ON THE MASS FORMULA OF SUPERSINGULAR ABELIAN VARIETIES WITH REAL MULTIPLICATIONS

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§1. Introduction

For a reductive group G over $\mathbb{Q}$ which is $\mathbb{R}$-anisotropic and an open compact subgroup K of $G(\mathbb{A}_f)$, one can define the mass of (G, K). The attached mass is a weighted class number. It measures the size of K as well as the class number; but it has better properties on relating different levels K and different groups G. The importance of the mass formula is its uniform organization of different arithmetic objects and the measurement of these arithmetic objects. The computation of the mass formula has a long history, dated from the pioneers Minkowski, Schmid, and Siegel. There are many important contributions on the Tamagawa numbers due to Weil, Ono, Langlands, Harder, Lai and Kottwitz, and on exact formulas due to Shimura, Hashimoto-Ibukiyama, G. Prasad, Gross, Gan-Yu and many others.

For a small class of groups G, the mass can also arise from special abelian varieties in characteristic p. The celebrated Deuring mass formula says

$$\sum_E \frac{1}{\#\operatorname{Aut} E} = \frac{p-1}{24},$$

where E runs over the isomorphism classes of supersingular elliptic curves. The viewpoint taken in this paper is to establish the connection between the geometrically defined mass and the arithmetically defined mass. Then one can take the advantage of the existing computation result of arithmetic mass formula, for example, to verify a mass formula by a different (geometric) method.

In this paper, we show that the mass for certain special polarized abelian varieties with real multiplications can be an arithmetic mass for some (G, K), see (2.9). The features are: these are supersingular points and need not to be superspecial; the polarizations can be inseparable; and the formulation does not require the existence or definition of the moduli space. The latter indicates that there is no difference or new difficulty to establish such connection when the moduli space has bad reduction or is not well-behaved.

As a beautiful application of Shimura's exact mass formula [Sh], we deduce the geometric mass in term of special values of the zeta function up to precise local terms. The special case when the totally real field F is $\mathbb{Q}$, the abelian varieties are superspecial, and the polarizations are principal, it reduces to a result of Ekedahl and van der Geer [G] obtained by the geometric method. Our formulation can be generalized to basic points in a PEL-Shimura variety modulo p, modulo the Hasse principle. Doing that will require much more work and we plan to carry it out in a subsequent paper.

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One part of this paper is to determine the local terms in the case of superspecial abelian varieties of Hilbert-Blumenthal type. Using the geometric mass formula (Theorem 3.9), we can determine the number of the irreducible components of the supersingular locus, following the methods in [LO] and [Y1]. In the last section we determine this number in some restricted cases, particularly when p is totally ramified in F, see Theorem 4.11. We note that our approach gives an explanation of the interesting result of Bachmat and Goren [BG] on the supersingular locus of Hilbert modular surfaces. Finally we note that there is interesting connection of supersingular points with the theory of modular forms modulo p, see [Se] and [Gr].

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§2. Supersingular points and the mass formula

(2.1) Shimura's exact mass formula. Let B be a quaternion division algebra over a totally real number field F, and let σ denote the standard involution of B. Let V be a left B-module of rank m, and let φ be a quaternion Hermitian form on V. That is,

$$\varphi : V \times V \to B$$

such that $\varphi(x,y) = \varphi(y,x)^\sigma$ and $\varphi(ax,by) = a\varphi(x,y)b^\sigma$ for all $x,y \in V$ and $a,b \in B$. Let G^φ denote the unitary group of φ. It is a reductive group over F and we will regard it as a reductive algebraic group over $\mathbb{Q}$ via the Weil restriction of scalars from F to $\mathbb{Q}$. Assume that $G^\varphi(\mathbb{R})$ is compact. For any open compact subgroup K of $G^\varphi(\mathbb{A}_f)$, the mass of K is defined as follows. Let $\{c_1, c_2, \dots, c_h\}$ be a set of (complete) representatives of the double coset space $G^\varphi(\mathbb{Q})\backslash G^\varphi(\mathbb{A}_f)/K$, and let $\Gamma_i := G^\varphi(\mathbb{Q}) \cap c_i K c_i^{-1}$. Note that each Γ_i is finite by the assumption of $G^\varphi(\mathbb{R})$. The mass of K is defined to be

$$\mathrm{mass}(K) := \sum_{i=1}^{h} \frac{1}{\#\Gamma_i}.$$

For the general definition of the mass of K (without the compactness assumption), we refer to [Sh, Introduction, p. 67]. It follows easily from the definition that for two open compact subgroups $K_1 \subset K_2$, one has $\mathrm{mass}(K_1) = [K_2 : K_1]\,\mathrm{mass}(K_2)$.

Choose a maximal order O_B of B. Let L be an O_B-lattice in V which is maximal along the lattices on which φ takes its value in O_B. Let K_0 be the maximal compact subgroup of $G^\varphi(\mathbb{A}_f)$ which stabilizes the adele lattice $L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$. In [Sh, Introduction, p. 68] Shimura gives an explicit formula

$$\operatorname{mass}(K_0) = |D_F|^{m^2} \cdot \prod_{k=1}^{m} \left\{ D_F^{1/2} \left[(2k-1)!(2\pi)^{-2k} \right]^d \zeta_F(2k) \right\} \prod_{i=1}^{s} \prod_{k=1}^{m} \left\{ (N(\mathfrak{p}_i)^k + (-1)^k \right\}, \quad (1)$$

where D_F is the discriminant of F over $\mathbb{Q}$, d is the degree of F, and $\mathfrak{p}_i$ are the primes of O_F at which B is ramified.

Note that the maximality of L is a local property. That is, L is maximal if and only if each $L_{\mathfrak{p}}$ is maximal for all primes $\mathfrak{p}$ of F.

(2.2) One can express the exact formula in terms of special values of zeta functions at negative integers. Let F be a totally real field of degree d. Set

$$\Lambda_F(s) := A^s \Gamma(s/2)^d \zeta_F(s), \quad A := D_F^{1/2} \pi^{-d/2}.$$

Then one has the functional equation for $\zeta_F(s)$:

$$\Lambda_F(1-s) = \Lambda_F(s).$$

It gives

$$\zeta_F(s) = D_F^{1/2} \cdot D_F^{-s} \pi^{-\frac{d}{2}+ds} \frac{\Gamma(\frac{1-s}{2})^d}{\Gamma(\frac{s}{2})^d} \cdot \zeta_F(1-s)$$

Let $s = 2k$, one gets

$$\zeta_F(2k) = D_F^{1/2} \cdot D_F^{-2k} \pi^{-\frac{d}{2}+2dk} \frac{\Gamma(\frac{1-2k}{2})^d}{\Gamma(k)^d} \cdot \zeta_F(1-2k).$$

On the other hand, one has

$$\frac{\Gamma(\frac{1-2k}{2})}{\Gamma(k)} = \frac{(-1)^k \cdot \frac{2^{2k-1}(k-1)!}{(2k-1)!} \cdot \Gamma(1/2)}{(k-1)!} = \frac{(-1)^k \cdot 2^{2k-1} \cdot \sqrt{\pi}}{(2k-1)!}.$$

