Subelliptic PDEs and
SubRiemannian Geometry (III):
Heat Kernel

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Harmonic Oscillator

We start with the 1-dimensional harmonic oscillator

\[ H = H_0 = \frac{1}{2} \left( \frac{d^2}{dx^2} - \lambda^2 x^2 \right), \]

where \( \lambda \in \mathbb{R}_+ \). The Hamiltonian function is

(1) \[ H(\xi, x) = \frac{1}{2} (\xi^2 - \lambda^2 x^2). \]

The Hamiltonian system is

\[ \dot{x} = H_\xi = \xi \quad \text{and} \quad \dot{\xi} = -H_x = \lambda^2 x \]

with boundary conditions

\[ x(0) = x_0, \quad x(\tau) = x. \]
The conservation law of energy is

\[ \frac{1}{2} \dot{x}^2(s) - \frac{1}{2} \lambda^2 x^2(s) = E \]

where \( E \) is the energy constant. This can be used to obtain \( x(s) \),

\[ \frac{dx}{ds} = \sqrt{2E + \lambda^2 x^2} \quad \Rightarrow \quad \frac{dx}{\sqrt{2E + \lambda^2 x^2}} = ds. \]

Integrating between \( s = 0 \) and \( s = \tau \), with \( x(0) = x_0 \) and \( x(\tau) = x \), yields

\[ \int_{x_0}^{x} \frac{du}{\sqrt{2E + \lambda^2 u^2}} = t \quad \Leftrightarrow \quad \int_{v_0}^{v} \frac{dw}{\sqrt{1 + w^2}} = \lambda \tau, \]
with \( v = \frac{\lambda x}{\sqrt{2E}} \) and \( v_0 = \frac{\lambda x_0}{\sqrt{2E}} \). Integrating yields

\[
\sinh^{-1}(v) - \sinh^{-1}(v_0) = \lambda \tau
\]

\( \iff v = \sinh \left( \sinh^{-1}(v_0) + \lambda \tau \right) \)

\( \iff v = v_0 \cosh(\lambda \tau) + \cosh \left( \sinh^{-1}(v_0) \right) \sinh(\lambda \tau) \)

\( \iff v = v_0 \cosh(\lambda \tau) + \sqrt{1 + v_0^2} \sinh(\lambda \tau) \)

\( \iff \frac{\lambda x}{\sqrt{2E}} = \frac{\lambda x_0}{\sqrt{2E}} \cosh(\lambda \tau) + \sqrt{1 + \frac{\lambda^2 x_0^2}{2E}} \sinh(\lambda \tau) \)

\( \iff \frac{\lambda (x - x_0 \cosh(\lambda \tau))}{\sinh(\lambda \tau)} = \sqrt{2E + \frac{\lambda^2 x_0^2}{2E} \sinh(\lambda \tau)} \)

Solving for \( E \) yields

**Proposition 1** The energy along a geodesic derived from the Hamiltonian (1) between the points \( x_0 \) and \( x \) is

\[
E = \frac{\lambda^2 \left( x^2 + x_0^2 - 2xx_0 \cosh(\lambda \tau) \right)}{2 \sinh(\lambda \tau)^2}.
\]
We note that if we take the limit $\lambda \to 0$ in (2), we obtain the Euclidean energy

$$\lim_{\lambda \to 0} E = \frac{(x - x_0)^2}{2\tau^2}.$$ 

Let $S = S(x_0, x, \tau)$ be the action with initial point $x_0$ and final point $x$, within time $\tau$. The action satisfies Hamilton-Jacobi equation

$$\partial_\tau S + H(x, \nabla S) = 0.$$

One may note that

$$H(\xi, x) = \frac{1}{2}(\xi^2 - \lambda^2 x^2) = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\lambda^2 x^2 = E,$$

and hence $\partial_\tau S = -E$. Using (2) yields

$$\frac{\partial S}{\partial \tau} = -\frac{\lambda^2 \left( x^2 + x_0^2 - 2xx_0 \cosh(\lambda \tau) \right)}{2 \sinh(\lambda \tau)^2}$$

$$= \frac{\partial}{\partial \tau} \left[ \frac{\lambda}{2} \left( x^2 + x_0^2 \right) \coth(\lambda \tau) - \frac{\lambda xx_0}{\sinh(\lambda \tau)} \right].$$
Hence we obtain the action function

$$S(x_0, x, \tau) = \frac{\lambda}{2 \sinh(\lambda \tau)} \left[ (x^2 + x_0^2) \cosh(\lambda \tau) - 2xx_0 \right].$$

It follows that

$$\lim_{\lambda \to 0} S(x_0, x, \tau) = \frac{(x - x_0)^2}{2 \tau},$$

which is the Euclidean action.
Heat Kernel of the Hermite Operator

From now on, we use $t$ as the time variable. Denote

$$(3) \quad K(x_0, x, t) = V(t)e^{kS(x_0, x, t)},$$

where $V(t)$ will satisfy a transport equation and $k \in \mathbb{R}$. Hence,

$$\partial_t K = V'(t)e^{kS} + V(t)ke^{kS}\partial_t S$$
$$= e^{kS}\left(V'(t) - kV(t)E\right).$$

Since

$$(\partial_x S)^2 = \lambda^2 x^2 + 2E$$

and

$$\partial_x^2 S = \lambda \coth(\lambda t),$$

one has

$$\partial_x e^{kS} = ke^{kS}\partial_x S.$$  
$$\partial_x^2 e^{kS} = ke^{kS}[k(\lambda^2 x^2 + 2E) + \lambda \coth(\lambda t)].$$
Let

\[ P = \partial_t - \partial_x^2 + \alpha \lambda^2 x^2, \]

where \( \alpha \) is a real multiplier, which will be determined such that \( PK(x_0, x, t) = 0, \forall t > 0. \)

\[
PK(x_0, x, t) = e^{ks} \left( V'(t) - kEV(t) \right) - ke^{ks} \left( k(\lambda^2 x^2 + 2E) + \lambda \coth(\lambda t) \right) \times \]
\[
\times V(t) \alpha \lambda^2 x^2 e^{ks}V(t) = e^{ks}V(t) \left[ \frac{V'(t)}{V(t)} - kE(2k + 1) \right]
\]
\[
+ (\alpha - k^2) \lambda^2 x^2 - k\lambda \coth(\lambda t) \right].
\]

