Gravitational radiation and the Bondi mass

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Structuring

1. Gravitational waves
2. Bondi's radiating space-time
3. Positive mass theorem at null infinity
    (a) Schoen-Yau's method
    (b) Witten's method

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Gravitational waves (GW)

- predicted by Einstein's general relativity
- time dependent solutions of Einstein's field equations which radiate or transport energy
- created by accelerated masses
- radiate with the light speed

Examples for GW

- Binary pulsars
- Collapsing black holes
- Double star
- Big bang

Detection of GW

Gravitational waves encode the history of the universe.

However, they are very weak and have not been detected yet.

How weak are they ?
For black holes that weigh about 10 times as much as the sun and which are a billion light-years away, the wave strength is about 10^{-21} when they arrive at the Earth.
Therefore the waves produce tides in the earth's ocean by $10^{-21} \cdot 10^{7} = 10^{-14}$ meter, or 10 times the diameter of an atom's nucleus.

There exist several GW detectors on the world.

GW Detectors

- LIGO – USA
- VIRGO – France + Italy
- Geo 600 – Germany and GB
- TAMAX – Japan
- AIGO – Australia

Detection of GW (2)

Although they have not been detected yet, the existence of gravitational waves has been proved indirectly from observations of the pulsar PSR 1913+16. This rapidly rotating binary system should emit gravitational radiation, hence lose energy and rotate faster. The observed relative change in period agrees remarkably with the theoretical value.

GW and energy

A fundamental conjecture is that gravitational waves can not carry away more energy than they have initially in an isolated gravitational system.

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Bondi's radiating space-time (1)

Historical notes

- Late 1950s-early 1960s:
  Pirani, Bondi, Robinson, Trautman and others
- Bondi, Van der Burg, and Metzner:
  - vacuum solutions of Einstein's field equations outside an isolated (i.e. spatially bounded) *axisymmetric* system
  - definition of mass along outgoing null hypersurfaces in the limit $r \to \infty$
  - the total mass, measured at null infinity, is non-increasing with respect to the retarded time

Ref.: H. Bondi, M. van der Burg, A. Metzner, *Gravitational waves in general relativity VII. Waves from axi-symmetric isolated systems*, Proc. Roy. Soc. London A 269(1962), 21-52.

Bondi's radiating space-time (2)

Historical notes

- Sachs 1962:
  Generalization of the work of Bondi, Van der Burg, Metzner to *asymptotically flat* space-times

Ref.: R. Sachs, *Gravitational waves in general relativity VIII. Waves in asymptotically flat space-time*, Proc. Roy. Soc. London, A 270(1962), 103-126.

Bondi's radiating space-time (3)

Historical notes

Interpretation of Bondi mass as *the total mass of the isolated physical system measured after the loss due to the gravitational radiation up to that time*

Bondi's radiating space-time (4)

Retarded time

Minkowski space-time:

- photon emitted when $t = t_0$ takes a certain amount of time to reach an observer located at distance $r \geq 0$ from the source, so the observer notices it when $t = t_1$,

$$t_1 = t_0 + \frac{r}{c} = t_0 + r \qquad \text{(where speed of light } c := 1\text{)}$$

- the time $t_0 = t_1 - r$ is defined as *retarded time.*
- set $u = t - r$:
  the hypersurface defined by $u = t - r = k$, k a constant, is the future directed null cone with vertex $r = 0$, $t = k$.

Bondi's radiating space-time (5)

Bondi's radiating space-time

Vacuum space-time ($T_{ij} = 0$) with *Bondi's radiating metric*

$$\begin{aligned} g_{Bondi} \;=\; & \Big(+ r^2 e^{2\gamma} U^2 \cosh 2\delta + r^2 e^{-2\gamma} W^2 \cosh 2\delta + 2r^2 UW \sinh 2\delta - \frac{V}{r} e^{2\beta} \Big)\, du^2 \\ & -2e^{2\beta} du\, dr - 2r^2 \Big(e^{2\gamma} U \cosh 2\delta + W \sinh 2\delta \Big)\, du\, d\theta \\ & -2r^2 \Big(e^{-2\gamma} W \cosh 2\delta + U \sinh 2\delta \Big) \sin\theta \; du\, d\varphi \\ & +r^2 \Big(e^{2\gamma} \cosh 2\delta \, d\theta^2 + e^{-2\gamma} \cosh 2\delta \sin^2\theta \; d\varphi^2 + 2 \sinh 2\delta \sin\theta \; d\theta\, d\varphi \Big) \end{aligned}$$