The term in (1) can be expressed

$$\begin{aligned} & D_F^{1/2} \left[(2k-1)!(2\pi)^{-2k} \right]^d \zeta_F(2k) \\ =& D_F^{1-2k} \left[(2k-1)!(2\pi)^{-2k} \right]^d \cdot \left[\frac{(-1)^k \cdot 2^{2k-1} \cdot \sqrt{\pi}}{(2k-1)!} \right]^d \cdot \pi^{-\frac{d}{2}+2dk} \cdot \zeta_F(1-2k) \\ =& D_F^{1-2k} \left[\frac{(-1)^k}{2} \right]^d \cdot \zeta_F(1-2k). \end{aligned} \quad (2)$$

Hence we get

$$\operatorname{mass}(K_0) = \prod_{k=1}^{m} \left\{ \left[\frac{(-1)^k}{2} \right]^d \cdot \zeta_F(1-2k) \right\} \cdot \prod_{i=1}^{s} \prod_{k=1}^{m} \left\{ (N(\mathfrak{p}_i)^k + (-1)^k \right\}. \quad (3)$$

(2.3) If $F=\mathbb{Q}$ and B is the quaternion algebra which is ramified at $\{\infty, p\}$, then we have

$$\operatorname{mass}(K_0)=\frac{(-1)^{m(m+1)/2}}{2^m}\left\{\prod_{k=1}^{m}\zeta(1-2k)\right\}\cdot\prod_{k=1}^{m}\{(p^k+(-1)^k\} \tag{4}$$

If $m=1$ and B is ramified only at the primes over p and infinite places, then we have

$$\operatorname{mass}(K_0)=\left[\frac{-1}{2}\right]^d\cdot\zeta_F(-1)\cdot\prod_{i=1}^{s}\{(N(\mathfrak{p}_i)+(-1)\} \tag{5}$$

(2.4) In the following, the ground field k is algebraically closed of characteristic p. Let $x=(A_0,\lambda_0,\iota_0)$ be a supersingular polarized abelian O_F-variety over k of dimension $g=md$. Let G_x denote the group scheme over $\operatorname{Spec}\mathbb{Z}$ whose group of R-points, for each commutative ring R, is

$$G_x(R)=\{\phi\in(\operatorname{End}_{O_F}(A_0)\otimes R)^\times;\phi'\phi=1\},$$

where the map $\phi\mapsto\phi'$ is the Rosati involution induced by λ_0.

Let Λ_x denote the set of the isomorphism classes of polarized abelian O_F-varieties (A,λ,ι) of dimension g over k such that

(i) the Dieudonné module $M(A)$ is isomorphic to $M(A_0)$ as quasi-polarized Dieudonné $O_F\otimes\mathbb{Z}_p$-modules, and

(ii) the Tate module $T_\ell(A)$ is isomorphic to $T_\ell(A_0)$ as non-degenerate alternating $O_F\otimes\mathbb{Z}_\ell$-modules for all $\ell\neq p$.

The condition (i) implies that A is supersingular. Let Λ'_x be the subset of Λ_x that consists of objects (A,λ,ι) such that

(iii) there exists an element $\phi\in\operatorname{Hom}_{O_F}(A_0,A)\otimes\mathbb{Q}$ such that $\phi^*\lambda=\lambda_0$.

(2.5) Theorem (Y1, Thm. 10.5) *There is a natural bijection of pointed sets between* Λ'_x *and* $G_x(\mathbb{Q})\backslash G_x(\mathbb{A}_f)/G_x(\hat{\mathbb{Z}})$.

(2.6) Proposition *Let* $(A_i,\lambda_i,\iota_i), i=1,2$ *be two supersingular polarized abelian* O_F*-varieties over* k. *Then there exists* $\varphi\in\operatorname{Hom}_{O_F}(A_1,A_2)\otimes\mathbb{Q}$ *such that* $\varphi^*\lambda_2=\lambda_1$. *That is, the condition (iii) of (2.4) is automatic.*

PROOF. Choose an isogeny $\varphi:A_1\to A_2$. By Noether-Skolem's theorem, φ can be chosen to be O_F-linear. Let $\lambda'_1:=\varphi^*\lambda_2$ and $*_1$ and $*'_1$ be the Rosati involutions induced by λ_1 and λ'_1 on $B:=\operatorname{End}(A_1)\otimes\mathbb{Q}$, respectively. By [Z, Satz 1.1], there exists a positive element $c\in B^\times$ with $c=c^{*_1}$ such that $x^{*'_1}=c^{-1}x^{*_1}c$ for all $x\in B$. As $x\in F$, c lies in $C:=\operatorname{End}_{O_F}(A_1)\otimes\mathbb{Q}$.

Let P be the the algebraic variety over $\mathbb{Q}$ defined by $\{X\in C;X^{*_1}X=c\}$. It is a torsor under the algebraic group $G_1:=\{g\in C^\times;g^{*_1}g=1\}$ over $\mathbb{Q}$ by left action. Hence it forms an element ξ in $H^1(\mathbb{Q}, G_1)$. By [K, Lemma 2.11], $P(\mathbb{R}) \neq \emptyset$. Since G_1 is semi-simple and simply-connected, $H^1(\mathbb{Q}_p, G_1) = 0$. Therefore ξ is locally trivial everywhere. By Hasse principle for G_1, ξ is trivial. Then there exists $g \in C(\mathbb{Q})$ such that $g^{*_1}g = c$. Replacing φ by φg, we have $*_1 = *_1'$, hence that $\lambda_1' = q\lambda_1$ for some $q \in \mathbb{Q}^\times$. Note that q is positive. This follows from the fact that the polarizations lie in the positive cone of the Néron-Severi group $\mathrm{NS}(A_1) \otimes \mathbb{R}$. Therefore q is a norm of C and we can find a $\varphi \in \mathrm{Hom}_{O_F}(A_1, A_2) \otimes \mathbb{Q}$ such that $\varphi^*\lambda_2 = \lambda_1$. This completes the proof.

(2.7) Corollary $\Lambda_x' = \Lambda_x$.

(2.8) Lemma *Let* $(A, \lambda, \iota) \in \Lambda_x$ *and* $[c]$ *be the corresponding double coset. Then* $\mathrm{Aut}(A, \lambda, \iota) \simeq \Gamma_c$*, where* $\Gamma_c := G_x(\mathbb{Q}) \cap cG_x(\hat{\mathbb{Z}})c^{-1}$.

PROOF. Let G' be the group scheme over $\mathrm{Spec}\,\mathbb{Z}$ attached to (A, λ, ι) defined as in (2.4). That is, for any commutative ring R,

$$G'(R) = \{\alpha \in (\mathrm{End}_{O_F}(A) \otimes R); \alpha'\alpha = 1\},$$

where the map $\alpha \mapsto \alpha'$ is the Rosati involution induced by λ.