In order to eliminate the middle two terms in the brackets, we choose \( k = -\frac{1}{2} \) and \( \alpha = \frac{1}{4}. \) Let \( \beta = \frac{\lambda}{2} > 0. \) Then the operator (4) becomes

\[ P = \partial_t - \partial_x^2 + \beta^2 x^2 \]

and

\[
PK(x_0, x, t) = K(x_0, x, t) \left( \frac{V'(t)}{V(t)} + \beta \coth(2\beta t) \right).
\]
We shall choose $V(t)$ such that

$$\frac{V'(t)}{V(t)} = -\beta \coth(2\beta t), \quad t > 0.$$ Integrating, yields

$$\ln V(t) = -\frac{1}{2} \ln \left( \sinh(2\beta t) \right)$$

which implies that

$$V(t) = \frac{C}{\sqrt{\sinh(2\beta t)}}.$$ Using the action function $S$, the fundamental solution formula (3) becomes

$$K(x_0, x, t) = \frac{C}{\sqrt{\sinh(2\beta t)}} e^{-\frac{2\beta}{4\sinh(2\beta t)} \left[ (x^2 + x_0^2) \cosh(2\beta t) - 2xx_0 \right]}$$

$$= \frac{C}{\sqrt{2\beta t} \sqrt{\sinh(2\beta t)}} \sqrt{\frac{2\beta t}{\sinh(2\beta t)}}$$

$$\times e^{-\frac{1}{4t} \cdot \frac{2\beta t}{\sinh(2\beta t)} \left[ (x^2 + x_0^2) \cosh(2\beta t) - 2xx_0 \right]}.$$
Let $\beta \to 0$, $P \to \partial_t - \partial^2_x$. As $\frac{2\beta t}{\sinh(2\beta t)} \to 1$, then

$$K(x_0, x, t) \sim \frac{C}{\sqrt{2\beta t}} e^{-\frac{1}{4t}(x-x_0)^2}, \quad \beta \to 0.$$ 

By comparison with the fundamental solution for the usual heat operator, which is

$$\frac{1}{\sqrt{4\pi t}} e^{-\frac{1}{4t}(x-x_0)^2},$$

we find $C = \sqrt{\frac{\beta}{2\pi}}$. We arrive at the following result.

**Theorem 2** Let $\beta \geq 0$. The fundamental solution for the operator $P = \partial_t - \partial^2_x + \beta^2 x^2$ is

$$K(x_0, x, t) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{\beta}{\sinh(2\beta t)}} \times e^{-\frac{\beta}{2}[(x+x_0)^2 \tanh(\beta t)+(x-x_0)^2 \coth(\beta t)]},$$

for $t > 0$. 

The $n$-dimensional version of Theorem ?? is

**Theorem 3** Let $\beta_j \geq 0$ for $j = 1, \ldots, n$. The fundamental solution for the operator

$$P_t = \partial_t - \sum_{j=1}^{n} \left( \partial_{x_j}^2 - \beta_j^2 x_j^2 \right)$$

is

$$K(x_0, x, t) = \frac{1}{(2\pi)^{n/2}} \left( \prod_{j=1}^{n} \frac{\beta_j}{\sinh(2\beta_j t)} \right)^{1/2} \times$$

$$e^{-\sum_{j=1}^{n} \frac{\beta_j}{2} \left\{ (x_j + x_j^0)^2 \tanh(\beta_j t) + (x_j - x_j^0)^2 \coth(\beta_j t) \right\}}$$

for $t > 0$.

Theorem 3 recovers a result of Hörmander.
Using Theorem 3, one can derive the fundamental solution of the Hermite operator by integration of the heat kernel.

**Theorem 4** For

\[ \alpha \notin \left\{ - \sum_{j=1}^{n} \lambda_j (2k_j + 1), \ k = (k_1, \ldots, k_n) \in (\mathbb{Z}_+)^n \right\}, \]

the Hermite operator

\[ H_{\alpha} = \alpha - \Delta + \sum_{j=1}^{n} \lambda_j^2 x_j^2 \]

has fundamental solution

\[
K_{\alpha}(x, y) = \int_{0}^{\infty} e^{-\alpha s} P_s(x, y) ds = \frac{1}{(2\pi)^{n/2}} \int_{0}^{\infty} e^{-\alpha s} \left( \prod_{j=1}^{n} \frac{\lambda_j}{\sinh(2\lambda_j s)} \right)^{1/2} \times \\
\times e^{-\frac{\lambda_j}{2} \left\{ (x_j+y_j)^2 \tanh(\lambda_j s) + (x_j-y_j)^2 \coth(\lambda_j s) \right\}} ds.
\]
The quartic oscillator

The name “quartic oscillator” refers to the following differential operator:

\[ -\frac{d^2}{dx^2} + x^4. \]

The mathematical description of the quantum mechanical problem of the double well potential leads to the differential operator

\[ -\frac{d^2}{dx^2} + (x^2 - a^2)^2. \]

Inverting (5) and (6) are equivalent problems. Let us consider the sub-elliptic Laplacian:

\[ \Delta_X = \left( \frac{\partial}{\partial x} - y^2 \frac{\partial}{\partial t} \right)^2 + \left( \frac{\partial}{\partial y} \right)^2 \]

which is induced by the vector fields

\[ X_1 = \frac{\partial}{\partial x} - \frac{1}{2} y^2 \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial y}. \]
Taking the Fourier transform of (7) in the \( x \) and \( t \) variables we obtain

\[-\frac{d^2}{dy^2} + (\xi^2 - y^2\tau^2)^2\]

which agrees with (6). To use these ideas on the quartic oscillator we regularize (5):

(9) \[-\frac{d^2}{dx^2} + x^4\eta^2,\]

and this is the Fourier transform in \( y \) of the subelliptic operator

(10) \[\Delta_X = \left(\frac{\partial}{\partial x}\right)^2 + \left(x^2\frac{\partial}{\partial y}\right)^2\]

subelliptic because at \( x = 0 \) we only have 1 vector field \( \frac{\partial}{\partial x} \) in \( \mathbb{R}^2 \). To invert (10), we introduce

\[R = \frac{1}{2}(x^6 + (x')^6 + 9|y - y'|^2),\]

\[\rho = \frac{|xx'|}{R},\]

\[\nu = \text{sgn}(xx') = \frac{xx'}{|xx'|}.\]
**Theorem 5** The operator (10) has the fundamental solution $F(x, y - y'; x')$, where

$$F = -\frac{1}{8\sqrt{\pi}} \frac{G}{R^\frac{1}{3}}$$

with

$$G = \frac{1}{\Gamma\left(\frac{1}{4}\right)} \int_0^1 \int_0^1 \frac{K(\nu, \sqrt{u_1 u_2}) du_1 du_2}{u_1^\frac{5}{6} u_2^\frac{1}{3} (1 - u_1)^\frac{3}{4} (1 - u_2)^\frac{3}{4}}.$$ 

$$K(\nu, z) = \frac{\phi_-(z^2)}{\sqrt{\phi_+(z^2) - 2z^\frac{1}{3} \nu}},$$

$$\phi_+(z^2) = \left(1 + \sqrt{1 - z^2}\right)^\frac{1}{3} + \left(1 - \sqrt{1 - z^2}\right)^\frac{1}{3},$$

$$\phi_-(z^2) = \frac{\left(1 + \sqrt{1 - z^2}\right)^\frac{1}{3} - \left(1 - \sqrt{1 - z^2}\right)^\frac{1}{3}}{\sqrt{1 - z^2}}.$$ 

**Corollary 6** The inverse kernel of the quartic oscillator (5) is

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iy} F(x, y; x') dy.$$
Missing Directions

What is the analogue of the Heisenberg $x_0$-axis, the center of the group, or the missing direction, for general subRiemannian geometries? Let us look at the vector fields:

$$X_1 = \frac{\partial}{\partial x} - \frac{1}{2} y^2 \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial y}$$

denote two vector fields in $\mathbb{R}^3$.

$$[X_2, X_1] = 2y \frac{\partial}{\partial t}, \quad [X_2, [X_2, X_1]] = -2 \frac{\partial}{\partial t}.$$ 

So, $\mathbb{R}^3$ naturally breaks into 2 domains, $y > 0$ and $y < 0$, and their boundary $y = 0$.

