TIME REVERSAL IN RANDOM MEDIA
SELF-AVERAGING, HYPERRESOLUTION AND DUALITY

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Main references


Arrays of transducers can re-create a sound and send it back to its source as if time had been reversed. The process can be used to destroy kidney stones, detect defects in materials and communicate with submarines.
Time reversibility and reciprocity

- Initial value problem:
  \[
  \frac{1}{c^2} \frac{\partial^2}{\partial t^2} u = \Delta u, \quad u(0, x) = u_0(x), \quad \partial_t u(0, x) = v_0(x)
  \]
  Solution at time $T$ is $u(T, x)$.

- Terminal value problem
  \[
  \frac{1}{c^2} \frac{\partial^2}{\partial t^2} u' = \Delta u', \quad u'(T, x) = u(T, x), \quad \partial_t u'(T, x) = \partial_t u(T, x).
  \]
  Solution at time 0 is $u_0(x)$. In other words, for $\tilde{u}(t, x) = u(T-t, x)$ then
  \[
  \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \tilde{u} = \Delta \tilde{u}, \quad \tilde{u}(0, x) = u(T, x), \quad \partial_t \tilde{u}(0, x) = -\partial_t u(T, x)
  \]
  and $\tilde{u}(T, x) = u_0(x)$.

- Time-harmonic solution $u(t, x) = e^{-iwt}v(x)$ satisfies the Helmholtz equation
  \[
  \Delta v + k^2 v = f(x), \quad k = \frac{w}{c}
  \]
where $k$ is the wavenumber and $w$ the frequency.

- General solution in the $3 - d$ free space:

$$v(x) = \int G(x, y) f(y) dy, \quad G(x, y) = \frac{e^{ikr}}{4\pi r}, \quad r = |x - y|$$

defines a unitary map.

- Reciprocity or symmetry: Green function $G(x, y)$ satisfies

$$G(x, y) = G(y, x).$$

This is a consequence of the self-adjointness of the Helmholtz operator.
Inversion and refocusing

• Inversion: volume measurement

\[
\int \bar{G}(x, y) G(y, x') dy = \delta(x - x') \implies f(x) = \int \bar{G}(x, y) v(y) dy.
\]

• Inversion: To completely recover the original signal \(u_0(x)\) or \(f(x)\) need **complete** information about

  – Time dependent case: the wave field \(u(T, x), \partial_t u(T, x), \forall x\).

  – Time-harmonic case: \(v(x), \forall x\).

  – Exception: \(3 - d\) wave satisfies the strong Huygens principle, need only the wave front data to recover perfectly.

• Refocusing: boundary measurement

  – Highly localized source \(u_0(x) = \delta(x - x_0)\) or \(f(x) = \delta(x - x_0)\).
– Not perfect inversion but seek highly localized refocusing.

– Example: Regardless of the intensity, refocusing can be achieved with

\[ \int G(T, x, y) \left( u(T, y), -\partial_t u(T, y) \right) \delta_A(y) dy \]  
(time dependent case)

or

\[ \int \bar{G}(x, y)v(y)\delta_A(y)dy \]  
(time harmonic case)

and \( A \) being the sphere surrounding the source.

• Questions of aperture in the remote sensing regime with small aperture compared to the distance of propagation.

– How large does the aperture need be in order to achieve certain resolution?

– How to trade space with time in achieving refocusing?

• Remote sensing regime: amplitude modulation, quasi-monochromatic waves.
ACOUSTIC TIME-REVERSAL MIRROR operates in two steps. In the first step (left) a source emits sound waves (orange) that propagate out, perhaps being distorted by inhomogeneities in the medium. Each transducer in the mirror array detects the sound arriving at its location and feeds the signal to a computer.

In the second step (right), each transducer plays back its sound signal in reverse in synchrony with the other transducers. The original wave is re-created, but traveling backward, retracing its passage back through the medium, untangling its distortions and refocusing on the original source point.
Range: $L$, Carrier wavelength $\lambda$, Array size $a = (N - 1)\lambda/2$. Source at $y$, Search point at $y_s$, Transducers at $x_p$. Remote sensing regime: $\lambda \ll a \ll L$. Random medium: Correlation length $l \ll L$, fluctuation strength $\sigma \ll 1$. 

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**Time Reversal Schematic**
Time reversal in homogeneous medium

- Transmission mode: \( \vec{x} = (z, x), x \in \mathbb{R}^2 \) where \( z \) is the propagation direction and \( x \) is the transverse dimensions.

