Gravitational radiation and the Bondi mass

National Center for Theoretical Sciences, Mathematics Division
March 16th, 2007

Wen-ling Huang
Department of Mathematics
University of Hamburg, Germany
Structuring

1. Gravitational waves
2. Bondi’s radiating space-time
3. Positive mass theorem at null infinity
   (a) Schoen-Yau’s method
   (b) Witten’s method
Structuring

1. Gravitational waves
2. Bondi’s radiating space-time
3. Positive mass theorem at null infinity
   (a) Schoen-Yau’s method
   (b) Witten’s method
Gravitational waves (GW)

- predicted by Einstein’s general relativity
- time dependent solutions of Einstein’s field equations which radiate or transport energy
- created by accelerated masses
- radiate with the light speed
Examples for GW

- Binary pulsars
- Collapsing black holes
- Double star
- Big bang
Detection of GW

Gravitational waves encode the history of the universe. However, they are very weak and have not been detected yet.

How weak are they? For black holes that weigh about 10 times as much as the sun and which are a billion light-years away, the wave strength is about $10^{-21}$ when they arrive at the Earth. Therefore the waves produce tides in the earth’s ocean by $10^{-21} \cdot 10^7 = 10^{-14}$ meter, or 10 times the diameter of an atom’s nucleus.

There exist several GW detectors on the world.
GW Detectors

- LIGO – USA
- VIRGO – France + Italy
- Geo 600 – Germany and GB
- TAMAX – Japan
- AIGO – Australia
Detection of GW (2)

Although they have not been detected yet, the existence of gravitational waves has been proved indirectly from observations of the pulsar PSR 1913+16. This rapidly rotating binary system should emit gravitational radiation, hence lose energy and rotate faster. The observed relative change in period agrees remarkably with the theoretical value.
GW and energy

A fundamental conjecture is that gravitational waves cannot carry away more energy than they have initially in an isolated gravitational system.
Structuring

1. Gravitational waves
2. Bondi’s radiating space-time
3. Positive mass theorem at null infinity
   (a) Schoen-Yau’s method
   (b) Witten’s method
Bondi’s radiating space-time (1)

Historical notes

- Late 1950s-early 1960s:
  Pirani, Bondi, Robinson, Trautman and others
- Bondi, Van der Burg, and Metzner:
  - vacuum solutions of Einstein’s field equations outside an isolated (i.e. spatially bounded) \textit{axisymmetric} system
  - definition of mass along outgoing null hypersurfaces in the limit \( r \to \infty \)
  - the total mass, measured at null infinity, is non-increasing with respect to the retarded time

Bondi’s radiating space-time (2)

Historical notes

- Sachs 1962:
  Generalization of the work of Bondi, Van der Burg, Metzner to asymptotically flat space-times

Bondi’s radiating space-time (3)

Historical notes
Interpretation of Bondi mass as the total mass of the isolated physical system measured after the loss due to the gravitational radiation up to that time
Bondi’s radiating space-time (4)

Retarded time

Minkowski space-time:

- photon emitted when $t = t_0$ takes a certain amount of time to reach an observer located at distance $r \geq 0$ from the source, so the observer notices it when $t = t_1$,

$$t_1 = t_0 + \frac{r}{c} = t_0 + r$$  \hspace{1cm} \text{(where speed of light $c := 1$)}

- the time $t_0 = t_1 - r$ is defined as *retarded time*.

- set $u = t - r$:
  the hypersurface defined by $u = t - r = k$, $k$ a constant, is the future directed null cone with vertex $r = 0$, $t = k$. 
Bondi’s radiating space-time (5)

**Bondi’s radiating space-time**

Vacuum space-time \( (T_{ij} = 0) \) with *Bondi’s radiating metric*

\[
g_{\text{Bondi}} = \left( + r^2 e^{2\gamma U^2} \cosh 2\delta + r^2 e^{-2\gamma W^2} \cosh 2\delta + 2r^2 U W \sinh 2\delta - \frac{V}{r} e^{2\beta} \right) du^2 \\
-2e^{2\beta} du dr - 2r^2 \left( e^{2\gamma U} \cosh 2\delta + W \sinh 2\delta \right) du d\theta \\
-2r^2 \left( e^{-2\gamma W} \cosh 2\delta + U \sinh 2\delta \right) \sin \theta du d\phi \\
+r^2 \left( e^{2\gamma} \cosh 2\delta d\theta^2 + e^{-2\gamma} \cosh 2\delta \sin^2 \theta d\phi^2 + 2 \sinh 2\delta \sin \theta d\theta d\phi \right)
\]