We shall say that a geodesic which joins 2 points in $y > 0$ is local if it stays in the domain $y > 0$. Geodesics that cross the boundary $y = 0$ cannot be localized so we refer to them as nonlocal.
Hamilton’s function is
\[ H = \frac{1}{2}(\xi^2 + y^2 \tau)^2 + \frac{1}{2} \eta^2. \]
The bicharacteristic curves
\[ (x(s), y(s), t(s), \xi(s), \eta(s), \tau(s)) \]
are solutions of Hamilton’s system:
\[
\begin{align*}
\dot{x}(s) &= \xi + \frac{1}{2}y^2 \tau, \\
\dot{y}(s) &= \eta, \\
\dot{t}(s) &= \frac{1}{2} \dot{x}y^2, \\
\dot{\xi}(s) &= 0 \Rightarrow \xi(s) = \xi, \text{ constant}, \\
\dot{\eta}(s) &= -\dot{x}y \tau, \\
\dot{\tau}(s) &= 0 \Rightarrow \tau(s) = \tau, \text{ constant},
\end{align*}
\]
where \( s \) denotes arclength. Given a point \( P(x, y, t) \) we fix the boundary conditions
\[
(x(0), y(0), t(0)) = (x_0, y_0, t_0),
\]
and
\[
(x(1), y(1), t(1)) = (x, y, t).
\]
The vector fields are translation invariant with respect to $x$ and $t$, so we may assume $x_0 = 0$ and $t_0 = 0$ with no loss of generality.

**Theorem 7** Assume $y_0 > 0$. Every point $P(x, y, t)$, $y > 0$, can be joined to $P(0, y_0, 0)$ by at least one local geodesic. The number of these local geodesics is finite if and only if

(1). $y \neq y_0$, or

(2). $y = y_0$ and $t + y_0^2 x \neq 0$.

When

$$y = y_0 \quad \text{and} \quad t + y_0^2 x = 0,$$

then $P(x, y_0, t)$ is joined to $P(0, y_0, 0)$ by a countable infinity of local geodesics.
Theorem 8 On the surface \( t = \frac{1}{6}xy^2, \) there is a unique subRiemannian geodesic between the origin and \( P(x, y, t) \) whose projection on the \((x, y)\)-plane is a straight line with length \( \ell = \sqrt{x^2 + y^2} \).

Theorem 9 There are countable infinity of geodesics connecting \( P(0, 0, t_0), t_0 \neq 0, \) to the origin. They are parametrized by \( m = 1, 2, \ldots \), and their length is given by \( d_m \), where
\[
d_m^3 = 12m^2K^2|m_0|.
\]
Here
\[
K_m = K(k_m) = \int_0^1 \frac{du}{\sqrt{(1 - u^2)(1 - k_m^2u^2)}}
\]
is the complete elliptic integral of the first kind.
**Theorem 10** Assume $y_0 = 0$. Every point $P(x, y, t)$ is connected to the origin by at least one geodesic. The number of geodesics joining $P(x, y, t)$ to the origin is finite if and only if $y \neq 0$. When $y = 0$, every point of the “canonical submanifold” $\{(x, 0, 0), x \neq 0\}$ is joined to the origin by a continuous infinity of geodesics, while every point of the complement $\{(x, 0, t), t \neq 0\}$ is joined to the origin by a discrete infinity of geodesics.
The line $y = y_0$, $t + y_0^2 x = 0$ is called the canonical curve and its tangent space may replace the missing direction not covered by the horizontal vector fields $X_1$ and $X_2$. Note that the canonical curve goes into the $x$-axis as $y_0 \to 0$.

We note that the canonical curve goes into the $x$-axis as $y_0 \to 0$. Thus our “natural coordinates” near $(0, y_0, 0)$ degenerate into the $x$ and $y$ axes at the origin. In particular this suggest that in higher step cases, higher than 2, we shall need to deal with singular coordinates, somewhat like polar coordinates in $\mathbb{R}^2$. 
The notion of the canonical submanifold is probably quite general. Its tangent space provides the missing directions not covered by the horizontal vector fields. We state this as follows:

“Given $m$ vector fields on an $n$-dimensional manifold $\mathcal{M}_n$ whose brackets generate $T\mathcal{M}$, for every points $P_0 \in \mathcal{M}_n$, there is an $n-m$ dimensional submanifold $\tilde{\mathcal{M}}_{n-m}$, $P \in \tilde{\mathcal{M}}_{n-m}$, characterized by having its points connected to $P$ by an infinite number of geodesics.”

The following is another project suggested by Siu: We first find the “canonical submanifold”, then use that to construct a new coordinate system. This will lead to the “normal form” of a defining function for a finite type domain in $\mathbb{C}^n$. 
Derivation of the Heat Kernel \( e^{\Delta x^t} \)

Recall that the Laplace operator on \( \mathbb{R}^n \),

\[
\Delta = \frac{1}{2} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}
\]

has the heat kernel

\[
P_t(x, x_0) = \frac{1}{(2\pi t)^{\frac{n}{2}}} e^{-\frac{|x-x_0|^2}{2t}}.
\]

Given a general second order elliptic operator in \( n \) dimensions,

\[
\Delta_X = \frac{1}{2} \sum_{j=1}^{n} X_j^2 + \cdots
\]

where the \( \{X_1, \ldots, X_n\} \) is a linearly independent set of vector fields and \( + \cdots \) stands for lower order terms, the heat kernel takes the form

\[
P_t(x, x_0) = \frac{1}{(2\pi t)^{\frac{n}{2}}} e^{-\frac{d^2(x,x_0)}{2t}} (a_0 + a_1 t + a_2 t^2 + \cdots).
\]
Here $d(x, x_0)$ stands for the Riemannian distance between $x$ and $x_0$ if the metric is induced by the orthonormal set $\{X_1, \ldots, X_n\}$. The $a_j$'s are functions of $x$ and $x_0$. Note that

$$\frac{\partial}{\partial t} \frac{d^2}{2t} + \frac{1}{2} \sum_{j=1}^{n} \left( X_j \frac{d^2}{2t} \right)^2 = 0,$$

i.e., $\frac{d^2}{2t}$ is a solution of the Hamilton-Jacobi equation.