- Time reversal is equivalent to phase conjugation in monochromatic waves, e.g.

\[
\begin{align*}
\text{forward} & \quad e^{ikz}e^{-iwt} \quad \rightarrow \quad \text{backward} \\
& \quad e^{-ikz}e^{-iwt} \quad \text{or} \quad e^{ikz}e^{iwt}.
\end{align*}
\]

- The wave field received at the time-reversal mirror: source \( \Psi_0 \)

\[
\nu_m(z, x_m) = I_A(x_m) \int G(0, x_m; z, x_s)f(x_s)dx_s
\]

\[
= I_A(x_m) \int G(z, x_s; 0, x_m)f(x_s)dx_s
\]

where \( I_A \) is the aperture function of the phase-conjugating mirror \( A \).
After phase conjugation and back-propagation:

\[ v^B(x) = \int G(L, x; 0, x_m) G(L, x_s; 0, x_m) I_A(x_m) \overline{f(x_s)} d x_m d x_s \]

\[ = \int \Gamma(L, x, x_s) f(x_s) d x_s \]

where \( \Gamma \) is the two-point function

\[ \Gamma(L, x, x_s) = \int G(L, x; 0, x_m) \overline{G(L, x_s; 0, x_m)} I_A(x_m) d x_m. \]

• Question: how close is \( \Gamma \) to a \( \delta \)-like function? Factors:
  
  – Random media
  
  – Aperture
  
  – Frequency band-width
FIG. 3. The left figure shows the spatial diffraction pattern of the amplitude of (25) at time \( t = 0 \). The right figure is a space-time contour plot of the amplitude of (25) and shows the parabolic shift in arrival time. Here, the pulse width is \( W = 0.5 \) ms, the TRM width \( W_t = 30 \) m, the propagation speed is \( V = 1500 \) m/s, the propagation distance is \( L = 1000 \) m, and the period of the carrier is 0.22 ms at 4.5 KHz.
ACOUSTIC CHAOTIC PINBALL occurs when an underwater ultrasonic pulse emitted by the transducer (at left in photograph) ricochets among 2,000 randomly placed steel rods before reaching the 96-element time-reversing mirror at right. Each element of the array receives a chaotic-seeming sound signal (a portion of one is shown in a) lasting much longer than the original one-microsecond pulse. When the mirror plays back the chaotic signals, reversed and in synchrony, they ricochet back through the maze of rods and combine to re-create a well-defined pulse (b) at the transducer.
FIG. 1. Top: Experimental setup. An ultrasonic array (central frequency 3.2 MHz, pitch 0.4 mm, total aperture 9.2 mm) communicates to five points simultaneously through a multiple scattering medium. The distance between neighboring receivers is 2 mm. Bottom: typical waveform scattered by the medium when a 1-µs pulse is sent through it by the array to the receivers.

FIG. 3. Directivity of the time-reversed waves around the desired focal point. Top: Comparison between the multiple scattering medium (solid line) and water (dashed line), for a 23-element array; the −6 dB widths are 1.1 mm and 10.3 mm. Bottom: Comparison between quasimonochromatic (3.2 MHz, dashed line) and wideband (solid line) focusing through the multiple scattering medium, for a single element.
Recently researchers from the Scripps Institution of Oceanography in La Jolla, Calif., and the SACLANT Undersea Research Center in La Spezia, Italy, built and tested a 20-element TRM in the Mediterranean Sea off the coast of Italy [see illustration above]. Led by Tuncay Akal, William Hodgkiss and William A. Kuperman, they showed in water about 120 meters deep that their mirror could focus sound 15 kilometers away from the source, reducing the previously required path length of 30 kilometers. The technology leaves little room for error, however, and more testing is needed. A solution might involve using multiple TRMs to increase the range.

RESULTS from an underwater experimental run. Color contours indicate intensity of sound. The transmitted signal pulse (red circle) is greatly distorted at the time-reversal mirror, but when the time-reversed signal is played back (at left) it reproduces the source signal (red circle) with high fidelity.
SINGLE TRANSUDER can time-reverse a wave in an enclosed "cavity." A source transducer emits a pulse at location A on a small silicon wafer (top). A transducer at location B records chaotic reverberations of the pulse reflected off the wafer edges hundreds of times. The transducer at B plays back a short segment of that signal in reverse (bottom). After many reflections, these recombine to re-create the short pulse focused again at location A, as was revealed by imaging the waves on the wafer near A (bottom left).
Applications

• Communication: under-water acoustics and wireless communication
  – Pilot wave to probe the medium; the point-spread function (the Green function) to encode the signal.
  – Use spatial refocusing to reduce interference and increase capacity.
  – Secrecy of channel.