- \( \beta, \gamma, \delta, U, V, W \): functions of \( (u, r, \theta, \varphi) \) which are smooth for \( 0 < r_0 \leq r, 0 \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi \)

- \( u \): retarded time, \( r \): radius function,
  \( \theta, \varphi \): spherical coordinates.

- \( u = \text{constant} \) are null hypersurfaces.
Bondi’s radiating space-time (6)

Examples

(i) Minkowski space-time:

\[
g_{\text{Mink}} = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2).
\]

Using the retarded time \( u = t - r \), the metric can be written as

\[
g_{\text{Mink}} = -du^2 - 2dudr + r^2(d\theta^2 + \sin^2 \theta d\varphi^2).
\]
Bondi’s radiating space-time (7)

Examples

(ii) Schwarzschild space-time:

\[ g_{\text{Schw}} = -(1 - \frac{2m}{r})dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2(d\theta^2 + \sin^2 \theta d\varphi^2), \]

where \( m \) is the mass.

Using the retarded time \( u = t - r - 2m \ln |r - 2m| \), the metric can be written as

\[ g_{\text{Schw}} = -(1 - \frac{2m}{r})du^2 - 2dudr + r^2(d\theta^2 + \sin^2 \theta d\varphi^2). \]
Bondi’s radiating space-time (8)

Examples

(iii) Kerr space-time:

$$g_{Kerr} = -\left(1 - \frac{2mr}{\Sigma}\right)dt^2 - \frac{4mar \sin^2 \theta}{\Sigma}dt d\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \left( r^2 + a^2 + \frac{2mra^2 \sin^2 \theta}{\Sigma^2} \right) \sin^2 \theta d\phi^2$$

where

$$\Sigma \equiv r^2 + a^2 \cos^2 \theta, \quad \Delta \equiv r^2 - 2mr + a^2,$$

$m$ is the mass, $ma$ is the angular momentum as measured from infinity.

Retarded time $u$? It is an interesting problem to obtain the Kerr solution in the form of a Bondi metric.
Bondi’s radiating space-time (9)

The **outgoing radiation conditions** imply, as \( r \to \infty \),

\[
\begin{align*}
\gamma &= \frac{c(u, \theta, \varphi)}{r} + O(r^{-3}), \\
\delta &= \frac{d(u, \theta, \varphi)}{r} + O(r^{-3}), \\
\beta &= -\frac{c^2 + d^2}{4r^2} + O(r^{-4}), \\
U &= -\frac{l(u, \theta, \varphi)}{r^2} + O(r^{-3}), \\
W &= -\frac{\bar{l}(u, \theta, \varphi)}{r^2} + O(r^{-3}), \\
V &= -r + 2M(u, \theta, \varphi) + O(r^{-1}),
\end{align*}
\]

where \( l = c_{,2} + 2c \cot \theta + d_{,3} \csc \theta, \bar{l} = d_{,2} + 2d \cot \theta - c_{,3} \csc \theta \).
Bondi’s radiating space-time (10)

Bondi’s radiating metric under the outgoing radiation conditions

\[ g_{\text{Bondi}} = -\left(1 - \frac{2M}{r} + O(r^{-2})\right)du^2 \]
\[ -2\left(1 - \frac{c^2 + d^2}{4r^2} + O(r^{-4})\right)dudr \]
\[ +2\left(l + \frac{2cl + 2dl}{r} + O(r^{-2})\right)dud\theta \]
\[ +2\left(\bar{l} - \frac{2c\bar{l} - 2dl}{r} + O(r^{-2})\right)\sin\theta dud\varphi \]
\[ +r^2\left(1 + \frac{2c}{r} + O(r^{-2})\right)d\theta^2 \]
\[ +r^2\left(1 - \frac{2c}{r} + O(r^{-2})\right)\sin^2\theta d\varphi^2 \]
\[ +r^2\left(\frac{4d}{r} + O(r^{-2})\right)\sin\theta d\theta d\varphi^2, \]

where \( M, c, d, l \) and \( \bar{l} \) are functions of \( u, \theta \) and \( \varphi \).
Bondi’s radiating space-time (11)

We have two regularity assumptions

**Condition A**
Each of the six functions $\beta$, $\gamma$, $\delta$, $U$, $V$, $W$ together with its derivatives up to the second orders are equal at $\varphi = 0$ and $2\pi$.