- β, γ, δ, U, V, W: functions of (u, r, θ, φ) which are smooth for $0 < r_0 \leq r$, $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$
- u: retarded time, r: radius function,
  θ, φ: spherical coordinates.
- $u = \mathit{constant}$ are null hypersurfaces.

Bondi's radiating space-time (6)

Examples

(i) Minkowski space-time:

$$g_{Mink} = -dt^2 + dr^2 + r^2\big(d\theta^2 + \sin^2\theta d\varphi^2\big).$$

Using the retarded time $u = t - r$, the metric can be written as

$$g_{Mink} = -du^2 - 2dudr + r^2\big(d\theta^2 + \sin^2\theta d\varphi^2\big).$$

Bondi's radiating space-time (7)

Examples

(ii) Schwarzschild space-time:

$$g_{Schw} = -\big(1-\frac{2m}{r}\big)dt^2 + \frac{dr^2}{1-\frac{2m}{r}} + r^2\big(d\theta^2 + \sin^2\theta d\varphi^2\big),$$

where m is the mass.

Using the retarded time $u = t - r - 2m\ln\big|r-2m\big|$, the metric can be written as

$$g_{Schw} = -\big(1-\frac{2m}{r}\big)du^2 - 2dudr + r^2\big(d\theta^2 + \sin^2\theta d\varphi^2\big).$$

Bondi's radiating space-time (8)

Examples

(iii) Kerr space-time:

$$g_{Kerr} = -\left(1 - \frac{2mr}{\Sigma}\right)dt^2 - \frac{4mar\sin^2\theta}{\Sigma}dtd\phi$$
$$+\frac{\Sigma}{\triangle}dr^2 + \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{2mra^2\sin^2\theta}{\Sigma^2}\right)\sin^2\theta d\phi^2$$

where

$\Sigma \equiv r^2 + a^2\cos^2\theta$, $\triangle \equiv r^2 - 2mr + a^2$,

m is the mass, ma is the angular momentum as measured from infinity.

Retarded time u ? It is an interesting problem to obtain the Kerr solution in the form of a Bondi metric.

Bondi's radiating space-time (9)

The **outgoing radiation conditions** imply, as $r \to \infty$,

$$\begin{aligned}
\gamma &= \frac{c(u,\theta,\varphi)}{r} + O(r^{-3}), \\
\delta &= \frac{d(u,\theta,\varphi)}{r} + O(r^{-3}), \\
\beta &= -\frac{c^2+d^2}{4r^2} + O(r^{-4}), \\
U &= -\frac{l(u,\theta,\varphi)}{r^2} + O(r^{-3}), \\
W &= -\frac{\bar{l}(u,\theta,\varphi)}{r^2} + O(r^{-3}), \\
V &= -r + 2M(u,\theta,\varphi) + O(r^{-1}),
\end{aligned}$$

where $l = c_{,2} + 2c\cot\theta + d_{,3}\csc\theta$, $\bar{l} = d_{,2} + 2d\cot\theta - c_{,3}\csc\theta$.

Bondi's radiating space-time (10)

Bondi's radiating metric under the outgoing radiation conditions

$$\begin{aligned} g_{Bondi} \;=\; & -\Big(1 - \frac{2M}{r} + O(r^{-2})\Big) du^2 \\ & -2\Big(1 - \frac{c^2 + d^2}{4r^2} + O(r^{-4})\Big) dudr \\ & +2\Big(l + \frac{2cl + 2dl}{r} + O(r^{-2})\Big) dud\theta \\ & +2\Big(\bar{l} - \frac{2c\bar{l} - 2dl}{r} + O(r^{-2})\Big) \sin\theta\Big) dud\varphi \\ & +r^2\Big(1 + \frac{2c}{r} + O(r^{-2})\Big) d\theta^2 \\ & +r^2\Big(1 - \frac{2c}{r} + O(r^{-2})\Big) \sin^2\theta d\varphi^2 \\ & +r^2\Big(\frac{4d}{r} + O(r^{-2})\Big) \sin\theta d\theta d\varphi^2, \end{aligned}$$

where M, c, d, l and $\bar{l}$ are functions of u, θ and φ.