• Detection and Imaging in strongly cluttered environment (Friday’s talk).

• Iterated time reversal: to create focused wave energy for medical purposes such as destroying kidney stones or for detection (of e.g. cracks). The strongest scatterer becomes the source as the process iterates.
medical imaging, where one wishes to send the ultrasound through fat, bone and muscle to targets such as tumors or kidney stones. Pulse-echo detection with a TRM can circumvent this problem.

First, one part of the array sends a brief pulse out through the distorting medium to illuminate the region of interest. Next, the wave that is reflected back to the array by a target is recorded, time-reversed and reemitted. The time-reversal process ensures that this reversed wave focuses on the target despite all the distortions of the medium.

When the region contains only one target, this self-focusing technique is highly effective. If there are several targets, the problem is more complicated, but a single target can be selected by repeating the procedure. Consider the simplest multtarget case, in which the medium contains two targets, one more reflective than the other. The echoes produced by the initial pulse will have a somewhat stronger component from the brighter target than from the weaker one. Therefore, the first time-reversed signal will focus a wave on each target but with a more powerful wave on the brighter target. The echoes from these waves will have an even greater bias toward the brighter target, and after a few more iterations one will have a signal that focuses primarily on that target. More complex techniques let one select the weaker reflectors.

Among the medical applications of pulse-echo TRM, the closest to intuition is the destruction of stones in kidneys and 96 Scientific American November 1999

KIDNEY STONES can be targeted and broken up with ultrasound by using the self-focusing property of a time-reversal mirror. An ultrasonic pulse emitted by one part of the array (a) produces a distorted echo from the stone (b). A powerful time-reverse of this echo passes through intervening tissues and organs, focuses back on the stone (c) and breaks it up. Iterating the procedure improves the focus and allows real-time tracking as the stone moves because of the patient’s breathing.
FIG. 2. Time-harmonic waves in random media. Propagation distance 1000 m, TRM width 50 m, width of numerical domain 150 m, width of random medium 112.5 m, contrast ±5%, 428 realizations. (a) Amplitude of the mean: Homogeneous (light) and average over random realizations (dark) case. (b) Relative variance, >O(1) except for a very small interval. (c) Individual realizations that show super-resolution (high) as well as no resolution at all (low).
Issues and approach

• Refocusing: How is it affected by the aperture and time-window?

• The effects of media: anomalous refocusing infractal media.
  
  – Localization and enhanced backscattering of waves in random media
  
  – Transition from the Bloch waves in a periodic medium to the diffusive wave regime when random impurities are added to the medium.

  – **Hyperresolution** instead of impedance. Examples: (1) Adaptive optics and space telescopes (such as the Hubble Space Telescope) (2) In satellite or wireless communications the multi-path effect of a random medium causes the inter-symbol-interference which induces errors in communication or slows down the rate of transmission.
– Duality in fractal media.

• Statistical stability: self-averaging effect of refocusing.

• The trade-off between the time window and the frequency bandwidth (the delay-bandwidth product).

• **Approaches**: asymptotics, approximations, efficient numerical schemes.
Modeling random media

- Discrete vs. continuous

- Weakly scattering objects: air (line-of-sight), fog (no line-of-sight).

- Continuous media: refractive index as random functions.

- Discrete media: randomly placed steel rods as Poisson point processes.
Figure 4.7 Measured time series of velocity, temperature, and absolute humidity fluctuations at a suburban site in Vancouver, Canada, during moderately unstable conditions. From Roth, 1990.
Fig. 1.9. Wake behind two identical cylinders at $R = 240$. Courtesy R. Dumas.

Fig. 1.10. Wake behind two identical cylinders at $R = 1800$. Courtesy R. Dumas.
Statistically stationary or homogeneous random fields

- $V(t)$ is a real-valued stationary process and admits the spectral representation:

$$V(t) = \int \exp[-iwt] \hat{V}(dw)$$

where $\hat{V}(dw)$ is a random orthogonal measure

$$\mathbb{E}[\hat{V}(dw)\hat{V}(dw')] = \delta(w + w')\Phi(w)dwdw'$$

and $\int$ is a stochastic integral converging in the mean square sense.