**Condition B**
For all $u$ and $\varphi$,

$$c(u, 0, \varphi) = c(u, \pi, \varphi) = 0.$$
Bondi’s radiating space-time (12)

At null infinity:
The Bondi energy-momentum of $u_0$-slice:

$$m_\nu(u_0) = \frac{1}{4\pi} \oint_{S^2} M(u_0, \theta, \varphi)n^\nu dS$$

for $\nu = 0, 1, 2, 3$, where

$$n^0 = 1, \quad n^1 = \sin \theta \cos \varphi, \quad n^2 = \sin \theta \sin \varphi, \quad n^3 = \cos \theta.$$

$m_0$: “Bondi energy” or “Bondi mass”
$m_i$: “Bondi momentum”

- Minkowski space-time: $m_\nu(u_0) = 0$.
- Schwarzschild space-time: $m_0(u_0) = m$ and $m_i(u_0) = 0$. 
Bondi’s radiating space-time (13)

We define

\[ M(u, \theta, \varphi) = M(u, \theta, \varphi) - \frac{1}{2}(l,2 + l \cot \theta + \bar{l},3 \csc \theta) \]

\[ = M(u, \theta, \varphi) - \frac{1}{2} \left[ -2c(u, \theta, \varphi) + c,22(u, \theta, \varphi) \right. \]
\[ - \csc^2 \theta c,33(u, \theta, \varphi) + 2 \csc \theta d,23(u, \theta, \varphi) \]
\[ + 3 \cot \theta c,2(u, \theta, \varphi) + 2 \cot \theta \csc \theta d,3(u, \theta, \varphi) \bigg]. \]

Its \( u \)-derivative is

\[ M,0 = -\left[ (c,0)^2 + (d,0)^2 \right]. \]
Bondi’s radiating space-time \((14)\)

Under condition A and B, we have

\[
\oint_{S^2} \left( l_2 + l \cot \theta + \bar{l}_3 \csc \theta \right) n^\nu dS = 0.
\]

This implies

\[
m_\nu(u_0) = \frac{1}{4\pi} \oint_{S^2} M(u_0, \theta, \varphi) n^\nu dS
\]

\[
= \frac{1}{4\pi} \oint_{S^2} M(u_0, \theta, \varphi) n^\nu dS.
\]

\(\nu = 0, 1, 2, 3.\)
Bondi’s radiating space-time (15)

Bondi mass loss formula

\[
\frac{d}{du} m_\nu = -\frac{1}{4\pi} \oint_{S^2} \left[ (c,0)^2 + (d,0)^2 \right] n^\nu dS, \quad \nu = 0, 1, 2, 3.
\]

When \( \nu = 0 \), this is the famous Bondi mass loss formula

\[
\frac{d}{du} m_0 = -\frac{1}{4\pi} \oint_{S^2} \left[ (c,0)^2 + (d,0)^2 \right] dS \leq 0.
\]

The Bondi mass is non-increasing in \( u \), i.e., more and more energy is radiated away.

Bondi mass-loss is measured by the news functions \( c,0 \) and \( d,0 \).
Bondi’s radiating space-time (16)

**Generalized Bondi energy-momentum loss formula**

Applying Hölder’s inequality and Cauchy-Schwarz’ inequality, we obtain

\[
\frac{d}{du} \left( m_0 - \sqrt{\sum_{1 \leq i \leq 3} m_i^2} \right) \leq 0.
\]
Bondi’s radiating space-time (17)

Positive Mass Conjecture at Null Infinity

*In Bondi’s vacuum radiating space-times, the Bondi mass must be nonnegative, i.e., a finite gravitational system cannot radiate away more energy than it has initially.*
Structuring

1. Gravitational waves
2. Bondi’s radiating space-time
3. Positive mass theorem at null infinity
   (a) Schoen-Yau’s method
   (b) Witten’s method
Positive mass theorem at null infinity (1)

The Positive mass theorem near infinity

1982, Schoen-Yau:
Solving the Jang’s equation–prescribing the mean curvatures.