Bondi's radiating space-time (11)

We have two regularity assumptions

Condition A
Each of the six functions β, γ, δ, U, V, W together with its derivatives up to the second orders are equal at $\varphi = 0$ and 2π.

Condition B
For all u and φ,

$$c(u, 0, \varphi) = c(u, \pi, \varphi) = 0.$$

Bondi's radiating space-time (12)

At null infinity:

The Bondi energy-momentum of u_0-slice:

$$m_\nu(u_0) = \frac{1}{4\pi} \oint_{S^2} M(u_0, \theta, \varphi) n^\nu dS$$

for $\nu = 0, 1, 2, 3$, where

$$n^0 = 1, \quad n^1 = \sin\theta\cos\varphi, \quad n^2 = \sin\theta\sin\varphi, \quad n^3 = \cos\theta.$$

m_0: "*Bondi energy*" or "*Bondi mass*"
m_i: "*Bondi momentum*"

- Minkowski space-time: $m_\nu(u_0) = 0$.
- Schwarzschild space-time: $m_0(u_0) = m$ and $m_i(u_0) = 0$.

Bondi's radiating space-time (13)

We define

$$\begin{aligned}\mathcal{M}(u,\theta,\varphi) &= M(u,\theta,\varphi) - \frac{1}{2}\big(l_{,2} + l\cot\theta\ + \bar{l}_{,3}\csc\theta\big) \\ &= M(u,\theta,\varphi) - \frac{1}{2}\big[-2c(u,\theta,\varphi) + c_{,22}(u,\theta,\varphi) \\ &\quad - \csc^2\theta\, c_{,33}(u,\theta,\varphi) + 2\csc\theta\, d_{,23}(u,\theta,\varphi) \\ &\quad + 3\cot\theta\, c_{,2}(u,\theta,\varphi) + 2\cot\theta\ \csc\theta\, d_{,3}(u,\theta,\varphi)\big].\end{aligned}$$

Its u-derivative is

$$\mathcal{M}_{,0} = -\big[(c_{,0})^2 + (d_{,0})^2\big].$$

Bondi's radiating space-time (14)

Under condition A and B, we have

$$\oint_{S^2} \left(l_{,2} + l \cot\theta + \bar{l}_{,3} \csc\theta\right) n^\nu dS = 0.$$

This implies

$$\begin{aligned} m_\nu(u_0) &= \frac{1}{4\pi}\oint_{S^2} M(u_0,\theta,\varphi)\, n^\nu\, dS \\ &= \frac{1}{4\pi}\oint_{S^2} \mathcal{M}(u_0,\theta,\varphi)\, n^\nu\, dS. \end{aligned}$$

$\nu = 0, 1, 2, 3.$

Bondi's radiating space-time (15)

Bondi mass loss formula

$$\frac{d}{du}m_\nu = -\frac{1}{4\pi}\oint_{S^2}\left[(c_{,0})^2 + (d_{,0})^2\right]n^\nu dS, \quad \nu = 0, 1, 2, 3.$$

When $\nu = 0$, this is the famous *Bondi mass loss formula*

$$\frac{d}{du}m_0 = -\frac{1}{4\pi}\oint_{S^2}\left[(c_{,0})^2 + (d_{,0})^2\right]dS \leq 0.$$

The Bondi mass is non-increasing in u, i.e., more and more energy is radiated away.

Bondi mass-loss is measured by the news functions $c_{,0}$ and $d_{,0}$.

Bondi's radiating space-time (16)

Generalized Bondi energy-momentum loss formula

Applying Hölder's inequality and Cauchy-Schwarz' inequality, we obtain

$$\frac{d}{du}\Big(m_0 - \sqrt{\sum_{1\le i\le 3} m_i^2}\Big) \le 0.$$

Bondi's radiating space-time (17)

Positive Mass Conjecture at Null Infinity

In Bondi's vacuum radiating space-times, the Bondi mass must be nonnegative, i.e., a finite gravitational gravitational system cannot radiate away more energy than it has initially.