- Stationarity implies $\delta$ correlation in $w$.

$$\mathbb{E}[V(t + s)V(s)] = \int e^{-iwt} e^{i(w' - w)s} \mathbb{E}[\hat{V}(dw)\hat{V}(dw')] = \int e^{-iwt} \Phi(w)dw$$
• $\Phi(w)$ is the power spectral density: the intensity of fluctuation in mode $w$.

• Discrete approximation (Riemann-Stieltjes sum): random Fourier series with increasing period.

• $V(\vec{x})$: $\vec{x}$-homogeneous random field

\[
V(\vec{x}) = \int \exp(i\vec{k} \cdot \vec{x}) \hat{V}(d\vec{k})
\]

with the $z$-stationary orthogonal spectral measure

\[
E[\hat{V}(d\vec{k})\hat{V}(d\vec{k}')] = \delta(\vec{k} + \vec{k}')\Phi(\vec{k})d\vec{k}d\vec{k}'.
\]
Fig. 6.20. Deformation of a plane wave front of star light by inhomogeneities in the atmosphere and origin of the speckle image in a telescope.
Fig. 5.6. Typical received signal fluctuations for a small detector 145 km from the transmitter. (From A. L. Buck, Applied Optics 6, 703, April 1967).
Fig. 5.5. Instantaneous intensity distribution of a laser beam at 15.5 km. Area of screen covered by photo is about 1.20 m by 1.55 m. (From D. H. Quine, AFIT Thesis, GEP/PH/72-17, 1971.)
Figure 8-12. (a) Long- and (b) short-exposure photographs of the star Lambda Cratis. (Courtesy of Gerd Weigelt and Gerhard Baier, University of Erlangen.)
Time Reversal

- \( \vec{x} = (z, x), x \in \mathbb{R}^d, d = 1, 2. \)

- Reciprocity:
  \[
  G(\vec{x}, \vec{y}) = G(\vec{y}, \vec{x})
  \]
  where \( G \) be the Green's function of the Helmholtz equation
  \[
  \Delta G(\vec{x}, \vec{y}) + k^2 n^2(\vec{x})G(\vec{x}, \vec{y}) = \delta(\vec{x} - \vec{y}).
  \]

- EM wave in dielectric media: refractive index \( n(\vec{x}) \) and electric susceptibility \( \chi(\vec{x}) \):
  \[
  n^2(\vec{x}) = \bar{n}^2(1 + \tilde{n}(\vec{x}))^2 = \chi(\vec{x}) = \bar{\chi}(1 + \tilde{\chi}(\vec{x})).
  \]

- The wave field received at the mirror:
  \[
  \psi_m(z, x_m) = I_A(x_m) \int G(0, x_m, z, x_s)\psi_0(x_s)dx_s
  = I_A(x_m) \int G(z, x_s, 0, x_m)\psi_0(x_s)dx_s
  \]
where $I_A$ is the aperture function of the phase-conjugating mirror $A$.

- After phase conjugation and back-propagation:
  \[
  \Psi^B(z, x) = \int G(z, x, 0, x_m)G(z, x_s, 0, x_m)I_A(x_m)\Psi_0(x_s)dx_mdx_s \\
  = \int e^{i\mathbf{p} \cdot (x - x_s)}W(z, \frac{x + x_s}{2}, p)\Psi_0(x_s)dpdx_s
  \]
  where $W$ is the mixed type Wigner distribution

  \[
  W(z, x, p) = \frac{1}{(2\pi)^d} \int e^{-i\mathbf{p} \cdot \mathbf{y}}G(z, x + \mathbf{y}/2; 0, x_m)G(z, x - \mathbf{y}/2; 0, x_m)I_A(x_m)d\mathbf{y}dx_m.
  \]

- Wigner distribution is a “statistically stable” quadratic functional of $\Psi$ after proper rescaling.
FIG. 2. Time-harmonic waves in random media. Propagation distance 1000 m, TRM width 50 m, width of numerical domain 150 m, width of random medium 112.5 m, contrast ±5%, 428 realizations. (a) Amplitude of the mean: Homogeneous (light) and average over random realizations (dark) case. (b) Relative variance, >O(1) except for a very small interval. (c) Individual realizations that show super-resolution (high) as well as no resolution at all (low).
FIG. 15. Homogeneous medium on the left, random medium on the right. Type of boundary conditions: DTBC with TRM width 60 m. We can clearly see super-resolution as the recompressed peak is narrower in the random medium case. The fluctuations in the sidelobes are partly due to the fact that we are pushing the paraxial approximation beyond its limit; the 10% contrast is stretching the “low-contrast” assumption.
FIG. 16. Homogeneous medium on the left, random medium on the right. Type of boundary conditions: Waveguide with TRM width 60 m. The waveguide effects are quite strong, but an argument for super-resolution can be made, since the peak is better defined. Randomness does not, in any case, degrade the results.
Wigner distribution