Choose a map $\phi \in \mathrm{Hom}_{O_F}(A_0, A) \otimes \mathbb{Q}$ such that $\phi^*\lambda = \lambda_0$. Then the element $c \in G_x(\mathbb{A}_f)$ has the property $(*)$: $\phi c_\ell \in \mathrm{Isom}((A_0(\ell), \lambda_0, \iota_0), (A(\ell), \lambda, \iota))$, $\forall\, \ell$. Note that $\alpha \in \mathrm{Aut}(A, \lambda, \iota)$ if and only if $\alpha \in G'(\mathbb{Q})$ and $\alpha_\ell \in \mathrm{Aut}(A(\ell), \lambda, \iota)$, $\forall\, \ell$.

The map ϕ gives an isomorphism $G_x(\mathbb{Q}) \to G'(\mathbb{Q})$ which sends β to $\phi\beta\phi^{-1} =: \alpha$. From $(*)$, we have $\alpha \in G'(\hat{\mathbb{Z}})$ if and only if $(\phi c)^{-1}\alpha(\phi c) \in G_x(\hat{\mathbb{Z}})$. Hence, $\alpha \in G'(\hat{\mathbb{Z}})$ if and only if $c^{-1}\beta c \in G_x(\hat{\mathbb{Z}})$. This completes the proof. ∎

(2.9) Corollary

$$\mathrm{mass}(\Lambda_x) := \sum_{(A,\lambda,\iota) \in \Lambda_x} \frac{1}{\#\,\mathrm{Aut}(A, \lambda, \iota)} = \mathrm{mass}(G_x(\hat{\mathbb{Z}})).$$

(2.10) The semi-simple algebra $C_x := \mathrm{End}_{O_F}(A_0) \otimes \mathbb{Q}$ is the centralizer of F into the simple algebra $M_g(\mathrm{End}^0(E))$, where E is a supersingular elliptic curve. We have $C_x \simeq M_m(B)$, where B is a quaternion algebra over the totally real field F ramified at all real places and unramified at all primes not dividing p. It follows from the local invariants of the commudant that $B \simeq F \otimes \mathrm{End}^0(E)$. Therefore,

$$\begin{aligned} &C_x \otimes \mathbb{R} \simeq M_m(\mathbb{H}) \\ &C_x \otimes \mathbb{Q}_\ell \simeq M_{2m}(F \otimes \mathbb{Q}_\ell), \quad \ell \neq p \\ &C_x \otimes \mathbb{Q}_p \simeq \prod_{\mathfrak{p}|p,\, g_\mathfrak{p}:\text{odd}} M_m(B_\mathfrak{p}) \times \prod_{\mathfrak{p}|p,\, g_\mathfrak{p}:\text{even}} M_{2m}(F_\mathfrak{p}), \end{aligned} \tag{6}$$

where $g_\mathfrak{p} := [F_\mathfrak{p} : \mathbb{Q}_p]$. The Rosati involution induces the standard involution on $M_m(\mathbb{H})$ and $M_m(B_\mathfrak{p})$, which we will denote by $*$. It also induces the symplectic involution on $M_{2m}(F\otimes\mathbb{Q}_\ell)$ and $M_{2m}(F_\mathfrak{p})$. Therefore, we have

$$\begin{aligned} G_x(\mathbb{R}) &\simeq \{g \in M_m(\mathbb{H}); g^*g = 1\}, \\ G_x(\mathbb{Q}_\ell) &\simeq \mathrm{Sp}_{2m}(F\otimes\mathbb{Q}_\ell), \quad \ell \neq p \\ G_x(\mathbb{Q}_p) &\simeq \prod_{\mathfrak{p}|p,\, g_\mathfrak{p}:\text{odd}} G_\mathfrak{p} \times \prod_{\mathfrak{p}|p,\, g_\mathfrak{p}:\text{even}} \mathrm{Sp}_{2m}(F_\mathfrak{p}), \end{aligned} \tag{7}$$

where $G_\mathfrak{p} = \{g \in M_m(B_\mathfrak{p}); g^*g = 1\}$. We will choose such isomorphisms and replace $\simeq$ by $=$.

(2.11) Lemma *There exist a left B-module V, a quaternion Hermitian from φ on V, and a maximal lattice L in the sense of Shimura (2.1) such that $G^\varphi \simeq G_x$ over $\mathbb{Q}$ and K_0 defined in (2.1) is*

$$K_0 = \prod_{\ell\neq p} \mathrm{Sp}_{2m}(O_F\otimes\mathbb{Q}_\ell) \times \prod_{\mathfrak{p}|p,\, g_\mathfrak{p}:odd} K_{0,\mathfrak{p}} \times \prod_{\mathfrak{p}|p,\, g_\mathfrak{p}:even} \mathrm{Sp}_{2m}(O_\mathfrak{p}), \tag{8}$$

where $O_\mathfrak{p}$ is the ring of integers in $F_\mathfrak{p}$ and $K_{0,\mathfrak{p}} = \{g \in M_m(O_{B_\mathfrak{p}}); g^*g = 1\}$.

PROOF. Let $V = B^{\oplus g}$ with $\varphi(\underline{x}, \underline{y}) = \sum x_i \bar{y}_i^\sigma$ and let $L = O_B^{\oplus g}$. ∎

(2.12) The computation of $\mathrm{mass}(G_x(\hat{\mathbb{Z}}))$ is to relate with the mass of the standard one K_0 and then to apply Shimura's mass formula.

For any two open compact subgroups K_1, K_2 of $G_x(\mathbb{A}_f)$, of $G_x(\mathbb{Q}_p)$, or of $G_x(\mathbb{Q}_\ell)$, we denote $\mu_\bullet(K_2/K_1) := [K_1 : K_1\cap K_2]^{-1}[K_2 : K_1 \cap K_2]$, for $\bullet = f,\ p$ or ℓ.

We have the following properties

- $\mu_\ell(K_2/K_1) = 1$ for almost all ℓ.
- $\mu_\bullet(K_2/K_1) = [K_2 : K_1]$ if $K_1 \subset K_2$.
- If $K_1 = \prod_\ell K_{1,\ell}$ and $K_2 = \prod_\ell K_{2,\ell}$, where ℓ runs over all primes of $\mathbb{Q}$, then $\mu_f(K_2/K_1) = \prod_\ell \mu_\ell(K_{2,\ell}/K_{1,\ell})$.
- $\mathrm{mass}(K_1) = \mu_f(K_2/K_1) \cdot \mathrm{mass}(K_2)$.