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Positive mass theorem at null infinity (1)

The Positive mass theorem near infinity

1982, Schoen-Yau:

Solving the Jang's equation–prescribing the mean curvatures.

1982-, Israel-Nester, Horowitz-Perry, Ashtekar-Horowitz, Renla-Tod, Ludvigsen-Vickers, etc.:

Witten's method–the Dirac operator.

2006, W.-l. Huang, S.T. Yau, and X. Zhang:

Detailed proof.

Ref.: W.-l. Huang, S.T. Yau, and X. Zhang, *Positivity of the Bondi mass in Bondi's radiating spacetimes*, Rend. Lincei. Mat. Appl. 17 (2006), 335-349.

Positive mass theorem at null infinity (2)

Asymptotically null initial data set

In Minkowski space-time, the spacelike hypersurface

$$t = \sqrt{1 + r^2}$$

has the hyperbolic metric $\breve{g}$ and the nontrivial second form $\breve{h}$,

$$\begin{aligned} \breve{g} &= \frac{dr^2}{1+r^2} + r^2\big(d\theta^2 + \sin^2\theta d\varphi^2\big), \\ \breve{h} &= \frac{dr^2}{1+r^2} + r^2\big(d\theta^2 + \sin^2\theta d\varphi^2\big) \end{aligned}$$

in polar coordinates (r, θ, φ) where $0 < r < \infty$, $0 \le \theta < \pi$, $0 \le \varphi < 2\pi$.

Positive mass theorem at null infinity (3)

Asymptotically null initial data set

Denote the associated orthonormal frame $\{\breve{e}_i\}$ and coframe $\{\breve{e}^i\}$ by

$$\begin{aligned}
\breve{e}_1 &= \sqrt{1+r^2}\frac{\partial}{\partial r}, & \breve{e}^1 &= \frac{dr}{\sqrt{1+r^2}},\\
\breve{e}_2 &= \frac{1}{r}\frac{\partial}{\partial \theta}, & \breve{e}^2 &= r d\theta,\\
\breve{e}_3 &= \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}, & \breve{e}^3 &= r\sin\theta d\varphi.
\end{aligned}$$

$\breve{\nabla}$: the Levi-Civita connection of $\breve{g}$, $\qquad \breve{\nabla}_i := \breve{\nabla}_{\breve{e}_i}$.

Positive mass theorem at null infinity (4)

Asymptotically null initial data set

An initial data set (M^3, g, p) (p is not necessarily symmetric) is asymptotically null of order τ if, outside a compact subset,

- M is diffeomorphic to $R^3 \backslash B_R$,
- the metric g and the 2-tensor p are

$$\begin{aligned} g(\breve{e}_i, \breve{e}_j) &= \breve{g}(\breve{e}_i, \breve{e}_j) + a_{ij}, \\ p(\breve{e}_i, \breve{e}_j) &= \breve{p}(\breve{e}_i, \breve{e}_j) + b_{ij} \end{aligned}$$

where a_{ij} and b_{ij} satisfy

$$\begin{aligned} a_{ij} &= O(r^{-\tau}), & b_{ij} &= O(r^{-\tau}), \\ \breve{\nabla}_k a_{ij} &= O(r^{-\tau}), & \breve{\nabla}_k b_{ij} &= O(r^{-\tau}), \\ \breve{\nabla}_k \breve{\nabla}_l a_{ij} &= O(r^{-\tau}). \end{aligned}$$

Positive mass theorem at null infinity (5)

In Bondi's radiating vacuum space-time, the spacelike hypersurface given by

$$u = \sqrt{1 + r^2} - r + \frac{(c^2 + d^2)_{u=0}}{12r^3} + \frac{a_3(\theta, \varphi)}{r^4} + o(r^{-4})$$

is asymptotically null of order 1.

