathbb{F}$, one stable vertex $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $q+1$ stable edges $\gamma_r := \langle \begin{pmatrix} r & 1 \\ 1 & 0 \end{pmatrix} \rangle$, $r \in \mathbb{F}$, and $\gamma_\infty := \langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rangle$.
- A harmonic cocycle for $\Gamma(T)$ is determined by its values at γ_r , $r \in \mathbb{F}$, since

$$\mathbf{c}(\gamma_\infty) + \sum_{r \in \mathbb{F}} \mathbf{c}(\gamma_r) = 0.$$

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$$\dim_{\mathbb{C}} S_{k,m}(\Gamma(T)) = (k-1)q,$$

$$\dim_{\mathbb{C}} S_{2,m}^2(\Gamma(T)) = 0$$

and

$$\dim_{\mathbb{C}} S_{k,m}^2(\Gamma(T)) = (k-2)q-1 \text{ for } k \geq 3.$$

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$$V(k, m)^{\Gamma(T)_{[\infty]}} = \langle Y^{k-2} \rangle$$

$$V(k, m)^{\Gamma(T)_{[r]}} = \langle (X - rY)^{k-2} \rangle.$$

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$$S_{k,m}^2(\Gamma(T)) = \{ \mathbf{c} \in S_{k,m}(\Gamma(T)) : \\ \mathbf{c}(\gamma_\infty)(Y^{k-2}) = 0 \text{ and} \\ \mathbf{c}(\gamma_r)((X - rY)^{k-2}) = 0 \text{ for all } r \in \mathbb{F} \}.$$

- In view of this, use the basis

$$(X - rY)^j Y^{k-2-j}, \quad 0 \leq j \leq k - 2,$$

for $V(k, m)$ to describe the value of $\mathbf{c}(\gamma_r)$.
(different bases on different directed edges)

- Compute the action of $T_{\mathfrak{P}}$ with resp. to the new bases.
- Let $\mathbf{c} \in S_{k,m}(\Gamma(T))$ be an eigenfunction of $T_{\mathfrak{P}}$, where $\mathfrak{P} \neq (T)$ has degree 1. If it is not a double cusp form, then the eigenvalue is 1.
- For $q \geq k \geq 2$, the functions \mathbf{c}_j in $S_{k,m}(\Gamma_1(T))$ lift to eigenfunctions in $S_{k,m}(\Gamma(T))$ with eigenvalues $\lambda_j(P)$.

Theorem 5. *Let $\mathfrak{P} \neq (T)$ be a degree one prime ideal of A . For $q \geq k \geq 2$ the distinct eigenvalues for the Hecke operator $T_{\mathfrak{P}}$ on $S_{k,m}(\Gamma(T))$ are the distinct $\lambda_j(P)$, $0 \leq j \leq k - 2$.*

This theorem is proved by using the following fact.

Let \mathbf{c} be an eigenfunction of $T_{\mathfrak{P}}$ with eigenvalue $\lambda \neq \lambda_n(P)$ for all $0 \leq n \leq k - 2$. Put

$$Z(r, j) = \mathbf{c}(\gamma_r)((X - rY)^j Y^{k-2-j}).$$

Then

$$\begin{aligned} & (\lambda - \lambda_n(P))Z(r, n) \\ &= \sum_{u=n+1}^{k-2} A_u^{(n)} \sum_{b \neq r} (b - r)^{n-u} Z(b, u), \end{aligned}$$

where $A_u^{(n)} \in K$ depends only u and n .

This expression also leads to a description of the $\lambda_n(P)$ -eigenspace. More precisely, we have

Theorem 6. *Suppose $q \geq k \geq 2$, $\mathfrak{P} = (P) \neq (T)$ degree one. Then $\lambda_i(P)$, $0 \leq i \leq k-2$, are the eigenvalues of $T_{\mathfrak{P}}$ on $S_{k,m}(\Gamma(T))$. For $0 \leq n \leq k-2$, set*

$$A_n = \{i : 0 \leq i \leq k-2 \text{ and } \lambda_i(P) = \lambda_n(P)\}$$

and denote the integers in $[0, k-2] \setminus A_n$ by $l_0 < \dots < l_t$. Let \mathbf{c} be an eigenfunction in $H_{k,m}(\Gamma(T))$ with eigenvalue $\lambda_n(P)$; write

$$Z(b, u) = \mathbf{c}(\gamma_b)((X - bY)^u Y^{k-2-u}).$$

Then \mathbf{c} is determined by $Z(b, u)$ with $u \in A_n$ and $b \in \mathbb{F}$ subject to the conditions