For the Heisenberg sub-Laplacian

$$\Delta_X = \frac{1}{2} \left( \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial y} \right)^2 + \frac{1}{2} \left( \frac{\partial}{\partial x_2} - 2x_1 \frac{\partial}{\partial y} \right)^2$$

on $H_1$ we shall try for a heat kernel in the form

$$\frac{1}{t^\alpha} e^{-\frac{f}{t}} \ldots$$

where $h = \frac{f}{t}$ is a solution of

$$\frac{\partial h}{\partial t} + \frac{1}{2} \left( \frac{\partial h}{\partial x_1} + 2x_2 \frac{\partial h}{\partial y} \right)^2 + \frac{1}{2} \left( \frac{\partial h}{\partial x_2} - 2x_1 \frac{\partial h}{\partial y} \right)^2 = 0.$$
Thus we start with

\[(12) \quad \frac{\partial z}{\partial t} + H(\nabla z) = 0,\]

where

\[
H = \frac{1}{2} \left[ (\xi_1 + 2x_2 \eta)^2 + (\xi_2 - 2x_1 \eta)^2 \right]
\]

\[(13) \quad = \frac{1}{2} [\zeta_1^2 + \zeta_2^2],\]

where \(\eta\) is dual to \(y\). Set

\[
F(x, y, t, z, \xi, \eta, \gamma) = \gamma + H(x, y, \xi, \eta) = 0,
\]

where \(\xi = \nabla_x z = (\frac{\partial z}{\partial x_1}, \frac{\partial z}{\partial x_2})\), \(\eta = \frac{\partial z}{\partial y}\) and \(\gamma = \frac{\partial z}{\partial t}\).
We shall find the bicharacteristic curves, solutions to

\[
\begin{align*}
\dot{x}_1 &= F_{\xi_1} = \xi_1 + 2x_2\eta = \zeta_1, \\
\dot{x}_2 &= F_{\xi_2} = \xi_2 - 2x_1\eta = \zeta_2, \\
\dot{y} &= F_{\eta} = 2\dot{x}_1x_2 - 2x_1\dot{x}_2, \\
\dot{t} &= F_{\gamma} = 1, \\
\dot{\xi}_1 &= -F_{x_1} - \xi_1 F_z = 2\eta\dot{x}_2, \\
\dot{\xi}_2 &= -F_{x_2} - \xi_2 F_z = -2\eta\dot{x}_1, \\
\dot{\eta} &= -F_y - \gamma F_z = 0, \\
\dot{\gamma} &= -F_t - \gamma F_z = 0, \\
\dot{z} &= \xi \cdot \nabla_\xi F + \eta F_{\eta} + \gamma F_{\gamma} \\
&= \xi \cdot \dot{x} + \eta \dot{y} - H
\end{align*}
\]

since \( \dot{t} = 1 \) and \( \gamma = -H \). With \( 0 \leq s \leq t \),

\[
\begin{align*}
\gamma(s) &= \gamma = -H = \text{constant}, \\
\eta(s) &= \eta = -H = \text{constant}, \\
t(s) &= s,
\end{align*}
\]
constant meaning “constant along the bicharacteristic curve”. Also

\[
H = \frac{1}{2}\dot{x}_1^2 + \frac{1}{2}\dot{x}_2^2 = E = \text{energy}.
\]
Another way to see that $E$ is constant along the bicharacteristic, note that

\begin{align*}
\ddot{x}_1 &= \dot{\xi}_1 + 2\eta \dot{x}_2 = +4\eta \dot{x}_2, \\
\ddot{x}_2 &= \dot{\xi}_2 - 2\eta \dot{x}_1 = -4\eta \dot{x}_1.
\end{align*}

Therefore, $\ddot{x}_1 \dot{x}_1 + \ddot{x}_2 \dot{x}_2 = 0$, and $E =$ constant.

Next from (14), we obtain

$$\ddot{x}_1 + 16\eta^2 \dot{x}_1 = 0.$$

Hence

$$\dot{x}_1(s) = \dot{x}_1(0) \cos(4\eta s) + \frac{\ddot{x}_1(0)}{4\eta} \sin(4\eta s) = \zeta_1(0) \cos(4\eta s) + \zeta_2(0) \sin(4\eta s)$$

which yields

$$x_1(s) = x_1(0) + \zeta_1(0) \frac{\sin(4\eta s)}{4\eta} + \zeta_2(0) \frac{1 - \cos(4\eta s)}{4\eta}.$$

At $s = t$ one has $x_1(t) = x_1$, so

$$\frac{1}{2} \zeta_1(0) \sin(4\eta t) + \frac{1}{2} \zeta_2(0) (1 - \cos(4\eta t)) = 2\eta \left( x_1 - x_1(0) \right),$$
or,  
(15)  
\[ \zeta_1(0) \cos(2\eta t) + \zeta_2(0) \sin(2\eta t) = \frac{2\eta(x_1 - x_1(0))}{\sin(2\eta t)}. \]

Hamilton's equations give  
\[ \xi_2(s) = -2\eta x_1(s) + \left( \xi_2(0) + 2\eta x_1(0) \right) \]
\[ = -2\eta x_1(0) - \frac{1}{2} \zeta_1(0) \sin(4\eta s) \]
\[ - \frac{1}{2} \zeta_2(0) \left( 1 - \cos(4\eta s) \right) + \zeta_2(0) + 4\eta x_1(0), \]
or,  
\[ \xi_2(s) = 2\eta x_1(0) \]
\[ - \frac{1}{2} \left[ \zeta_1(0) \sin(4\eta s) - \zeta_2(0) \left( 1 - \cos(4\eta s) \right) \right]. \]

We need to find the classical action integral  
\[ S(t) = \int_0^t \left( \xi \cdot \dot{x} + \eta \cdot y - H \right) ds, \]
and so far we have found \( \dot{x}_1(s) \) and \( \xi_2(s) \). Using (14) again,  
\[ \ddot{x}_2 + 16\eta^2 \dot{x}_2 = 0, \]
which yields
\[
\dot{x}_2(s) = \dot{x}_2(0) \cos(4\eta s) + \frac{\ddot{x}_2(0)}{4\eta} \sin(4\eta s)
\]
\[
= \dot{x}_2(0) \cos(4\eta s) - \dot{x}_1(0) \sin(4\eta s)
\]
\[
= -\zeta_1(0) \sin(4\eta s) + \zeta_2(0) \cos(4\eta s).
\]
Continuing one has
\[
x_2(s) = x_2(0) - \zeta_1(0) \frac{1 - \cos(4\eta s)}{4\eta} + \zeta_2(0) \frac{\sin(4\eta s)}{4\eta}.
\]
Set \( s = t \), we obtain
\[
-\frac{1}{2} \zeta_1(0) \left(1 - \cos(4\eta t)\right) + \frac{1}{2} \zeta_2(0) \sin(4\eta t)
\]
\[
= 2\eta \left(x_2 - x_2(0)\right),
\]
or,
\[
(16)
\]
\[
-\zeta_1(0) \sin(2\eta t) + \zeta_2(0) \cos(2\eta t) = \frac{2\eta \left(x_2 - x_2(0)\right)}{\sin(2\eta t)}.
\]
Now we are ready to find $S$. The above calculations imply

$$
\xi_1 \dot{x}_1 + \xi_2 \dot{x}_2 \\
= -2\eta \dot{x}_1(s)x(0) + 2\eta x(0)\dot{x}_2(s) \\
+ \frac{1}{2}(\zeta_1^2(0) + \zeta_2^2(0))(1 + \cos(4\eta s)) \\
= -2\eta(\dot{x}_1(s)s(0) + x(0)\dot{x}_2(s)) \\
+ (1 + \cos(4\eta s))E,
$$

and

$$
\int_0^t (\xi \cdot \dot{x} + \eta \dot{y} - H) ds \\
= \eta \left[ y - y(0) + 2(x(0)x - x_1x_2(0)) + \frac{\sin(4\eta t)}{4\eta^2} E \right].
$$

To find $E$ we square and add (15) and (17),

$$
E = \frac{1}{2}\zeta_1^2(0) + \frac{1}{2}\zeta_2^2(0) = 2\eta^2 \frac{|x - x_0|^2}{\sin^2(2\eta t)}.
$$
Hence,

\[ S(t) = \int_0^t (\xi \cdot \dot{x} + \eta \dot{y} - H) \, ds \]

\[ = \eta \left[ y - y(0) + 2(x_1(0)x_2 - x_1x_2(0)) \right. \]

\[ \left. + |x - x_0|^2 \cot(2\eta t) \right]. \]

We note that \( x, y, t, x_0 \) and \( \eta = \eta(0) \) are free parameters while \( y(0) = y(0; x, t, y, x_0, \eta) \) is not.