(Since Bondi's radiating metric is complicated, we have calculated the induced metric g and the second fundamental form h using *Mathematica 5.0.*)

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Schoen-Yau's method (1)

Jang's equation:

$$\Big(g^{ij} - \frac{f^i f^j}{1+|\nabla f|^2}\Big)\Big(\frac{f_{,ij}}{\sqrt{1+|\nabla f|^2}} - h_{ij}\Big) = 0.$$

If Jang's equation has a solution f which has the asymptotic expansion

$$f = \sqrt{1+r^2} + p(\theta,\varphi)\ln r + o(1)$$

for r sufficiently large, then $p(\theta,\varphi)$ and $\mathcal{M}(0,\theta,\varphi)$ must be constant.

Schoen-Yau's method (2)

Proof. For r sufficiently large,

$$J(f) \approx \frac{\ln r}{r^3} \triangle_{S^2} p(\theta,\varphi) + \frac{p(\theta,\varphi) - 2\mathcal{M}(0,\theta,\varphi)}{r^3},$$

where

$$\triangle_{S^2} = \frac{\partial^2}{\partial\theta^2} + \cot\theta \frac{\partial}{\partial\theta} + \csc^2\theta \ \frac{\partial^2}{\partial\varphi^2}.$$

$J(f) = 0$ implies

$$\triangle_{S^2} p(\theta,\varphi) = 0, \quad p(\theta,\varphi) - 2\mathcal{M}(0,\theta,\varphi) = 0.$$

There is no nonconstant harmonic function on S^2
$\Rightarrow p(\theta,\varphi)$ and $\mathcal{M}(0,\theta,\varphi) = \frac{p(\theta,\varphi)}{2}$ are constant.

Schoen-Yau's method (3)

Theorem 1. *Let* $\left(\mathbb{L}^{3,1}, \tilde{g}\right)$ *be a vacuum Bondi's radiating space-time with Bondi-metric* $\tilde{g}$. *Suppose that Condition A and Condition B hold.*

If there exists u_0 *such that* $\mathcal{M}(u_0, \theta, \varphi)$ *is constant, then*

$$m_0(u) \geq \sqrt{\sum_{1 \leq i \leq 3} m_i^2(u)}$$

for all $u \leq u_0$.

Schoen-Yau's method (4)

Idea of the proof:

- Let $\mathcal{M}(u_0, \theta, \varphi) = \frac{p}{2}$ be constant for some u_0.
- Without loss of generality, $u_0 = 0$.
- On the hypersurface

$$u = \sqrt{1 + r^2} - r + \frac{(c^2 + d^2)_{u=0}}{12r^3} + \frac{a_3(\theta, \varphi)}{r^4} + o(r^{-4}),$$

Jang's equation has a solution

$$f = \sqrt{1 + r^2} + p \ln r + q,$$

and the metric $\bar{g}_{ij} = g_{ij} + f_{,i} f_{,j}$ is asymptotic flat.

Schoen-Yau's method (5)

Idea of the proof:

- ADM total energy $E(\bar{g}) = p$.
- Scalar curvature $\bar{R} \geq 2\,|Y|^2_{\bar{g}} - 2\,\mathrm{div}_{\bar{g}} Y$.
- $\bar{g}$ can be transformed conformally to a metric $\hat{g}$ with
$$\hat{R} = 0 \quad \text{and} \quad E(\bar{g}) \geq E(\hat{g}).$$
- The positive mass theorem at spatial infinity implies
$$p = E(\bar{g}) \geq E(\hat{g}) \geq 0.$$
- Bondi mass on the $u = 0$ slice:
$$m_0(0) = \frac{p}{2} \geq 0, \;\; m_1(0) = m_2(0) = m_3(0) = 0.$$

Schoen-Yau's method (6)

Idea of the proof:

- From the Bondi mass-loss formula,

$$m_0(u) \geq m_0(u_0) \geq 0 \qquad \forall u \leq u_0.$$

- From the generalized Bondi energy-momentum-loss formula,

$$m_0(u) \geq \sqrt{\sum_{1\leq i\leq 3} m_i^2(u)} \qquad \forall u \leq u_0.$$

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Witten's method (1)

Let $(\mathbb{X}, g, h)$ be an asymptotically null spacelike hypersurface.