From Corollary 2.9, we have

$$\mathrm{mass}(\Lambda_x) = \mu_f(K_0/G_x(\hat{\mathbb{Z}})) \cdot \mathrm{mass}(K_0). \tag{9}$$

If the polarization λ_0 has prime-to-ℓ degree, then $G_x(\mathbb{Z}_\ell)$ is isomorphic to a product of $\mathrm{Sp}_{2m}(O_v)$ and $\mu_\ell(K_{0,\ell}/G_x(\mathbb{Z}_\ell)) = 1$. Therefore, we have

$$\mathrm{mass}(\Lambda_x) = \left\{ \prod_{\ell=p,\ \text{or } \ell|\deg\lambda_0} \mu_\ell(K_{0,\ell}/G_x(\mathbb{Z}_\ell)) \right\} \cdot \mathrm{mass}(K_0). \tag{10}$$

(2.13) Theorem *Let notations be as above. If* $m=1$ *or* $\deg\lambda_0$ *is a power of* p*, then we have the simplified mass formula*

$$\mathrm{mass}(\Lambda_x)=\mu_p(K_{0,p}/K_{M_0})\cdot\mathrm{mass}(K_0).$$

where K_{M_0} *is the group of automorphisms of the quasi-polarized Dieudonné* $O_F\otimes\mathbb{Z}_p$*-module* M_0 *attached to* x*.*

As a consequence, the following corollary is a generalization of the Deuring mass formula. Recently, Ekedahl and van der Geer [G] compute the intersections of cycle classes on the moduli space $\mathcal{A}_g$. They obtain this as a consequence by the Hirzebruch-Mumford proportionality principle.

(2.14) Corollary *Let* Λ *be the set of the isomorphism classes of principally polarized superspecial abelian varieties over* k *of dimension* g*. Then*

$$\sum_{(A,\lambda)\in\Lambda}\frac{1}{\#\,\mathrm{Aut}(A,\lambda)}=\frac{(-1)^{g(g+1)/2}}{2^g}\left\{\prod_{k=1}^{g}\zeta(1-2k)\right\}\cdot\prod_{k=1}^{g}\left\{(p^k+(-1)^k\right\}. \tag{11}$$

PROOF. Take $x=(A_0,\lambda_0)$ to be a principally polarized superspecial abelian variety over k of dimension g. By [E, Prop. 5.1], the Dieudonné module M_0 of x is isomorphic to the product of g copies of separably quasi-polarized rank two supersingular Dieudonné modules. It follows that $\Lambda=\Lambda_x$ and the group K_{M_0} of the automorphisms of M_0 is

$$K_{M_0}=\{\alpha\in M_g(O_{B_p});\alpha^*\alpha=1\},$$

where B_p is the quaternion division algebra over $\mathbb{Q}_p$. Therefore, we obtain $\mathrm{mass}(\Lambda)=\mathrm{mass}(K_0)$ and finish the proof by (4). ∎

§3. Superspecial points of HB type and the mass formula

We keep the notations in the previous section except that we denote the totally real field by $\mathbf{F}$ and reserve F for the Frobenius operator on a Dieudonné module. Let $\mathfrak{p}$ be a prime of $O_{\mathbf{F}}$ over p, and let $e_\mathfrak{p}$ and $f_\mathfrak{p}$ denote the ramification degree and residue degree of $\mathfrak{p}$ respectively. In the rest of this paper, we will only treat the Hilbert-Blumenthal cases, namely $m=1$ and $d=g$. Let $\mathcal{O}:=O_{\mathbf{F}}\otimes\mathbb{Z}_p=\oplus_{\mathfrak{p}|p}\mathcal{O}_\mathfrak{p}$.

(3.1) We first recall the classification of superspecial quasi-polarized Dieudonné $\mathcal{O}$-modules [Y2]. Let $x=(A_0,\lambda_0,\iota_0)$ be a superspecial polarized abelian $O_{\mathbf{F}}$-variety over k and let M_0 be its covariant Dieudonné module. We have $M_0=\oplus_{\mathfrak{p}|p}M_\mathfrak{p}$ and will describe $M_\mathfrak{p}$ for each $\mathfrak{p}$.

Let $\underline{e}$ be the Lie type of $M_\mathfrak{p}$ and $\underline{a}$ be the a-type of $M_\mathfrak{p}$. We recall the definition in [Y2, 1.8, 1.9] that

$$\underline{e}(M_\mathfrak{p})=(\{e_1^i,e_2^i\})_{i\in\mathbb{Z}/f_\mathfrak{p}\mathbb{Z}}\iff\mathrm{Lie}(M_\mathfrak{p})\simeq\bigoplus_{i\in\mathbb{Z}/f_\mathfrak{p}\mathbb{Z}}k[\pi]/(\pi^{e_1^i})\oplus k[\pi]/(\pi^{e_2^i})$$

$$\underline{a}(M_{\mathfrak{p}}) = (\{a_1^i, a_2^i\})_{i\in\mathbb{Z}/f_{\mathfrak{p}}\mathbb{Z}} \iff M_{\mathfrak{p}}/(F,V)M_{\mathfrak{p}} \simeq \bigoplus_{i\in\mathbb{Z}/f_{\mathfrak{p}}\mathbb{Z}} k[\pi]/(\pi^{a_1^i}) \oplus k[\pi]/(\pi^{a_2^i}),$$

where the right hand sides are isomorphisms of $O_{\mathfrak{p}} \otimes_{\mathbb{Z}_p} k$-modules.

As $M_{\mathfrak{p}}$ is superspecial, $\underline{e} = \underline{a}$ and it has the form

$$(\{e_1, e_2\}, \{e_{\mathfrak{p}} - e_1, e_{\mathfrak{p}} - e_2\}, \{e_1, e_2\}, \cdots)$$

for some integers e_1, e_2 with $0 \le e_1 \le e_2 \le e_{\mathfrak{p}}$, see [Y2, 2.22]. When $f_{\mathfrak{p}}$ is odd, it has an additional condition $e_1 + e_2 = e_{\mathfrak{p}}$. We say $M_{\mathfrak{p}}$ is of type (e_1, e_2) for convenience.

Let $\mathcal{D}_{\mathfrak{p}}^{-1} = (\pi^{-d})$ be the inverse of the different of $\mathcal{O}_{\mathfrak{p}}$ over $\mathbb{Z}_p$. There is a unique $W \otimes \mathcal{O}_{\mathfrak{p}}$-bilinear pairing $(\,,) : M_{\mathfrak{p}} \times M_{\mathfrak{p}} \to W \otimes \mathcal{O}_{\mathfrak{p}}$ such that $\langle x, y\rangle = \mathrm{Tr}_{W\otimes\mathcal{O}_{\mathfrak{p}}/W}(\pi^{-d}(x,y))$. We have $O_{\mathfrak{p}} \otimes W = \oplus_{i\in\mathbb{Z}/f_{\mathfrak{p}}\mathbb{Z}} W^i$ by the action of $\mathcal{O}_{\mathfrak{p}}^{\mathrm{ur}}$ and this also gives the decomposition $M_{\mathfrak{p}} = \oplus_{i\in\mathbb{Z}/f_{\mathfrak{p}}\mathbb{Z}} M_{\mathfrak{p}}^i$.

(3.2) Lemma *(1) If $f_{\mathfrak{p}}$ is even, then there is a W^i-basis X_i, Y_i for $M_{\mathfrak{p}}^i$ for each $i \in \mathbb{Z}/f_{\mathfrak{p}}\mathbb{Z}$ such that*

(i) $(X_i, Y_i) = \begin{cases} \pi^n & \text{if } i \text{ is odd,} \\ \pi^{n+e_{\mathfrak{p}}-e_1-e_2} & \text{if } i \text{ is even,} \end{cases}$
for all $i \in \mathbb{Z}/f_{\mathfrak{p}}\mathbb{Z}$ and some $n \in \mathbb{Z}$.