**Question.** How about the solution \( z(t) = z(0) + S(t) \) of the Hamilton-Jacobi equation (12)?
The problem is $z(0)$. To find it we shall substitute $S$ into (12). Straightforward computation shows that

$$\frac{\partial h}{\partial t} + H(x, y, \xi(t), \eta(t)) = 0$$

where

$$(17) \quad h = \eta(0)y(0) + S.$$  

This yields

$$\frac{\partial h}{\partial t} + H(x, y, \nabla xh, \frac{\partial h}{\partial y}) = 0.$$  

We have the following theorem.

**Theorem 11** We have shown that

$$h = \eta(0)y(0) + \int_0^t (\xi \cdot \dot{x} + \eta \dot{y} - H) \, ds$$

$$= \eta y + 2\eta (x_1(0)x_2 - x_1x_2(0))$$

$$+ \eta |x - x_0|^2 \cot(2\eta t)$$

is a “complete integral” of (12) and (13), i.e., a solution of (12) and (13) which depends on 3 free parameters $x_1(0)$, $x_2(0)$ and $\eta$. 
**Remark 1.** We note that the derivation that (17) satisfies the Hamilton-Jacobi equation was complete general, not restriction to $H(x, y, \nabla_x h, \frac{\partial h}{\partial y})$ being (13). In particular we did not assume that $\eta(s) = \text{constant}$. 

**Remark 2.** The action integral is not a solution of the Hamilton-Jacobi equation because one of our free parameters is a dual variable $\eta(0)$ instead of $y(0)$. On the model case, $\eta(0) = \eta$ cannot be switched to $y(0)$. Hence we need to calculate $y(0)$. 
From the Hamilton’s system and the above calculations,

\[
\dot{y} = 2 \left( \dot{x}_1 x_2 - x_1 \dot{x}_2 \right) \\
= 2 \left( \dot{x}_1 x_2(0) - x_1(0) \dot{x}_2 \right) \\
+ \left( \zeta_1^2(0) + \zeta_2^2(0) \right) \frac{1 - \cos(4 \eta s)}{2 \eta}
\]

and

\[
y(s) = 2 \left( x_1(s)x_2(0) - x_1(0)x_2(s) \right) \\
+ \frac{E}{\eta} \frac{4 \eta s - \sin(4 \eta s)}{4 \eta} + C.
\]

At \( s = t \),

\[
y = 2 \left( x_1 x_2(0) - x_1(0) x_2 \right) + \frac{E}{\eta} \frac{4 \eta t - \sin(4 \eta t)}{4 \eta} + C,
\]

and therefore, one has

\[
y(s) = y - 2 \left[ \left( x_1 - x_1(0) \right) x_2(0) - x_1(0) \left( x_2 - x_2(s) \right) \right] \\
- \frac{E}{4 \eta^2} \left[ 4 \eta (t-s) - (\sin(4 \eta t) - \sin(4 \eta s)) \right].
\]

At \( s = 0 \),

\[
y(0) = y + 2 \left( x_1(0) x_2 - x_1 x_2(0) \right) + |x-x_0|^2 \mu(2 \eta t),
\]
where we set
\[ \mu(\phi) = \frac{\phi}{\sin^2 \phi} - \cot \phi. \]

To replace \( \eta \) by \( y(0) \), one needs to invert \( \mu \),
\[ \mu(2\eta t) = \frac{y - y(0) + 2(x_1(0)x_2 - x_1x_2(0))}{|x - x_0|^2}. \]

This is impossible since for most of the values on the right hand side \( \mu^{-1} \) is a many valued function. Therefore we must leave \( \eta \) as one of the free parameters which does not permit \( S \) to be a solution of the Hamilton-Jacobi equation.

Set
\[ f(x, y, x_0, \eta(0)) = h(x, y, x_0, \eta(0), t) \bigg|_{t=1}. \]

Then

**Proposition 12** \( f \) is a solution of the eiconal equation

\[ (18) \quad \eta(0) \frac{\partial f}{\partial \eta(0)} + H(x, y, \nabla_x f, \frac{\partial f}{\partial y}) = f. \]
Proof. By homogeneity property of the function $h$, one has

$$h(x, y, x_0, \eta(0), t) = \frac{1}{t} h(x, y, x_0, t\eta(0), 1)$$

$$= \frac{1}{t} f(x, y, x_0, t\eta(0)),$$

so,

$$\frac{\partial h}{\partial t} = -\frac{1}{t^2} f + \frac{1}{t} \frac{\partial f}{\partial \eta(0)}$$

on one hand. On the other hand,

$$\frac{\partial h}{\partial t} = -\frac{1}{2} \left( \frac{\partial h}{\partial x_1} + 2x_2 \frac{\partial h}{\partial y} \right)^2 - \frac{1}{2} \left( \frac{\partial h}{\partial x_2} - 2x_1 \frac{\partial h}{\partial y} \right)^2.$$

The two right hand sides agree $\forall t$ so we may as well set $t = 1$ which yields the proposition.
We return to our heat kernel:

\[(19) \quad \frac{1}{t^\alpha} e^{-\frac{f}{t}} \ldots ,\]

where \( f = h(x, y, x_0, t\eta(0), 1) \). We still need an "origin" for our heat kernel with \( x_0 \) for its \( x \)-component. We choose an arbitrary \( y_0 \neq y(0) \), for its \( y \)-component, and note that the homogeneity condition

\[
h(x, y, x_0, \eta(0), t) = \lambda h(x, y - y_0, x_0, \frac{\eta(0)}{\lambda}, \lambda).
\]

still holds for

\[
h(x, y, x_0, \eta(0), t) - y_0\eta(0) = h(x, y - y_0, x_0, \eta(0), t).
\]

Consequently (18) also holds for

\[
f(x, y - y_0, x_0, \eta(0)) = h(x, y - y_0, x_0, \eta(0), t) \bigg|_{t=1}.
\]

So, \( f \) in (19) stands for \( f(x, y - y_0, x_0, \eta(0)) \) from now on. (19) depends on the free parameter \( \eta(0) \) although the heat kernel \( P \) should depend only on \( x, y, x_0, y_0 \) and \( t \). To this end we shall sum over \( \eta(0) \), or for convenience over
\[ u = t\eta(0); \text{ an extra } t \text{ can always be absorbed in the power } \alpha, \text{ still unknown. Thus we write} \]

\[
P = \frac{1}{(2\pi t)^\alpha} \int_\gamma e^{-\frac{f(x,y-y_0,x_0,u)}{t}} V(u)du.
\]

The path of integration \( \gamma \subset \mathbb{C} \) still to be chosen. \( V(u) \) is thrown in for good measure.