1982-, Israel-Nester, Horowitz-Perry, Ashtekar-Horowitz, Renla-Tod, Ludvigsen-Vickers, etc.:
Witten’s method–the Dirac operator.

2006, W.-l. Huang, S.T. Yau, and X. Zhang:
Detailed proof.

Positive mass theorem at null infinity (2)

Asymptotically null initial data set

In Minkowski space-time, the spacelike hypersurface

\[ t = \sqrt{1 + r^2} \]

has the hyperbolic metric \( \tilde{g} \) and the nontrivial second form \( \tilde{h} \),

\[
\tilde{g} = \frac{dr^2}{1 + r^2} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2),
\]

\[
\tilde{h} = \frac{dr^2}{1 + r^2} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)
\]

in polar coordinates \((r, \theta, \varphi)\) where \(0 < r < \infty\), \(0 \leq \theta < \pi\), \(0 \leq \varphi < 2\pi\).
Positive mass theorem at null infinity (3)

Asymptotically null initial data set

Denote the associated orthonormal frame \{\tilde{e}_i\} and coframe \{\tilde{e}^i\} by

\[
\tilde{e}_1 = \sqrt{1 + r^2} \frac{\partial}{\partial r}, \\
\tilde{e}_2 = \frac{1}{r} \frac{\partial}{\partial \theta}, \\
\tilde{e}_3 = \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi},
\]

\[
\tilde{e}^1 = \frac{dr}{\sqrt{1 + r^2}}, \\
\tilde{e}^2 = r d\theta, \\
\tilde{e}^3 = r \sin \theta d\varphi.
\]

\nabla: the Levi-Civita connection of \tilde{g},

\nabla_i := \nabla \tilde{e}_i.
Positive mass theorem at null infinity (4)

Asymptotically null initial data set

An initial data set \((M^3, g, p)\) \((p \text{ is not necessarily symmetric})\) is asymptotically null of order \(\tau\) if, outside a compact subset,

- \(M\) is diffeomorphic to \(R^3 \setminus B_R\),
- the metric \(g\) and the 2-tensor \(p\) are

\[
\begin{align*}
g(\check{e}_i, \check{e}_j) &= \check{g}(\check{e}_i, \check{e}_j) + a_{ij}, \\
p(\check{e}_i, \check{e}_j) &= \check{p}(\check{e}_i, \check{e}_j) + b_{ij}
\end{align*}
\]

where \(a_{ij}\) and \(b_{ij}\) satisfy

\[
\begin{align*}
a_{ij} &= O(r^{-\tau}), \\
\nabla_k a_{ij} &= O(r^{-\tau}), \\
\nabla_k \nabla_l a_{ij} &= O(r^{-\tau}).
\end{align*}
\]

\[
\begin{align*}
b_{ij} &= O(r^{-\tau}), \\
\nabla_k b_{ij} &= O(r^{-\tau}), \\
\nabla_k \nabla_l b_{ij} &= O(r^{-\tau}).
\end{align*}
\]
Positive mass theorem at null infinity (5)

In Bondi’s radiating vacuum space-time, the spacelike hypersurface given by

\[ u = \sqrt{1 + r^2} - r + \frac{(c^2 + d^2)_{u=0}}{12r^3} + \frac{a_3(\theta, \varphi)}{r^4} + o(r^{-4}) \]

is asymptotically null of order 1.