(ii) $FX_i = \begin{cases} -\pi^{e_1} Y_{i+1} & \text{if } i \text{ is odd,} \\ -\pi^{e_{\mathfrak{p}}-e_2} Y_{i+1} & \text{if } i \text{ is even,} \end{cases} \quad FY_i = \begin{cases} v\pi^{e_2} X_{i+1} & \text{if } i \text{ is odd,} \\ v\pi^{e_{\mathfrak{p}}-e_1} X_{i+1} & \text{if } i \text{ is even,} \end{cases}$
where $v\pi^{e_{\mathfrak{p}}} = p$.

(2) If $f_{\mathfrak{p}}$ is odd, then there is a W^i-basis X_i, Y_i for $M_{\mathfrak{p}}^i$ for each $i \in \mathbb{Z}/f_{\mathfrak{p}}\mathbb{Z}$ such that

(i) $(X_i, Y_i) = \pi^n$ for some $n \in \mathbb{Z}$.

(ii) $FX_i = -\pi^{e_1} Y_{i+1}, FY_i = v\pi^{e_2} Y_{i+1}$ for $i \in \mathbb{Z}/f_{\mathfrak{p}}\mathbb{Z}$, where $v\pi^{e_{\mathfrak{p}}} = p$.

(3) In particular, if the pairing $(\,,)$ is perfect and $M_{\mathfrak{p}}$ satisfies the Rapoport condition (in this case, $M_{\mathfrak{p}}$ is of type $(0, e_{\mathfrak{p}})$), then there exists a W^i-basis $\{X_i, Y_i\}$ of $M_{\mathfrak{p}}^i$ for each $i \in \mathbb{Z}/f_{\mathfrak{p}}\mathbb{Z}$ such that

- *$Y_i \in (VM)^i$ and $(X_i, Y_i) = 1$,*
- *$FX_i = -Y_{i+1},\ FY_i = pX_{i+1}$,*

for all $i \in \mathbb{Z}/f_{\mathfrak{p}}\mathbb{Z}$.

PROOF. See Lemma 4.5 and 4.6 of [Y2].

Let $\mathbf{F}_{\mathfrak{p}}'$ be the unique quadratic unramified extension of $\mathbf{F}_{\mathfrak{p}}$ and τ be the generator of $\mathrm{Gal}(\mathbf{F}_{\mathfrak{p}}'/\mathbf{F}_{\mathfrak{p}})$.

(3.3) Lemma *Let $K_{M_\mathfrak{p}}$ be the group of automorphisms of the quasi-polarized Dieudonné $\mathcal{O}_\mathfrak{p}$-module $M_\mathfrak{p}$.*

(1) If the residue degree $f_\mathfrak{p}$ is even, then

$$K_{M_\mathfrak{p}} \simeq \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(O_{\mathbf{F}_\mathfrak{p}}); c \equiv 0 \mod \pi^{e_2-e_1} \right\}.$$

(2) If the residue degree $f_\mathfrak{p}$ is odd, then

$$K_{M_\mathfrak{p}} \simeq \left\{ \begin{pmatrix} a & b \\ \pi^{e_2-e_1} b^\tau & a^\tau \end{pmatrix} \in \mathrm{SL}_2(O_{\mathbf{F}'_\mathfrak{p}}) \right\}.$$

Proof. Let $\phi \in K_{M_\mathfrak{p}}$. Choose a W^i-basis $\{X_i, Y_i\}$ for $M^i_\mathfrak{p}$ as in Lemma 3.2. Write

$$\begin{pmatrix} \phi(X_i) \\ \phi(Y_i) \end{pmatrix} = A_i \begin{pmatrix} X_i \\ Y_i \end{pmatrix}, \quad A_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \mathrm{GL}_2(W^i).$$

It follows from $(\phi(X_i), \phi(Y_i)) = (X_i, Y_i)$ that $A_i \in \mathrm{SL}_2(W^i)$. Note that we have $F^2(X_i) = -pX_{i+2}$ and $F^2(Y_i) = -pY_{i+2}$ for all i. The condition $\phi F^2 = F^2 \phi$ gives $A_{i+2} = A_i^{(2)}$ for all $i \in \mathbb{Z}/f_\mathfrak{p}\mathbb{Z}$, where we write $A^{(n)}$ for A^{σ^n}.

(1) If $f_\mathfrak{p}$ is even, then $\{A_i\}$ are determined by A_0 and A_1, and $A_0, A_1 \in \mathrm{SL}_2(O_{\mathbf{F}_\mathfrak{p}})$. We have

$$\begin{pmatrix} F(X_0) \\ F(Y_0) \end{pmatrix} = J_1 \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}, \quad J_1 = \begin{pmatrix} 0 & -\pi^{e_\mathfrak{p}-e_2} \\ v\pi^{e_\mathfrak{p}-e_1} & 0 \end{pmatrix}$$

$$\begin{pmatrix} F(X_1) \\ F(Y_1) \end{pmatrix} = J_0 \begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}, \quad J_0 = \begin{pmatrix} 0 & -\pi^{e_1} \\ v\pi^{e_2} & 0 \end{pmatrix}.$$

The condition $F\phi = \phi F$ gives $J_1 A_1 = A_0^{(1)} J_1$ and $J_0 A_0^{(1)} = A_1 J_0$. These give the relations

$$a_1 = d_0^{(1)}, \quad d_1 = a_0^{(1)}, \quad c_0^{(1)} = -v\pi^{e_2-e_1} b_1, \quad c_1 = -v\pi^{e_2-e_1} b_0^{(1)}.$$

We see that A_1 is determined by A_0 and $c_0 \equiv 0 \mod \pi^{e_2-e_1}$.

(2) If $f_\mathfrak{p}$ is odd, then $\{A_i\}$ are determined by A_0 and $A_0 \in \mathrm{SL}_2(O_{\mathbf{F}'_\mathfrak{p}})$. We have

$$\begin{pmatrix} F(X_0) \\ F(Y_0) \end{pmatrix} = J \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -\pi^{e_1} \\ v\pi^{e_2} & 0 \end{pmatrix}.$$

Applying $F\phi = \phi F$, we have $JA_1 = A_0^{(1)} J$. As $A_1 = A_0^{(f_\mathfrak{p}+1)}$, this equation gives

$$d_0 = a_0^\tau, \quad c_0 = -v\pi^{e_2-e_1} b_0^\tau.$$

As $-v$ is a norm of $O^\times_{\mathbf{F}'_\mathfrak{p}}/O^\times_{\mathbf{F}_\mathfrak{p}}$, we choose an isomorphism

$$K_{M_\mathfrak{p}} \simeq \left\{ \begin{pmatrix} a & b \\ \pi^{e_2-e_1} b^\tau & a^\tau \end{pmatrix} \in \mathrm{SL}_2(O_{\mathbf{F}'_\mathfrak{p}}) \right\}.$$

This completes the proof. ∎

(3.4) It remains to compute the local term $\mu_{\mathfrak{p}}(K_{0,\mathfrak{p}}/K_{M_{\mathfrak{p}}})$. We write q for $N(\mathfrak{p})$, the cardinality of the residue field $k(\mathfrak{p})$.