To see whether (20) is a representation of the heat kernel we apply the heat operator to \( P \) and take it across the integral:

\[
\left( \Delta_X - \frac{\partial}{\partial t} \right) \frac{e^{-\frac{f(u)}{t}}}{t^{\alpha}} = \frac{e^{-\frac{f(u)}{t}}}{t^{\alpha+2}} \left( H(\nabla_X f) - f \right) - \frac{e^{-\frac{f(u)}{t}}}{t^{\alpha+1}} (\Delta_X f - \alpha),
\]

and the eiconal equation (18) implies that

\[
\left( \Delta_X - \frac{\partial}{\partial t} \right) e^{-\frac{f(u)}{t}} \frac{V(u)}{t^\alpha}
\]

\[
\overset{\text{=}}{=} \frac{e^{-\frac{f(u)}{t}}}{t^{\alpha+1}} u \left( -\frac{1}{t} \frac{\partial f}{\partial u} \right) V
\]

\[
- \frac{e^{-\frac{f(u)}{t}}}{t^{\alpha+1}} (\Delta_X f - \alpha) V
\]

\[
= - \frac{e^{-\frac{f(u)}{t}}}{t^{\alpha+1}} \left[ u \frac{dV}{du} + (\Delta_X f - \alpha + 1) V \right]
\]

\[
+ \frac{\partial}{\partial u} \frac{e^{-\frac{f(u)}{t}} uV(u)}{t^{\alpha+1}}.
\]

Assuming

\[
\frac{e^{-\frac{f(u)}{t}} uV(u)}{t^{\alpha+1}} \rightarrow 0
\]
as \( u \to \) the ends of \( \gamma \), one has

\[
\left( \Delta X - \frac{\partial}{\partial t} \right) \frac{1}{(2\pi t)^\alpha} \int_\gamma e^{-\frac{f(x,y-y_0,x_0,u)}{t}} V(u)\,du
\]

\[
= -\frac{1}{(2\pi t)^{\alpha+1}} \int e^{-\frac{f(u)}{t}} \left[ u \frac{dV}{du} + (\Delta_X f - \alpha + 1) V \right] du
\]

\[
= 0
\]

if \( t \neq 0 \) and

(21) \[ u \frac{dV}{du} + (\Delta_X f - \alpha + 1) V = 0. \]

With

\[ f = uy + |x|^2 \cot(2u) \Rightarrow \Delta_X f = 2u \cot(2u), \]

(21) becomes

\[ u \frac{dV}{du} + (2u \cot(2u) - \alpha + 1) V = 0 \iff \]

\[ \frac{dV}{V} = \left( \frac{\alpha - 1}{u} - 2 \cot(2u) \right) du \Rightarrow \]

\[ \log V = (\alpha - 1)(\log u + \log C) - \log(\sin(2u)) \Rightarrow \]

\[ V(u) = \frac{(Cu)^{\alpha-1}}{\sin(2u)}. \]
Set
\[ u = \rho \theta = \rho (\theta_1 + i \theta_2), \quad \rho = |u|, \quad \theta_1, \theta_2 \in \mathbb{R}. \]

Then
\[ f = \rho \theta \left[ y + |x|^2 \cot(2\rho \theta) \right] \]
\[ = \rho (\theta_1 + i \theta_2) \left[ y + \frac{|x|^2}{2} \frac{\sin(4\rho \theta_1)}{\cosh^2(2\rho \theta_2) - \cos^2(2\rho \theta_1)} \right. \]
\[ - \left. \frac{i|x|^2}{2} \frac{\sinh(4\rho \theta_2)}{\cosh^2(2\rho \theta_2) - \cos^2(2\rho \theta_1)} \right]. \]

We may look at \( f \) in three cases:

1. \( \theta = \pm i \), or \( \theta_1 = 0, \theta_2 = \pm 1 \). Then
   \[ \text{Re}(f) = \rho \theta_2 |x|^2 \coth(2\rho \theta_2) \to \infty \]
   as \( \rho \to \infty \) if \( |x|^2 \neq 0 \).
2. $\theta = \pm 1$, or $\theta_1 = \pm 1$, $\theta_2 = 0$. Then

$$\text{Re}(f) = \rho \theta_1 y + \rho \theta_1 |x|^2 \cot(2\rho \theta_1)$$

is singular if $|x|^2 \neq 0$, otherwise

$$\text{Re}(f) = \rho \theta_1 y \rightarrow \text{sgn}(\theta_1 y)\infty$$
as $\rho \rightarrow \infty$; depending on $y$ this could be $\pm\infty$.

3. $\theta_1 \neq 0$, $\theta_2 \neq 0$. For large $\rho$,

$$f \sim \rho(\theta_1 + i\theta_2) \left[ y - i|x|^2 \tanh(2\rho \theta_2) \right],$$

and

$$\text{Re}(f) \sim \rho \left[ \theta_1 y + \theta_2 |x|^2 \tanh(2\rho \theta_2) \right] \sim \rho \left[ \theta_1 y + |\theta_2||x|^2 \right].$$

We choose $y$ so that $|\theta_1 y| > |\theta_2||x|^2$. Then

$$\text{Re}(f) \rightarrow \text{sgn}(\theta_1 y)\infty$$
as $\rho \rightarrow \infty$; again, depending on $y$, $\text{sgn}(\theta_1 y) = \pm 1$. 
Hence, if \((x_0, y_0) = (0, 0)\) and \(|x| \neq 0\), then
\[
\text{Re}(f)(u) \to \infty \quad \text{as} \quad |u| \to \infty, \quad u \in \mathbb{C}
\]
\(\Leftrightarrow u \to \infty\) on the imaginary axis. Replacing \(u\) by \(-iu\) one has

\[
f = -iuy + u|x|^2 \coth(2\tau)
\]
\[
= -iuy + u(x_1^2 + x_2^2) \coth(2\tau).
\]
(22)

Consequently,

\[
P = \frac{A}{(2\pi t)^\alpha} \int_{-\infty}^{+\infty} e^{-f(u)/t} V(u) \, du,
\]
(23)

where \(f\) is given by (22) and

\[
V(u) = \frac{(2u)^{\alpha-1}}{\sinh(2u)};
\]

for suitable constant \(A\).