(Since Bondi’s radiating metric is complicated, we have calculated the induced metric \( g \) and the second fundamental form \( h \) using Mathematica 5.0.)
Structuring

1. Gravitational waves
2. Bondi’s radiating space-time
3. Positive mass theorem at null infinity
   (a) Schoen-Yau’s method
   (b) Witten’s method
Schoen-Yau’s method (1)

Jang’s equation:

\[
\left( g^{ij} - \frac{f^i f^j}{1 + |\nabla f|^2} \right) \left( \frac{f_{,ij}}{\sqrt{1 + |\nabla f|^2}} - h_{ij} \right) = 0.
\]

If Jang’s equation has a solution \( f \) which has the asymptotic expansion

\[
f = \sqrt{1 + r^2} + p(\theta, \varphi) \ln r + o(1)
\]

for \( r \) sufficiently large, then \( p(\theta, \varphi) \) and \( M(0, \theta, \varphi) \) must be constant.
Schoen-Yau’s method (2)

Proof. For $r$ sufficiently large,

$$ J(f) \approx \frac{\ln r}{r^3} \Delta_{S^2} p(\theta, \varphi) + \frac{p(\theta, \varphi) - 2M(0, \theta, \varphi)}{r^3}, $$

where

$$ \Delta_{S^2} = \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \csc^2 \theta \frac{\partial^2}{\partial \varphi^2}. $$

$J(f) = 0$ implies

$$ \Delta_{S^2} p(\theta, \varphi) = 0, \quad p(\theta, \varphi) - 2M(0, \theta, \varphi) = 0. $$

There is no nonconstant harmonic function on $S^2$

$$ \Rightarrow p(\theta, \varphi) \text{ and } M(0, \theta, \varphi) = \frac{p(\theta, \varphi)}{2} \text{ are constant.} $$
Schoen-Yau’s method (3)

Theorem 1. Let $(\mathbb{L}^{3,1}, \tilde{g})$ be a vacuum Bondi’s radiating space-time with Bondi-metric $\tilde{g}$. Suppose that Condition A and Condition B hold.

If there exists $u_0$ such that $\mathcal{M}(u_0, \theta, \varphi)$ is constant, then

$$m_0(u) \geq \sqrt{\sum_{1 \leq i \leq 3} m_i^2(u)}$$

for all $u \leq u_0$. 

Schoen-Yau’s method (4)

Idea of the proof:

- Let $\mathcal{M}(u_0, \theta, \varphi) = \frac{p}{2}$ be constant for some $u_0$.
- Without loss of generality, $u_0 = 0$.
- On the hypersurface

$$u = \sqrt{1 + r^2} - r + \frac{(c^2 + d^2)u=0}{12r^3} + \frac{a_3(\theta, \varphi)}{r^4} + o(r^{-4}),$$

Jang’s equation has a solution

$$f = \sqrt{1 + r^2} + p \ln r + q,$$

and the metric $\bar{g}_{ij} = g_{ij} + f,i \cdot f,j$ is asymptotic flat.
Schoen-Yau’s method (5)

Idea of the proof:

• ADM total energy $E(\bar{g}) = p$.

• Scalar curvature $\bar{R} \geq 2|Y|^2_\bar{g} - 2 \text{div}_\bar{g} Y$.

• $\bar{g}$ can be transformed conformally to a metric $\hat{g}$ with

$$\hat{R} = 0 \quad \text{and} \quad E(\bar{g}) \geq E(\hat{g}).$$

• The positive mass theorem at spatial infinity implies

$$p = E(\bar{g}) \geq E(\hat{g}) \geq 0.$$

• Bondi mass on the $u = 0$ slice:

$$m_0(0) = \frac{p}{2} \geq 0, \quad m_1(0) = m_2(0) = m_3(0) = 0.$$
Schoen-Yau’s method (6)

Idea of the proof:

- From the Bondi mass-loss formula,
  \[ m_0(u) \geq m_0(u_0) \geq 0 \quad \forall u \leq u_0. \]
- From the generalized Bondi energy-momentum-loss formula,
  \[ m_0(u) \geq \sqrt{\sum_{1 \leq i \leq 3} m_i^2(u)} \quad \forall u \leq u_0. \]
Structuring

1. Gravitational waves
2. Bondi’s radiating space-time
3. Positive mass theorem at null infinity
   (a) Schoen-Yau’s method
   (b) Witten’s method
Witten’s method (1)

Let \((X, g, h)\) be an asymptotically null spacelike hypersurface.