When $f_{\mathfrak{p}}$ is even, $K_{M_{\mathfrak{p}}}=\Gamma_0(\pi^{e_2-e_1})$ is an open compact subgroup of $\mathrm{SL}_2(\mathbf{F}_{\mathfrak{p}})$. If $e_1=e_2$, then $\mu_{\mathfrak{p}}(K_{0,\mathfrak{p}}/K_{M_{\mathfrak{p}}})=1$; if $e_2>e_1$, then

$$\mu_{\mathfrak{p}}(K_{0,\mathfrak{p}}/K_{M_{\mathfrak{p}}})=\#\mathbf{P}^1(O_{\mathbf{F}_{\mathfrak{p}}}/\pi^{e_2-e_1})=q^{e_2-e_1-1}(q+1).$$

Consider now the case where $f_{\mathfrak{p}}$ is odd. Put

$$K(n):=\left\{\begin{pmatrix} a & b \\ \pi^n b^\tau & a^\tau\end{pmatrix}\in \mathrm{SL}_2(O_{\mathbf{F}'_{\mathfrak{p}}})\right\},$$

If n is odd, then $K(n)$ is an open compact subgroup of the group $B^\times_{\mathfrak{p},1}$ of reduced norm one and $K(1)=K_{0,\mathfrak{p}}$. If n is even, then $K(n)$ is an open compact subgroup of $\mathrm{SL}_2(\mathbf{F}_{\mathfrak{p}})$.

Let $\epsilon=0$ or 1 and put

$$A_\epsilon:=\{\,a+b\xi; a,b\in O_{\mathbf{F}'_{\mathfrak{p}}}, \xi^2=\pi^\epsilon, \xi a=a^\tau\xi\,\}$$

equipped with the usual reduced norm and trace. For $d\geq 1$, let $A_{\epsilon,d}:=\{a+\pi^d b\xi\}\subset A_\epsilon$. Then we have $(A_\epsilon)^\times_1=K(\epsilon)$ and $(A_{\epsilon,d})^\times_1\simeq K(2d+\epsilon)$, where $(\ )_1$ denotes the set of elements of reduced norm one. It is clear that

$$(1+\pi^k A_\epsilon)^\times_1/(1+\pi^{k+1}A_\epsilon)^\times_1\simeq\{a\in A_\epsilon/\pi A_\epsilon; \mathrm{Trd}(a)=0\},\quad k\geq 1$$

hence that

$$\#(1+\pi^k A_\epsilon)^\times_1/(1+\pi^{k+1}A_\epsilon)^\times_1=q^3. \tag{12}$$

If $\epsilon=1$, then $(A_\epsilon/\pi)^\times_1\simeq(O_{\mathbf{F}'_{\mathfrak{p}}}/\pi)^\times_1\ltimes(O_{\mathbf{F}'_{\mathfrak{p}}}/\pi)$ and its has $q^2(q+1)$ elements. When $\epsilon=0$, $(A_\epsilon/\pi)^\times_1\simeq\{a,b\in O_{\mathbf{F}'_{\mathfrak{p}}}/\pi; N(a)-N(b)=1\}$. Write $\alpha=N(a)$ and $\beta=N(b)$. There are $q+1$ solutions for $(\alpha,\beta)=(1,0)$ or $(0,-1)$ and $(q+1)^2$ solutions for others. This group has $(q+1)2+(q-2)(q+1)^2=q(q^2-1)$ elements. Namely, we have

$$\#(A_\epsilon/\pi)^\times_1=\begin{cases} q^2(q+1), & \epsilon=1\\ q(q^2-1), & \epsilon=0\end{cases} \tag{13}$$

and hence that

$$\#(A_\epsilon/\pi^d)^\times_1=\begin{cases} q^{3d-1}(q+1), & \epsilon=1\\ q^{3d-2}(q^2-1), & \epsilon=0.\end{cases} \tag{14}$$

On the other hand, $(A_{\epsilon,d}/\pi^d A_\epsilon)^\times_1\simeq(O_{\mathbf{F}'_{\mathfrak{p}}}/\pi^d)^\times_1$ and it has $q^{d-1}(q+1)$ elements. We conclude

(3.5) Lemma *Notations being as above, then*

$$\mu_{\mathfrak{p}}(K(\epsilon)/K(2d+\epsilon))=\begin{cases} q^{2d}, & \epsilon=1\\ q^{2d-1}(q-1), & \epsilon=0,\end{cases}$$

where q is $N(\mathfrak{p})$, the cardinality of the residue field $k(\mathfrak{p})$.

Finally we need to compute $\mu_{\mathfrak{p}}(K_{0,\mathfrak{p}}/K(0))$ when $e_{\mathfrak{p}}$ is even.

(3.6) Lemma *Notations being as above, let* $O_{\mathbf{F}'_{\mathfrak{p}}} = O_{\mathbf{F}_{\mathfrak{p}}}[\sqrt{c}]$ *for some non-square residue* c *in* $O^{\times}_{\mathbf{F}_{\mathfrak{p}}}$. *Then for* $\epsilon = 0$,

$$A_\epsilon \simeq \left\{ \begin{pmatrix} \alpha & \beta \\ c\gamma & \delta \end{pmatrix} \in M_2(O_{\mathbf{F}_{\mathfrak{p}}});\ \alpha \equiv \delta \mod 2,\ \beta \equiv \gamma \mod 2 \right\}.$$

In particular, if the residue characteristic p *is not* 2, *then* $K(0) \simeq \mathrm{SL}_2(O_{\mathbf{F}_{\mathfrak{p}}})$ *and* $\mu_{\mathfrak{p}}(K_{0,\mathfrak{p}}/K(0)) = 1$.

PROOF. The $O_{\mathbf{F}_{\mathfrak{p}}}$-order A_ϵ is generated by x, ξ with relations $x^2 = c, \xi^2 = 1$, and $\xi x = -x\xi$. Choose an isomorphism $A \otimes \mathbf{F}_{\mathfrak{p}} \simeq M_2(\mathbf{F}_{\mathfrak{p}})$ with

$$x \mapsto \begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix}, \quad \xi \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then the image of A_ϵ in $M_2(\mathbf{F}_{\mathfrak{p}})$ is

$$\left\{ \begin{pmatrix} a_0 + b_0 & a_1 - b_1 \\ c(a_1 + b_1) & a_0 - b_0 \end{pmatrix};\ a_0, a_1, b_0, b_1 \in O_{\mathbf{F}_{\mathfrak{p}}} \right\},$$

and the assertion follows. ∎

(3.7) Lemma *If* $f_{\mathfrak{p}}$ *is odd,* $e_{\mathfrak{p}}$ *is even, and* $p = 2$, *then* $\mu_{\mathfrak{p}}(K_{0,\mathfrak{p}}/K(0)) = q^{2e_{\mathfrak{p}}-1}(q-1)$.