We clearly have

\[
\frac{\partial P}{\partial t} - \nabla X P = 0, \quad t > 0.
\]

What we need to show is

\[
\lim_{t \to 0} P(x, y, t) = \delta(x)\delta(y).
\]
To simplify the calculation we write

\[(24) \quad P = \frac{A}{(2\pi t)^\alpha} \int_{+\infty}^{\infty} e^{-\frac{f(u)}{t}} V(u) du,\]

and recall that when \(0 < \nu < \frac{\pi}{2}\) and \(u \in \mathbb{R},\)

\[
\text{Re}(f(x, y, u + i\nu \text{sgn}(y)) \geq \varepsilon(\nu)(|x|^4 + y^2)^{\frac{1}{2}}
\]

for some \(\varepsilon(\nu) > 0\). Furthermore \(V(u)\) is exponential decreasing on the contour in (24). We have

**Theorem 13** *The equation (23) represents the heat kernel for \(\Delta_X\) if and only if \(\alpha = 2\), in which case \(A = 1\).*

**Proof.** One has

\[
\lim_{t \to 0} \left( \frac{d}{dt} \right)^m P(x, y, t) = 0, \quad m \in \mathbb{Z}_+
\]

uniformly in \((x, y)\) in compact subsets of \(H_1 \setminus \{(0, 0)\}\). So we only need to show that

\[
\lim_{t \to 0} \int_{\mathbb{R}^3} P(x, y, t) dx dy = 1.
\]
Straightforward calculation shows that

\[
\int_{-\infty}^{+\infty} dy \int_{-\infty+i\nu}^{+\infty+i\nu} \text{sgn}(y) e^{-\frac{f(u)}{t}} V(u) du \\
= \int_{0}^{+\infty} dy \int_{-\infty+i\nu}^{+\infty+i\nu} e^{-\frac{f(u)}{t}} V(u) du \\
+ \int_{-\infty}^{0} dy \int_{-\infty-i\nu}^{+\infty-i\nu} e^{-\frac{f(u)}{t}} V(u) du \\
= \int_{-\infty}^{+\infty} V(u + i\nu) du \int_{0}^{+\infty} e^{-\frac{f(u + i\nu)}{t}} dy \\
+ \int_{-\infty}^{+\infty} V(u - i\nu) du \int_{0}^{+\infty} e^{-\frac{f(u - i\nu)}{t}} dy \\
= \int_{-\infty}^{+\infty+i\nu} V(u)e^{-\frac{|u|^{2}}{t} \coth(2u)} \frac{tdu}{\nu - iu} \\
+ \int_{-\infty}^{+\infty-i\nu} V(u)e^{-\frac{|u|^{2}}{t} \coth(2u)} \frac{tdu}{\nu + iu} \\
= t \int_{|u| = \varepsilon} \frac{V(u)}{iu} e^{-\frac{|u|^{2}}{t} \coth(2u)} \frac{tdu}{iu} \\
\rightarrow 2\pi t V(0)e^{-\frac{|x|^{2}}{2t}} \quad \text{as} \quad \varepsilon \rightarrow 0.
\]
Therefore,

\[
\int_{\mathbb{R}^3} P(x, y, t) \, dx \, dy = 2\pi t V(0) \int_{\mathbb{R}^2} e^{-\frac{|x|^2}{2t}} \, dx
\]

\[
= (2\pi t)^2 V(0).
\]

Hence

\[
V(u) = \frac{2u}{\sinh(2u)} (2u)^{\alpha-2}.
\]

**Remark.** If it is always true that \( V(u) \) is analytic neat \( u = 0 \) but \( V(0) \neq 0 \), then substituting

\[
V(u) = a_0 + a_1 u + a_2 u^2 + \cdots
\]

into

\[
u \frac{dV}{du} + \left( \Delta_X f - \alpha + 1 \right) V = 0,
\]

one obtains

\[
a_1 u + \left( 2u \cot(2u) - \alpha + 1 \right) \left( \sum_{k=0}^{\infty} a_k u^k \right) + \cdots = 0,
\]
or,

\[ a_1 u + \left( \frac{2u}{2u(1 - \frac{4u^2}{6} + \cdots)} - \alpha + 1 \right) \left( \sum_{k=0}^{\infty} a_k u^k \right) + \cdots = 0 \]

\[ \Rightarrow \]

\[ a_1 u + \left( 1 + \frac{4u^2}{6} - \alpha + 1 + \cdots \right) \left( \sum_{k=0}^{\infty} a_k u^k \right) + \cdots = 0 \]

\[ \Rightarrow (2 - \alpha)a_0 + (3 - \alpha)a_1 u + O(u^2) = 0 \]

\[ \Rightarrow \alpha = 2. \]
On the volume element

When $\Delta_X f$ is a function of the variables $x_1$, $x_2$ and $y$ the volume element $V$ may also have to be allowed to depend on $x$ and $y$, although not on $t$. To achieve this we consider

$$
\left( \Delta_X - \frac{\partial}{\partial t} \right) e^{-\frac{f(u)}{t}} W
$$

where $W = W(x, y, u)$. Then

$$
\Delta_X e^{-\frac{f}{t}} = \left( \frac{H(\nabla_X f)}{t^2} - \frac{\Delta_X f}{t} \right) e^{-\frac{f}{t}}.
$$

and

$$
\Delta_X \left( e^{-\frac{f}{t}} W \right)
= \Delta_X \left( e^{-\frac{f}{t}} \right) W + \sum_{j=1}^{m} X_j \left( e^{-\frac{f}{t}} \right) X_j W + e^{-\frac{f}{t}} \Delta_X W
$$

$$
= \Delta_X \left( e^{-\frac{f}{t}} \right) W - \frac{e^{-\frac{f}{t}}}{t} Xf \cdot XW + e^{-\frac{f}{t}} \Delta_X W,
$$
so,
\[
\Delta X \left( \frac{e^{-\frac{f}{t} W}}{t^\alpha} \right) = e^{-\frac{f}{t} W} \left[ \left( \frac{H(\nabla_X f)}{t^2} - \frac{\Delta_X f}{t} \right) W \right. \\
- \frac{1}{t} X f \cdot X W + \Delta_X W \right].
\]

Also,
\[
\frac{\partial}{\partial t} e^{-\frac{f(u)}{t} W} = e^{-\frac{f(u)}{t} W} \left( \frac{f}{t^2} - \frac{\alpha}{t} \right).
\]

Consequently,
\[
\left( \Delta_X - \frac{\partial}{\partial t} \right) e^{-\frac{f(u)}{t} W} = e^{-\frac{f(u)}{t} W} \left[ \left( \frac{H(\nabla_X f)}{t^2} - \frac{\Delta_X f}{t^2} \right) W \right. \\
- \left. \frac{\Delta_X f - \alpha}{t} W + X f \cdot X W \right] + \Delta_X W \right].
\]

Next we make use of the eiconal equation (18)
\[
u \frac{\partial f}{\partial u} + H(\nabla_X f) = f
\]
to obtain
\[ e^{-\frac{f}{t}} t^\alpha \left( \frac{H(\nabla_X f)}{t^2} - \frac{\Delta_X f}{t} \right) W \]
\[ = e^{-\frac{f}{t}} \left( -\frac{1}{t} \frac{\partial f}{\partial u} \right) u W \]
\[ = \frac{1}{t} \left[ \frac{\partial}{\partial u} \left( e^{-\frac{f}{t}} u W \right) - e^{-\frac{f}{t}} \frac{\partial}{\partial u} (uW) \right] \]
which yields
\[ \left( \Delta_X - \frac{\partial}{\partial t} \right) e^{-\frac{f}{t}} \frac{W}{t^\alpha} \]
\[ = - \frac{e^{-\frac{f}{t}}}{t^{\alpha+1}} \left[ u \frac{\partial W}{\partial u} + Xf \cdot XW + (\Delta_X f - \alpha + 1) W \right] \]
\[ + \frac{e^{-\frac{f}{t}}}{t^\alpha} \Delta_X W + \frac{\partial}{\partial u} \left( \frac{e^{-\frac{f}{t}} u W}{t^{\alpha+1}} \right). \]