- total energy:

$$\mathcal{E} = \breve{\nabla}^j a_{1j} - \breve{\nabla}_1 \mathrm{tr}_{\breve{g}}(a) - \big[a_{11} - \delta_{11}\mathrm{tr}_{\breve{g}}(a)\big],$$

$$E_\nu(\mathbb{X}) = \frac{1}{16\pi} \lim_{r\to\infty} \oint_{S_r} \mathcal{E}\, n^\nu\, r\, dS$$

- total linear momentum:

$$\mathcal{P} = b_{11} - \delta_{11}\mathrm{tr}_{\breve{g}}(b),$$

$$P_\nu(\mathbb{X}) = \frac{1}{8\pi} \lim_{r\to\infty} \oint_{S_r} \mathcal{P}\, n^\nu\, r\, dS$$

S_r: sphere of radius r in $\mathbb{R}^3$, $\nu = 0, 1, 2, 3$.

Witten's method (2)

Positive mass theorem (X. Zhang, 2002):

Let $(\mathbb{X}, g_{ij}, p_{ij})$ be a 3-dimensional asymptotically null initial data set of order $\tau = 3$. Denote

$$\mu := \frac{1}{2}\big(R + (p_i{}^i)^2 - p_{ij}p^{ij}\big), \quad \varphi_j := \nabla^i p_{ji} - \nabla_j p_i{}^i, \quad \sigma_j := 2\nabla^i\big(p_{ij} - p_{ji}\big).$$

If the initial data set satisfies the dominant energy condition

$$\mu \geq \max\Big\{\sqrt{\sum \varphi_j^2}, \sqrt{\sum (\varphi_j + \sigma_j)^2}\Big\},$$

then

$$E_0(\mathbb{X}) - P_0(\mathbb{X}) \geq \sqrt{\sum_{i=1,2,3} \big[E_i(\mathbb{X}) - P_i(\mathbb{X})\big]^2}.$$

If equality holds, then

$$R_{ijkl} + p_{ik}p_{jl} - p_{il}p_{jk} = 0, \quad \nabla_i p_{jk} - \nabla_j p_{ik} = 0, \quad \nabla^j\big(p_{ij} - p_{ji}\big) = 0.$$

Witten's method (3)

Remark: The proof of the theorem can still go through if the order $\tau > \frac{3}{2}$ and the $E_\nu - P_\nu$ are finite for $\nu = 0, 1, 2, 3$.

Witten's method (4)

Theorem 2. *Let* $\left(\mathbb{L}^{3,1}, \tilde{g}\right)$ *be a vacuum Bondi's radiating space-time with Bondi-metric* $\tilde{g}$*. Suppose that Condition A and Condition B hold.*

If there exists u_0 *such that*

$$c\big|_{u=u_0} = d\big|_{u=u_0} = 0,$$

then

$$m_0(u) \geq \sqrt{\sum_{1\leq i\leq 3} m_i^2(u)}$$

for all $u \leq u_0$*.*

Witten's method (5)

Proof.

- Without loss of generality, assume $u_0 = 0$.
- Choose an asymptotically null spacelike hypersurface $\mathbb{X}$ with

$$u = \sqrt{1+r^2} - r + \frac{(c^2+d^2)_{u=0}}{12r^3} + O(r^{-4}).$$

- $\mathcal{E} - 2\mathcal{P} = -\frac{3}{2r^3}(l_{,2} + l\cot\theta + \bar{l}_{,3}\csc\theta)_{u=0} + \frac{4M(0,\theta,\varphi)}{r^3} + O(r^{-4}).$
- $E_\nu(\mathbb{X}) - P_\nu(\mathbb{X}) = m_\nu(0).$
- $E_0(\mathbb{X}) - P_0(\mathbb{X}) \geq \sqrt{\sum_{i=1}^3 \left[E_i(\mathbb{X}) - P_i(\mathbb{X})\right]^2}.$
- $m_0(u) \geq \sqrt{\sum_{i=1}^3 m_i^2(u)}$

Open questions

1. What does it mean in physics that $M(u_0, \theta, \varphi) = \text{constant}$?
2. What is the physical meaning of the condition

$$c(u_0, \theta, \varphi) = 0 = d(u_0, \theta, \varphi) \ ?$$

   Does it preclude gravitational waves ?

Thank you for your attention !