PROOF. It is clear that $(A_\epsilon/2)^{\times}_1 \simeq (O_{\mathbf{F}'_{\mathfrak{p}}}/2)^{\times}_1$, hence that this group has $q^{e_{\mathfrak{p}}-1}(q+1)$ elements. It follows from $\#\,\mathrm{SL}_2(O_{\mathbf{F}_{\mathfrak{p}}}/2) = q^{3(e_{\mathfrak{p}}-1)}(q+1)(q^2-q)$ that $\mu_{\mathfrak{p}}(K_{0,\mathfrak{p}}/K(0)) = q^{2e_{\mathfrak{p}}-1}(q-1)$. ∎

(3.8) Proposition *Notations are as above.*

(1) If $f_{\mathfrak{p}}$ *is even, then*

$$\mu_{\mathfrak{p}}(K_{0,\mathfrak{p}}/K_{M_{\mathfrak{p}}}) = \begin{cases} 1, & e_1 = e_2 \\ q^{e_2-e_1-1}(q+1), & e_1 < e_2. \end{cases}$$

(2) If $f_{\mathfrak{p}}$ *is odd, then*

$$\mu_{\mathfrak{p}}(K_{0,\mathfrak{p}}/K_{M_{\mathfrak{p}}}) = \begin{cases} q^{e_2-e_1-1}, & e_{\mathfrak{p}} \text{ is odd} \\ o_{\mathfrak{p}}, & e_{\mathfrak{p}} \text{ is even and } e_1 = e_2 \\ q^{e_2-e_1-1}(q-1)\cdot o_{\mathfrak{p}}, & e_{\mathfrak{p}} \text{ is even and } e_1 < e_2, \end{cases}$$

where

$$o_{\mathfrak{p}} = \begin{cases} 1, & p \neq 2 \\ q^{2e_{\mathfrak{p}}-1}(q-1), & p = 2. \end{cases}$$

(3.9) Theorem *Let $x = (A_0, \lambda_0, \iota_0)$ be a superspecial polarized abelian $O_{\mathbf{F}}$-variety and let Λ_x be the set defined in (2.4). The attached Dieudonné module M_0 decomposes as $\oplus_{\mathfrak{p}|p} M_{\mathfrak{p}}$ and suppose that each $M_{\mathfrak{p}}$ has type $(e_{1,\mathfrak{p}}, e_{2,\mathfrak{p}})$ (3.1). Then*

$$\sum_{(A,\lambda,\iota)\in\Lambda_x} \frac{1}{\#\operatorname{Aut}(A,\lambda,\iota)} = \left[\frac{-1}{2}\right]^g \cdot \zeta_{\mathbf{F}}(-1) \cdot \prod_{\mathfrak{p}|p} c_{\mathfrak{p}} \cdot o_{\mathfrak{p}}, \tag{15}$$

where

$$c_{\mathfrak{p}} = \begin{cases} 1, & f_{\mathfrak{p}} \textit{ is even, } e_{1,\mathfrak{p}} = e_{2,\mathfrak{p}}, \\ q_{\mathfrak{p}}^{e_{2,\mathfrak{p}}-e_{1,\mathfrak{p}}-1}(q_{\mathfrak{p}}+1), & f_{\mathfrak{p}} \textit{ is even, } e_{1,\mathfrak{p}} < e_{2,\mathfrak{p}}, \\ q_{\mathfrak{p}}^{e_{2,\mathfrak{p}}-e_{1,\mathfrak{p}}-1}(q_{\mathfrak{p}}-1), & f_{\mathfrak{p}} \textit{ is odd, } e_{\mathfrak{p}} \textit{ is odd,} \\ 1, & f_{\mathfrak{p}} \textit{ is odd, } e_{\mathfrak{p}} \textit{ is even, } e_{1,\mathfrak{p}} = e_{2,\mathfrak{p}}, \\ q_{\mathfrak{p}}^{e_{2,\mathfrak{p}}-e_{1,\mathfrak{p}}-1}(q_{\mathfrak{p}}-1) & f_{\mathfrak{p}} \textit{ is odd, } e_{\mathfrak{p}} \textit{ is even, } e_{1,\mathfrak{p}} < e_{2,\mathfrak{p}}. \end{cases} \tag{16}$$

and

$$o_{\mathfrak{p}} = \begin{cases} q_{\mathfrak{p}}^{2e_{\mathfrak{p}}-1}(q_{\mathfrak{p}}-1), & f_{\mathfrak{p}} \textit{ is odd, } e_{\mathfrak{p}} \textit{ is even, and } p = 2, \\ 1, & \textit{otherwise.} \end{cases} \tag{17}$$

(3.10) We say $(A_0, \lambda_0, \iota_0)$ is *minimal* if it reaches the minimal mass among all the types of superspecial points. In this case the mass formula above is simplified as

$$\sum_{(A,\lambda,\iota)\in\Lambda_x} \frac{1}{\#\operatorname{Aut}(A,\lambda,\iota)} = \left[\frac{-1}{2}\right]^g \cdot \zeta_{\mathbf{F}}(-1) \cdot \prod_{\mathfrak{p}|p,\ g_{\mathfrak{p}}:\text{odd}} (q_{\mathfrak{p}}-1) \cdot \prod_{\mathfrak{p}|p} o_{\mathfrak{p}}, \tag{18}$$

where $o_{\mathfrak{p}}$ is as above.

§4. Supersingular locus

(4.1) Let p be a fixed prime number. Let $\mathcal{M}^{(p)}$ denote the moduli stack over $\operatorname{Spec} \hat{\mathbb{Z}}_{(p)}$ of polarized abelian $O_{\mathbf{F}}$-varieties (A, λ, ι) of dimension $g = [\mathbf{F} : \mathbb{Q}]$ with the polarization λ of prime-to-p degree. It is a separated Deligne-Mumford algebraic stack over $\operatorname{Spec} \hat{\mathbb{Z}}_{(p)}$ locally of finite type. In [DP], Deligne and Pappas showed that the algebraic stack $\mathcal{M}^{(p)}$ is flat and a locally complete intersection over $\operatorname{Spec} \hat{\mathbb{Z}}_{(p)}$ of relative dimension g, and that the closed fiber $\mathcal{M}^{(p)} \otimes \mathbb{F}_p$ is geometrically normal and has singularities of codimension at least two. It follows from Deligne-Pappas's results and the compactification of Rapoport [R] that the irreducible components of geometric special fiber $\mathcal{M}^{(p)} \otimes \overline{\mathbb{F}}_p$ are in bijection correspondence with those of geometric generic fiber $\mathcal{M}^{(p)} \otimes \overline{\mathbb{Q}}$. Those are parameterized by the isomorphism classes of non-degenerate skew-symmetric $O_{\mathbf{F}}$-modules $H_1(A(\mathbb{C}), \mathbb{Z})$ for all $(A, \lambda, \iota) \in \mathcal{M}^{(p)}(\mathbb{C})$.