We define \( g \) by
\[ f(u) = u g(u), \]
and set
\[ T = \frac{\partial}{\partial u} + Xg \cdot X. \]
$T$ is derivation along a geodesic. From the eiconal equation (18) one has

$$\frac{\partial g}{\partial u} + H(\nabla_X g) = 0 \iff g_u = -H(\nabla_X g).$$

Also,

$$Tg = \frac{\partial g}{\partial u} + Xg \cdot Xg = H(\nabla_X g) \Rightarrow Tg = -g_u.$$

Again,

$$Tg_u = \frac{\partial g_u}{\partial u} + Xg \cdot Xg_u = \frac{\partial g_u}{\partial u} + \frac{\partial}{\partial u} \left( \frac{1}{2} Xg \cdot Xg \right) = \frac{\partial}{\partial u} \left( g_u + H(\nabla_X g) \right) = 0,$$

and this we emphasize, $Tg_u = 0$. Similar (18) gives

$$f_u + uH(\nabla_X g) = g, \quad f_u - u g_u = g,$$

and $Tf_u - g_u = -g_u$

$$\Rightarrow Tf_u = 0.$$
Thus $g_u$ and $f_u$ are constants along the geodesics; $-g_u = E$, the energy. Again,

$$Tf = \frac{\partial}{\partial u}(ug) + Xg \cdot X(ug) = g + u g_u + 2uH(\nabla_X g),$$

so,

$$Tf = g - u g_u.$$ 

This implies

$$\left(\Delta_X - \frac{\partial}{\partial t}\right)e^{-\frac{f}{t}}\frac{W}{t^\alpha}$$

$$= - \frac{e^{-\frac{f}{t}}}{t^{\alpha+1}}\left[uTW + (\Delta_X f - \alpha + 1)W\right]$$

$$+ \frac{e^{-\frac{f}{t}}}{t^\alpha}\Delta_X W + \frac{\partial}{\partial u}\left(\frac{e^{-\frac{f}{t}}uW}{t^{\alpha+1}}\right).$$

To simplify the matters we set

$$W = -f_u w$$
which gives

\[
\left( \Delta X - \frac{\partial}{\partial t} \right) e^{-\frac{f}{t}} W \frac{e^{-\frac{f}{t}} W}{t^\alpha} \\
= e^{-\frac{f}{t}} f_u \left[uTw + (\Delta_X f - \alpha + 1)w\right] \\
+ \frac{e^{-\frac{f}{t}}}{t^\alpha} \Delta_X W + \frac{\partial}{\partial u} \left( \frac{e^{-\frac{f}{t}} uW}{t^\alpha + 1} \right).
\]

Then

\[
\frac{e^{-\frac{f}{t}} f_u}{t^\alpha + 1} \left[uTw + (\Delta_X f - \alpha + 1)w\right] \\
= \left( \frac{\partial}{\partial u} \left( -\frac{e^{-\frac{f}{t}}}{t^\alpha} \right) \right) \left(uTw + (\Delta_X f - \alpha + 1)w\right) \\
= \frac{\partial}{\partial u} \left( -\frac{e^{-\frac{f}{t}}}{t^\alpha} (uTw + (\Delta_X f - \alpha + 1)w) \right) \\
+ \frac{e^{-\frac{f}{t}}}{t^\alpha} \frac{\partial}{\partial u} \left(uTw + (\Delta_X f - \alpha + 1)w\right),
\]
and
\[
\left(\Delta X - \frac{\partial}{\partial t}\right) \frac{e^{-\frac{f}{t}} W}{t^{\alpha}}
\]
\[
= \frac{e^{-\frac{f}{t}}}{t^{\alpha}} \left[ \frac{\partial}{\partial u} \left( uT w + (\Delta X f - \alpha + 1) w \right) + \Delta X (-fuw) \right]
\]
\[- \frac{\partial}{\partial u} \left( \frac{e^{-\frac{f}{t}} fuw}{t^{\alpha+1}} \right)
\]
\[- \frac{\partial}{\partial u} \left[ \frac{e^{-\frac{f}{t}}}{t^{\alpha}} \left( uT w + (\Delta X f - \alpha + 1) w \right) \right].
\]
Now
\[
\Delta (-fuw) = -(\Delta X fu) w - X fu \cdot X w - fu \Delta X w,
\]
so,
\[
\frac{\partial}{\partial u} \left( uT w + (\Delta X f - \alpha + 1) w \right) + \Delta X (-fuw)
\]
\[
= \left( u(T + \Delta X g) + 2 - \alpha \right) \frac{\partial w}{\partial u} - fu \Delta X w.
\]
Therefore one has

\[
\left( \Delta_X - \frac{\partial}{\partial t} \right) e^{-f \frac{t}{\alpha}} \frac{e^{-f \frac{t}{\alpha}} (-f_w w)}{t^\alpha} = e^{-f \frac{t}{\alpha}} \left\{ [u(T + \Delta_X g) + 2 - \alpha] \frac{\partial w}{\partial u} - f_u \Delta_X w \right\} \\
- \frac{\partial}{\partial u} \left( e^{-f \frac{t}{\alpha}} u f_w w \right) \\
- \frac{\partial}{\partial u} \left[ e^{-f \frac{t}{\alpha}} (uT w + (\Delta_X f - \alpha + 1) w) \right].
\]

**Theorem 14**

\[
P = \int_{\gamma} e^{-f \frac{t}{\alpha}} w (-f_u du)
\]

is a solution of

\[
\left( \Delta_X - \frac{\partial}{\partial t} \right) P = 0, \quad t > 0,
\]

whenever \(w\) is a solution of

\[
(25) \left[ u(T + \Delta_X g) + 2 - \alpha \right] \frac{\partial w}{\partial u} - f_u \Delta_X w = 0,
\]
and

\[ (26) \quad \frac{e^{-\frac{f}{t} u f u w}}{t}, \quad e^{-\frac{f}{t} (u T w + (\Delta_X f - \alpha + 1) w)} \]

both vanish at the end points of the contour \( \gamma \).

Thus it is reasonable to suppose that \( P \) represents the heat kernel,

\[ P = \ker e^{\Delta_X t} \]

if \( w \) is a solution of (25) and \( \gamma \) is chosen so that (15) vanish at its end points. The choice of \( \alpha \) and the behavior of \( w \) near \( u = 0 \) are intimately connected.

**Remark 1.** So far Theorem 14 has been worked out explicitly and proved to be correct only for the Heisenberg sub-Laplacian and the Grusin operator. One does expect the theorem to hold in much greater generality.
**Remark 2.** Suppose we have a local heat kernel of the form

\[ P_t(x, x_0) = \frac{1}{n} e^{-\frac{d^2(x, x_0)}{2t}} (a_0 + a_1 t + a_2 t^2 + \cdots) \]

for a Laplace-Beltrami operator in some neighborhood of a point \( O \). Question: how far can one extend this domain and still have the above represent the heat kernel? Clearly, only as far as the first cut or conjugate points. As soon as we reach one of these points one has more than one geodesic connection to \( O \), so we cannot use the coordinates of \( O \) as free parameters and we may have to switch to a dual variable. This would again give an integral representation for the heat kernel.