- Pure-state version

\[ W(x, p) = \frac{1}{(2\pi)^d} \int e^{-ip \cdot y} u(x + \frac{y}{2}) \overline{u(x - \frac{y}{2})} dy \]

- Real-valued and satisfies the marginal distributions

\[
\int W(x, p) dp = |u(x)|^2 \\
\int W(x, p) dx = (2\pi)^d |\hat{u}(p)|^2
\]

- Fourier transform becomes $\pi/2$-rotation on the phase space

\[ W[\hat{u}](x, p) = W[u](−p, x). \]

- Generalization to fractional Fourier transform with applications to optics.
Recover from the Wigner distribution the wave amplitude up to a constant phase factor

$$\Psi(z, x_1)\Psi^*(z, x_2) = \int W(z, \frac{1}{2}(x_1 + x_2), q) \exp [iq \cdot (x_1 - x_2)]dq.$$
• In the homogeneous medium, the parabolic (paraxial) approximation is the Fresnel approximation of the Kirchhoff solution. For \((z, x), z \gg |x|\)

\[
    r = z \sqrt{1 + \frac{|x|^2}{z^2}} \approx z + \frac{|x|^2}{2z}
\]

\[
    \frac{e^{ikr}}{r} \approx e^{ikz}z^{-1}e^{ik|x|^2/(2z)}
\]

which, up to a constant factor and the phase \(\exp(ikz)\), is the Green function of the free Schrödinger equation.

• Approximating the spherical wave front near the optical axis by a parabolic wave front. In fact, this approximation is valid for any other wave fronts.

• Ansatz: \(u(z, x) = e^{ikz}\Psi(z, x)\) for the Helmholtz equation. Then \(\Psi\) satisfies

\[
    \partial_z^2 \Psi + 2ik \partial_z \Psi + \Delta_\perp \Psi = f(x)e^{-ikz}
\]
and the parabolic approximation is equivalent to dropping $\partial_{\bar{z}}^2 \Psi$.

- **Forward approximation**: the ansatz
  \[
  u(t, z, x) = \Psi(z, x) \exp i(\bar{n}kz - wt)), \quad \bar{x} = (z, x) \in \mathbb{R}^{d+1}, d = 1, 2
  \]

- **$\Psi$**: slowly varying on the length scale $k^{-1}$ $\implies$ neglect $\partial_{\bar{z}}^2 \Psi$.

- The wave equation becomes the Parabolic wave (Schrödinger) equation
  \[
  ik\bar{n} \frac{\partial}{\partial z} \Psi(z, x) + \frac{1}{2} \nabla^2 \Psi(z, x) + k^2 \bar{n}^2 \tilde{n}(z, x) \Psi(z, x) = 0
  \]
  with one-sided boundary condition.

- Normalized refractive index:
  \[
  \tilde{n}(z, x) = \sigma V \left( \frac{z}{L_0}, \frac{x}{L_0} \right)
  \]
  where $\sigma$ is the standard deviation of $\tilde{n}$ and $L_0$ is the correlation length.
Non-dimensionalization by the aperture $L_x$ and the propagation distance $L_z$

$$x' = x/L_x, \quad z' = z/L_z$$

and dropping the prime

$$i\frac{\partial}{\partial z} \Psi(z, x) + \frac{\gamma}{2} \nabla^2 \Psi(z, x) + \mu V(z\ell_z, x\ell_x)\Psi(z, x) = 0$$

with

$$\ell_z = \frac{L_z}{L_0}, \quad \ell_x = \frac{L_x}{L_0}$$

and

(Fresnel number) $\gamma = \frac{L_z}{kL_x^2}, \quad \mu = k\bar{n}L_z\sigma.$

All are dimensionless.
Random refractive index

- $V(z, x)$: $z$-stationary, $x$-homogeneous random field
  \[ V(z, x) = \int \exp (i \vec{k} \cdot \vec{x}) \hat{V}(d\vec{k}) \]
  with the $z$-stationary orthogonal spectral measure
  \[ \mathbb{E}[\hat{V}(d\vec{k})\hat{V}(d\vec{k}')] = \delta(\vec{k} + \vec{k}') \Phi(\vec{k}) d\vec{k} d\vec{k}' . \]