- **total energy:**
  \[
  \mathcal{E} = \bar{\nabla}^j a_{1j} - \bar{\nabla}_1 \text{tr} \bar{g}(a) - \left[ a_{11} - \delta_{11} \text{tr} \bar{g}(a) \right],
  \]
  \[
  E_\nu(X) = \frac{1}{16\pi} \lim_{r \to \infty} \int_{S_r} \mathcal{E} \, n^\nu \, r \, dS
  \]

- **total linear momentum:**
  \[
  \mathcal{P} = b_{11} - \delta_{11} \text{tr} \bar{g}(b),
  \]
  \[
  P_\nu(X) = \frac{1}{8\pi} \lim_{r \to \infty} \int_{S_r} \mathcal{P} \, n^\nu \, r \, dS
  \]

\(S_r\): sphere of radius \(r\) in \(\mathbb{R}^3\), \(\nu = 0, 1, 2, 3\).
Witten’s method (2)

Positive mass theorem (X. Zhang, 2002):

Let \((X, g_{ij}, p_{ij})\) be a 3-dimensional asymptotically null initial data set of order \(\tau = 3\). Denote

\[
\mu := \frac{1}{2}(R + (p_i)^2 - p_{ij} p^{ij}), \quad \varphi_j := \nabla^i p_{ji} - \nabla_j p^i, \quad \sigma_j := 2\nabla^i (p_{ij} - p_{ji}).
\]

If the initial data set satisfies the dominant energy condition

\[
\mu \geq \max \left\{ \sqrt{\sum \varphi_j^2}, \sqrt{\sum (\varphi_j + \sigma_j)^2} \right\},
\]

then

\[
E_0(X) - P_0(X) \geq \sqrt{\sum_{i=1,2,3} \left[ E_i(X) - P_i(X) \right]^2}.
\]

If equality holds, then

\[
R_{ijkl} + p_{ik} p_{jl} - p_{il} p_{jk} = 0, \quad \nabla_i p_{jk} - \nabla_j p_{ik} = 0, \quad \nabla^j (p_{ij} - p_{ji}) = 0.
\]
Witten’s method (3)

Remark: The proof of the theorem can still go through if the order $\tau > \frac{3}{2}$ and the $E_\nu - P_\nu$ are finite for $\nu = 0, 1, 2, 3$. 
Witten’s method (4)

Theorem 2. Let \((\mathbb{L}^{3,1}, \tilde{g})\) be a vacuum Bondi’s radiating space-time with Bondi-metric \(\tilde{g}\). Suppose that Condition A and Condition B hold.

If there exists \(u_0\) such that

\[
\begin{align*}
&c|_{u=u_0} = d|_{u=u_0} = 0, \\
&\text{then} \\
&m_0(u) \geq \sqrt{\sum_{1 \leq i \leq 3} m_i^2(u)},
\end{align*}
\]

for all \(u \leq u_0\).
Witten’s method (5)

Proof.

- Without loss of generality, assume $u_0 = 0$.
- Choose an asymptotically null spacelike hypersurface $\mathbf{X}$ with

$$u = \sqrt{1 + r^2} - r + \frac{(c^2 + d^2)_{u=0}}{12r^3} + O(r^{-4}).$$

- $\mathcal{E} - 2\mathcal{P} = -\frac{3}{2r^3}(l,2 + l\cot\theta + \bar{\ell},3\csc\theta)_{u=0} + \frac{4M(0,\theta,\phi)}{r^3} + O(r^{-4}).$

- $E_{\nu}(\mathbf{X}) - P_{\nu}(\mathbf{X}) = m_{\nu}(0)$.

- $E_0(\mathbf{X}) - P_0(\mathbf{X}) \geq \sqrt{\sum_{i=1}^{3} \left[ E_i(\mathbf{X}) - P_i(\mathbf{X}) \right]^2}$.

- $m_0(u) \geq \sqrt{\sum_{i=1}^{3} m_i^2(u)}$.
Open questions

1. What does it mean in physics that $M(u_0, \theta, \varphi) = \text{constant}$?

2. What is the physical meaning of the condition

$$c(u_0, \theta, \varphi) = 0 = d(u_0, \theta, \varphi)$$

Does it preclude gravitational waves?
Thank you for your attention!