$$0 = \sum_{\substack{0 \leq u \leq k-2 \\ \lambda_u(P) = \lambda_n(P)}} C_u(i, P) \sum_{b \neq r} (b-r)^{i-u} Z(b, u)$$

for $i \in A_n$ and $r \in \mathbb{F}$. The remaining $Z(b, l)$'s are determined by

$$\begin{aligned} & Z(b, l_v) \\ = & \sum_{\substack{0 \leq u < l_v \\ \lambda_u(P) = \lambda_n(P)}} A_u^{(l_v)} \sum_{s \in \mathbb{F}} (s-b)^{l_v-u} Z(s, u) \\ & + \sum_{u=l_v+1}^{k-2} A_u^{(l_v)} \sum_{s \neq b} (s-b)^{l_v-u} Z(s, u) \end{aligned}$$

from $v = t$ to $v = 0$. Here $C_u(i, P)$ and $A_u^{(l)}$ are explicitly given elements in K .

Example. $S_{3,m}(\Gamma(T))$

- $\lambda_0(P) = \lambda_1(P) = 1$,
 $\dim_{\mathbb{C}} S_{3,m}(\Gamma(T)) = 2q$,
 $\dim_{\mathbb{C}} S_{3,m}^2(\Gamma(T)) = q - 1$.
- Only one relation

$$\sum_{b \in \mathbb{F}} Z(b, 0) = \sum_{b \neq r} \frac{Z(b, 1)}{r - b} \text{ for all } r \in \mathbb{F}$$

- Solve the system

$$M \begin{pmatrix} Z(0, 1) \\ Z(a, 1) \\ Z(a^2, 1) \\ \vdots \\ Z(a^{q-1}, 1) \end{pmatrix} = \begin{pmatrix} c \\ c \\ c \\ \vdots \\ c \end{pmatrix},$$

where M is the coefficient matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \frac{1}{a} & 0 & \dots & 0 \\ 0 & 0 & \frac{1}{a^2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{a^{q-1}} \end{pmatrix} C$$

with

$$C = \begin{pmatrix} 0 & -\frac{1}{a} & -\frac{1}{a^2} & -\frac{1}{a^3} & \cdots & -\frac{1}{a^{q-1}} \\ 1 & 0 & \frac{1}{1-a} & \frac{1}{1-a^2} & \cdots & \frac{1}{1-a^{q-2}} \\ 1 & \frac{1}{1-a^{q-2}} & 0 & \frac{1}{1-a} & \cdots & \frac{1}{1-a^{q-3}} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \frac{1}{1-a} & \frac{1}{1-a^2} & \frac{1}{1-a^3} & \cdots & 0 \end{pmatrix}$$

and $c = \sum_{b \in \mathbb{F}} Z(b, 0)$.

- $\mathbf{v}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ and $\mathbf{v}_j = \begin{pmatrix} 0 \\ 1 \\ a^j \\ \vdots \\ a^{(q-2)j} \end{pmatrix}$,

$j = 1, \dots, q-1$, are q linearly independent eigenvectors of C with the eigenvalues 0 and j , resp.

- Since \mathbb{F} has characteristic p , this shows that the nullity of M is q/p .
- If $c = \sum_{b \in \mathbb{F}} Z(b, 0) = 0$, then we obtain $(q-1) + q/p$ linearly independent eigenvectors for $T_{\mathfrak{A}}$.

- When $c \neq 0$, $\mathbf{v} = \begin{pmatrix} 0 \\ ca \\ ca^2 \\ \vdots \\ ca^{q-1} \end{pmatrix}$ is a solution of the system.

Proposition 7. *Suppose $q \geq 3$ and characteristic p . For a degree one prime ideal $\mathfrak{P} \neq (T)$, 1 is the only eigenvalue of $T_{\mathfrak{P}}$ on $S_{3,m}(\Gamma(T))$. The eigenspace of $T_{\mathfrak{P}}$ has dimension $q + q/p$, hence $T_{\mathfrak{P}}$ is not diagonalizable on $S_{3,m}(\Gamma(T))$. But $T_{\mathfrak{P}}$ is diagonalizable on $S_{3,m}^2(\Gamma(T))$.*

Example. $S_{4,m}(\Gamma(T))$

- $\lambda_0(P) = \lambda_2(P) = 1$ and $\lambda_1(P) = 2 - P$.
 $\dim_C S_{4,m}(\Gamma(T)) = 3q$,
 $\dim_C S_{4,m}^2(\Gamma(T)) = 2q - 1$.
- Two relations on 1-eigenspace

$$\sum_{b \in \mathbb{F}} Z(b, 0) = \sum_{b \neq r} \frac{Z(b, 2)}{(r - b)^2}$$

and

$$Z(r, 1) = \sum_{b \in \mathbb{F}} (b - r) Z(b, 0) + \sum_{b \neq r} \frac{Z(b, 2)}{(b - r)}$$

for all $r \in \mathbb{F}$.