- Power spectrum $\Phi \in C^\infty$: $\vec{k} = (\xi, k)$
  \[ \Phi(\xi, k) \leq K \left( 1 + \ell_1^2 |\vec{k}|^2 \right)^{-\zeta/2} , \quad \zeta > d + 2 \]

- Atmospheric turbulence: von Kármán spectrum
  \[ \Phi_{vk}(\vec{k}) \sim (L_0^{-2} + |\vec{k}|^2)^{-H-1/2-d/2} \exp (-|\vec{k}|^2 \ell_0^2) , \quad H = 1/3 \]
  with inner scale $\ell_0$, outer scale $L_0$, Hurst exponent $H \in (0, 1)$. 

Self-averaging limits

- Radiative transfer: a small Fresnel number $\gamma$ and large observation scales
  
  $$\gamma \ll 1, \quad \ell_z, \ell_x \gg 1$$

  such that
  
  $$\gamma \ell_x = O(1).$$

- Wigner distribution:
  
  $$W^\gamma(x, p) = \frac{1}{(2\pi)^d} \int e^{-ip \cdot y} \psi\left(z, x + \frac{\gamma y}{2}\right) \overline{\psi\left(z, x - \frac{\gamma y}{2}\right)} dy$$

- Wigner-Moyal equation is exact
  
  $$\frac{\partial W^\gamma}{\partial z} + p \cdot \nabla W^\gamma + \mu \mathcal{L}W^\gamma = 0,$$

  and

  $$\mathcal{L}W^\gamma(z, x, p) = i \int e^{iq \cdot x \ell_x} \left[ W^\gamma(z, x, p + \ell_x \gamma q/2) - W^\gamma(z, x, p - \ell_x \gamma q/2) \right] \hat{V}(z \ell_z, dq)$$
where $\hat{V}(z, dq)$ is the partial spectral measure

$$\hat{V}(z, dq) = \int e^{iz\xi} \hat{V}(d\xi, dp)$$

- The mixed state Wigner distribution arising from the time reversal calculation satisfies the same equation.

- The simplicity of the Wigner-Moyal equation for the parabolic waves is a main advantage.
Exactly solvable, deterministic limit

- Suppose $\mu \sim \sqrt{\ell_z}, \ell_x \ll \ell_z$ and $\gamma \ell_x = \lambda > 0$ fixed.

  The Wigner distribution converges in probability as a generalized function on $\mathbb{R}^{2d}$ to the solution of the transport equation

  $$
  \frac{\partial}{\partial z} W(z, x, p) + p \cdot \nabla W(z, x, p) = 2\pi \lambda^{-2} \int \Phi(0, q) [W(z, x, p + \lambda q) - W(z, x, p)] dq.
  $$

- Convergence in probability:

  $$
  \lim_{\gamma \to 0} \mathbb{P} (| \langle (W^\gamma - W), \theta \rangle | > \delta) = 0, \quad \forall \delta > 0.
  $$

- Point: significant amount of transverse diversity leads to self-averaging; interchangeability of ensemble and spatial averages.
Green function is given by

\[ G_W(z, x, p; \bar{x}, \bar{p}) = \frac{1}{(2\pi)^{2d}} \int e^{i(w \cdot (x - \bar{x}) + r \cdot (p - \bar{p}) - zw \cdot \bar{p})} \times \exp \left[ -1/(2\lambda^2) \int_0^z D_*(\lambda(r + w(z - s))) ds \right] dr dw \]

with the structure function

\[ D_*(x) = \int \Phi(0, q) [1 - e^{ix \cdot q}] \, dq \]
Medium with power-law spectral density

- The power-law spectral density

\[ \Phi(\vec{k}) \sim (1 + |\vec{k}|^2)^{-H-1/2-d/2} \exp(-|\vec{k}|^2 \ell_0^2), \quad H \in (0,1) \]

where \( \ell_0 \) is the (normalized) inner scale.

- We focus on the intermediate (or inertial) regime

\[ \ell_0 \ll r = |x| \ll 1. \]

- The structure function with the power-law spectral density

\[ D_*(r) \approx C_*^2 r^{2H_*}, \quad \text{for } \ell_0 \ll r \ll 1 \]

\[ H_* = \begin{cases} 
H + 1/2 & \text{for } H \in (0,1/2) \\
1 & \text{for } H \in (1/2,1]
\end{cases} \]

where \( C_* > 0 \) is a structure parameter.
Anomalous superresolution in time-reversed refocusing

- Probe the refocused field near a point source located at $x_0$, i.e.
  $\Psi_0(x_s) = \delta(x_s - x_0)$, and write
  $$x = x_0 + \ell_x^{-1}y = x_0 + \frac{\gamma}{\lambda}y.$$  

- Recall the time-reversed, back-propagated field
  $$\Psi^B(x_0 + \ell_x^{-1}y) = \int e^{i\mathbf{p} \cdot \mathbf{y}/\lambda} W(z, x_s + \gamma y/2, p) \Psi_0(x_s) dp dx_s$$
  where $W$ is the mixed type Wigner distribution
  $$W(z, x, p) = \frac{1}{(2\pi)^d} \int e^{-i\mathbf{p} \cdot \mathbf{y}} G(z, x + \gamma y/2; 0, x_m) \overline{G(z, x - \gamma y/2; 0, x_m)} I_A(x_m) dy dx$$

- In the scaling limit $\gamma \to 0, \ell_x, \ell_z \to \infty$, $\Psi^B$ has a deterministic limit.
The Green function for time-reversal field:

\[ P_{tr}(x_0, x) = (z \gamma)^{-d} e^{i|x|^2/2z} e^{ix \cdot x_0} \hat{I}_A(\frac{|x|}{\lambda z}) T_{tr}(x) \]

with

\[ T_{tr}(x) = \exp \left[ -z/(2\lambda^2) \int_0^1 D_*(-sx)ds \right]. \]

Here \( \hat{I}_A \) is the Fourier transform of the indicator function \( I_A \) and is related to the Bessel function \( J_1 \) when \( A \) is the circular disk of diameter \( a \).

Rayleigh diffraction limit in homogeneous medium: \( D_* = 0, T_{tr} = 1 \). With circular aperture of diameter \( a \)

\[ P_{tr}(z, x) = \frac{a}{4\pi \lambda z |x|} J_1(\frac{|x| a}{2\lambda z}) \exp \left[ i|x|^2/(2\lambda z) \right] \]

\[ \rho_{tr} \sim \frac{\lambda z}{a} \sim \lambda \]

where \( a \) and \( z \) can both be taken as 1 since they are respectively normalized by \( L_x \) (the back-propagated beam width) and \( L_z \) (the propagation distance).
- Turbulent medium effect is described by

\[ T_{tr}(x) = \exp \left[ -C_*^2 z |x|^{2H_*} \lambda^{-2} / (4H_* + 2) \right] \]

which, in the intermediate regime yields a sharper turbulence-induced resolution

\[ \rho_{tr} = \sqrt[\mu]{\frac{\int |x|^2 T_{tr}^2(z, x) dx}{\int T_{tr}^2(z, x) d\xi}} \rho_{tr} \sim C_*^{-1/H_*} z^{-1/(2H_*)} \lambda^{1/H_*}, \quad \lambda \ll 1 \]

for \( H_* \in (1/2, 1] \).

- To be consistent with the intermediate range assumption (1) impose the condition

\[ C_* \ell_0^{H_*} \ll \lambda \ll C_* . \]

As a result the lower bound of the resolution achievable in the intermediate regime is

\[ \rho_{tr} \sim \ell_0 , \]

i.e. the smallest scale of the medium.
The turbulence-induced resolution is clearly smaller than the diffraction limit when either $C_*$ is sufficiently large (i.e. sufficiently strong medium fluctuation) or $H \in (0, 1/2)$ with sufficiently small $\lambda$ since $H_* < 1$. The latter case includes the Kolmogorov spectrum of $H = 1/3$.

Short-wavelength limit

$$\lambda \ll C_* \ell_0^{H_*}.$$  

The structure function has the asymptotic

$$D_*(r) \approx C_* r^2$$

and hence the turbulence-induced resolution has the asymptotic

$$\rho_{tr} \sim C_*^{-1} \lambda.$$
forward beam spread

- Wave energy in position and wavevector:

\[ \Gamma_2(z, x) = E \left[ \Psi(z, x) \overline{\Psi(z, x)} \right] = \int E[W(z, x, p)] dp \]

\[ \hat{\Gamma}_2(z, p) = E[|\hat{\Psi}(z, p)|^2] = \left( \frac{\lambda}{2\pi} \right)^d \int E[W(z, x, \lambda p)] dx \]