- $(2 - P)$ -eigenspace contains only double cusp forms.
- Relations are $Z(r, 2) = 0$ for all $r \in \mathbb{F}$ and

$$Z(r, 0) = -2 \sum_{b \neq r} \frac{Z(b, 1)}{b - r} \text{ for all } r \in \mathbb{F}.$$

- So $(2 - P)$ -eigenspace is q -dim'l.

Proposition 8. *Suppose $q \geq 4$ and characteristic p . For a degree one prime ideal $\mathfrak{P} = (P) \neq (T)$, 1 and $2 - P$ are the two distinct eigenvalues of $T_{\mathfrak{P}}$ on $S_{4,m}(\Gamma(T))$. The 1 -eigenspace has dimension $q + 2q/p$ if $p > 2$ and dimension $q + q/p$ if $p = 2$. The $(2 - P)$ -eigenspace has dimension q . Thus $T_{\mathfrak{P}}$ is not diagonalizable on $S_{4,m}(\Gamma(T))$. But $T_{\mathfrak{P}}$ on $S_{4,m}^2(\Gamma(T))$ is diagonalizable.*

Example. $S_{5,m}(\Gamma(T))$

- $\lambda_0(P) = \lambda_3(P) = 1$ and
 $\lambda_1(P) = \lambda_2(P) = 3 - 2P$,
 $\dim_C S_{5,m}(\Gamma(T)) = 4q$,
 $\dim_C S_{5,m}^2(\Gamma(T)) = 3q - 1$.
 First assume $p > 2$ so that $1 \neq 3 - 2P$.
- Three relations on 1-eigenspace

$$Z(r, 1) = \frac{P+1}{2} \sum_{b \in \mathbb{F}} (b-r) Z(b, 0) - \frac{P-3}{2} \sum_{b \neq r} \frac{Z(b, 2)}{b-r} - \frac{P-1}{2} \sum_{b \neq r} \frac{Z(b, 3)}{(b-r)^2},$$

$$Z(r, 2) = \frac{1}{2} \sum_{b \in \mathbb{F}} (b-r)^2 Z(b, 0) + \frac{1}{2} \sum_{b \neq r} \frac{Z(b, 3)}{b-r},$$

and

$$\sum_{b \in \mathbb{F}} Z(b, 0) = \sum_{b \neq r} \frac{Z(b, 3)}{(r-b)^3}$$

for all $r \in \mathbb{F}$.

- $(3 - 2P)$ -eigenspace contains only double cusp forms.

- Relations are $Z(r, 3) = 0$,

$$\sum_{b \in \mathbb{F}} Z(b, 1) = 2 \sum_{b \neq r} \frac{Z(b, 2)}{r - b}$$

and

$$\sum_{b \in \mathbb{F}} Z(b, 2) = 0$$

for all $r \in \mathbb{F}$.

Proposition 9. *Suppose $q \geq 4$ and characteristic $p > 2$. For a degree one prime ideal $\mathfrak{P} = (P) \neq (T)$, 1 and $3 - 2P$ are the two distinct eigenvalues of $T_{\mathfrak{P}}$ on $S_{5,m}(\Gamma(T))$. The 1-eigenspace has dimension $q + 3q/p$ if $p > 3$ and dimension $q + q/p$ if $p = 3$. The $(3 - 2P)$ -eigenspace has dimension $q + q/p$. Further, neither on $S_{5,m}(\Gamma(T))$ nor on $S_{5,m}^2(\Gamma(T))$ is $T_{\mathfrak{P}}$ diagonalizable.*

Proposition 10. *When $p = 2$, 1 is the only eigenvalue of $T_{\mathfrak{P}}$ on $S_{5,m}(\Gamma(T))$ and neither on $S_{5,m}(\Gamma(T))$ nor on $S_{5,m}^2(\Gamma(T))$ is $T_{\mathfrak{P}}$ diagonalizable.*