- Gaussian source or pupil function:

\[ \psi_0(x) = \exp \left[ -|x|^2/(2\alpha^2) \right] \]

with \( \alpha > 0 \): the aperture of the incident beam.

\[ \Gamma_2(z, x) = \left( \frac{\alpha}{2\sqrt{\pi}} \right)^d \int e^{-|w|^2[\alpha^2/4+\lambda^2 z^2/(4\alpha^2)]} e^{i w \cdot x} T(z, w) dw \]

\[ \hat{\Gamma}_2(z, p) = \left( \frac{\alpha \lambda z}{2\sqrt{\pi}} \right)^d \int e^{-|y|^2 \lambda^2 z^2/(4\alpha^2)} e^{-i p \cdot y} \hat{T}(z, y) dy. \]
Turbulent Transfer functions:

\[ T(z, y) = e^{-\int_0^z D_*(-\lambda y) ds/(2\lambda^2)}, \quad \hat{T}(z, y) = e^{-z D_*(-y)/(2\lambda^2)} \]

Turbulence-induced spreads

\[ s_* \equiv \left( \frac{\int |x - \tilde{x}|^2 F[T]^2(z, x) dx}{\int F[T]^2(z, x) dx} \right)^{1/(H_* - 1)} \sim C_*^{-1/(H_* - 1)} z^{1+1/(2H_*) - 1/H_*} \]

\[ \hat{s}_* \equiv \left( \frac{\int |p - \tilde{p}|^2 F[\hat{T}]^2(z, p, p) dp}{\int F[\hat{T}]^2(z, p) dp} \right)^{1/(H_* - 1)} \sim C_*^{-1/(H_* - 1)} z^{1/(2H_*) - 1/H_*} \]

Coherence length:

\[ \Gamma_2(x + y/2, x - y/2) = E \left[ \Psi(z, x + y/2) \overline{\Psi(z, x - y/2)} \right] \]

\[ = \left( \frac{\alpha}{\sqrt{2\pi}} \right)^d \int e^{-|w|^2/(4\alpha^2)} e^{-|y - \lambda z w|/(4\alpha^2)} e^{i w \cdot x} T_2(z, w, y) dw \]

with

\[ T_2(z, w, y) = \exp \left[ -1/(2\lambda^2) \int_0^z D_*(-y + \lambda w(z - s)) ds \right] \]

\[ \approx \exp \left( -D_*(-y) z/(2\lambda^2) \right), \quad \lambda \ll 1. \]
• (De)coherence length:

\[ d_* = \sqrt{\frac{\int |y|^2 \mathcal{F}[T_2]^2(z, 0, y)dy}{\int \mathcal{F}[T_2]^2(z, 0, y)dy}} \sim C_*^{-1/H_*} z^{-1/(2H_*)} \lambda^{1/H_*} \]

• Rytov method yields the same coherence function.

• D. Fried’s coherence disk \( \sim 10 - 20cm \) basic mirror component in adaptive optical telescopes.
Duality

- Scale-invariance: \( f_\ell(x) = f(\ell x) \), \( \ell > 0 \)

\[
u_H \equiv \frac{||x||f||_2}{||f||_2} \times \frac{||p|\mathcal{F}[f]|_2}{||\mathcal{F}[f]|_2} = \frac{||x||f_\ell||_2}{||f_\ell||_2} \times \frac{||p|\mathcal{F}[f_\ell]|_2}{||\mathcal{F}[f_\ell]|_2}
\]

- Substitution:

\[
f(x) = \exp \left[ -|x|^{2H^*_z}/2 \right], \quad \ell = C_*^z \zeta/2 \lambda^{-\zeta}.
\]

\[
\hat{T}(z, x) \approx \exp \left[ -C_*^2 z|x|^{2H^*_z}/(2\ell^2) \right] \approx f_\ell(x)
\]

- Duality:

\[
\rho_{tr} \approx d_* \approx \frac{u_H}{\hat{s}_*}
\]

where \( u_H \) is a constant depending only on \( H \).
Conclusions

• Self-averaging regimes: large aperture in TRM or broad-band signals.

• Hyperresolution of time-reversed focal spot: superlinear dependence on the wave length. Motivation for experimental realization.

• Application in time reversal: duality between the turbulence enhanced spread $\hat{s}_*$ and the coherence length $d_*$, the time-reversal spot size $\rho_{tr}$

\[ \rho_{tr} \approx d_* \approx \frac{uH}{\hat{s}_*}. \]