(4.2) Let n be an integer such that $n \geq 3$ and $(n, p) = 1$ and we choose a primitive n-th root of unity ζ_n in $\overline{\mathbb{Q}} \subset \mathbb{C}$. For any geometric point $\operatorname{Spec} k \to \operatorname{Spec} \hat{\mathbb{Z}}_{(p)}[\zeta_n]$, the choice of ζ_n determines a $(1 + n\hat{\mathbb{Z}}^{(p)})^\times$-orbit of isomorphisms $\alpha(k) : \hat{\mathbb{Z}}^{(p)} \xrightarrow{\sim} \prod_{\ell \neq p} \mu_{\ell^\infty}(k)$. Let

S be a connected $\mathbb{Z}_{(p)}[\zeta_n]$-scheme and (A,λ,ι) be polarized abelian $O_{\mathbf{F}}$-scheme over S. A (full) symplectic n-level structure on (A,λ,ι) w.r.t. ζ_n we understand is a $\pi_1(S,s)$-invariant K_n-orbit of $O_{\mathbf{F}} \otimes \hat{\mathbb{Z}}^{(p)}$-linear isomorphisms $\eta : V_{\mathbb{Z}} \otimes \hat{\mathbb{Z}}^{(p)} \xrightarrow{\sim} \prod_{\ell \neq p} T_\ell(A_s)$ for some non-degenerate skew-symmetric $O_{\mathbf{F}}$-module $(V_{\mathbb{Z}},\psi,i)$ such that the pull-back of the Weil pairing is $\alpha(k(s)) \circ \psi$, where K_n is the kernel of $G(\hat{\mathbb{Z}}^{(p)}) \to G(\mathbb{Z}/n\mathbb{Z})$, G is the automorphism group scheme $\mathrm{Aut}_{O_{\mathbf{F}}}(V_{\mathbb{Z}},\psi)$ over $\operatorname{Spec}\mathbb{Z}$ and s is a geometric point of S. Note that $(V_{\mathbb{Z}} \otimes \mathbb{Z}[\frac{1}{p}],\psi,i)$ is uniquely determined by (A,λ,ι) by the strong approximation. When $\deg\lambda$ is prime to n, it is the same to an $O_{\mathbf{F}}/nO_{\mathbf{F}}$-isomorphism $V_{\mathbb{Z}}/nV_{\mathbb{Z}} \xrightarrow{\sim} A[n](S)$ such that the pull-back of the Weil pairing is $\alpha \circ \psi$.

(4.3) Let $\mathcal{M}_n^{(p)}$ be the moduli stack over $\operatorname{Spec}\mathbb{Z}_{(p)}[\zeta_n]$ of objects in $\mathcal{M}^{(p)}$ together with a symplectic n-level structure w.r.t. ζ_n, and let $\mathcal{M}_n$ be an irreducible component of $\mathcal{M}_n^{(p)}$. If $\mathcal{M}_n$ is the one that classifies the principally polarized objects in $\mathcal{M}_n^{(p)}$, then it is the connected (and irreducible) component of the moduli space $\mathcal{M}_n^{O_{\mathbf{F}}}$ denoted in [DP, Sect. 2] by the choice of the element ζ_n in $\mathrm{Isom}(\mu_n,\mathbb{Z}/n)$.

Let $(V_{\mathbb{Z}},\psi,i)$ be the non-degenerate skew-symmetric $O_{\mathbf{F}}$-module corresponding to $\mathcal{M}_n$. Let G denote the automorphism group scheme $\mathrm{Aut}_{O_{\mathbf{F}}}(V_{\mathbb{Z}},\psi)$ over $\mathbb{Z}$. We have

$$\mathcal{M}_n(\mathbb{C}) \simeq \Gamma(n)\backslash G(\mathbb{R})/SO_2(\mathbb{R})^g,$$

where $\Gamma(n)$ is the kernel of the map $G(\mathbb{Z}) \to G(\mathbb{Z}/n)$.

(4.4) Proposition *Any $\mathcal{M}_n$ is isomorphic to $\mathcal{M}[L,L^+]_n$ for some (L,L^+), where (L,L^+) is a projective rank one $O_{\mathbf{F}}$-module together with a notion of positivity, and $\mathcal{M}[L,L^+]$ is the Deligne-Pappas space corresponding to the class (L,L^+). Conversely, any Deligne-Pappas space $\mathcal{M}[L,L^+]_n$ is isomorphic to some $\mathcal{M}_n$.*

PROOF. Recall that $\mathcal{M}[L,L^+]$ classifies the objects (A,i,ι), where (A,ι) is an abelian $O_{\mathbf{F}}$-variety, and $i : (L,L^+) \to (\mathcal{P}(A),\mathcal{P}(A)^+)$ such that the $L \otimes A \cong A^t$ (the DP condition). By the weak approximation, there is $\lambda_0 \in L^+$ such that $(\#L/O_{\mathbf{F}}\lambda_0, p) = 1$. The map $(A,i,\iota,\eta) \mapsto (A,i(\lambda_0),\iota,\eta)$ gives a morphism $\mathcal{M}[L,L^+]_n \to \mathcal{M}_n^{(p)}$, which factors through an irreducible component $\mathcal{M}_n$ by the irreducibility of $\mathcal{M}[L,L^+]_n$. It follows from $\mathrm{Aut}(A,i,\iota,\eta) = \mathrm{Aut}(A,i(\lambda_0),\iota,\eta)$ that it is isomorphic.

Conversely, let $\widetilde{A}$ be the universal family over $\mathcal{M}_n$. By [Y2, Thm. 2.12], the polarization sheaf $\mathcal{P}(\widetilde{A})$ is constant and $\widetilde{A}$ satisfies the DP condition. Take $(L,L^+) = (\mathcal{P}(A),\mathcal{P}(A)^+)$ and this finishes the proof. ∎

(4.5) In this section we treat the supersingular locus $\mathcal{S}$ of $\mathcal{M}_n \otimes \overline{\mathbb{F}}_p =: \mathcal{M}_{n,p}$ in the restricted case that *all residue degrees $f_\mathfrak{p}$ are one.* The method of computing the number of irreducible components and describing each component has been documented in [LO]. Based on loc. cit. and earlier works, the work [Y1] indicates that one needs to find out all possible types of isogenies such that models constructed have the right dimension. In this restricted case, on one hand, we know the right dimension as the Grothendieck conjecture has been proved [Y2, Theorem 7.3]. On the other hand, the isogeny type of the generic supersingular point is simple, just one step. Hence the recipe in [Y1] gives the direct connection to a class number. Combining the Shimura mass formula and the computation of local factors in the previous section, we express the number $\mathrm{irrd}(\mathcal{S})$ of irreducible components of $\mathcal{S}$ in terms of special values of the zeta function.

We make the assumption above through this section.

(4.6) Proposition *The points that satisfy the Rapoport locus are dense in each Newton stratum of* $\mathcal{M}_{n,p}$.

Proof. We may assume that there is one prime over p as the problem is local. The Lie stratum $\mathcal{N}_i$ of type $\{i, g-i\}$ has dimension $g-2i$ and each generic point has slope sequence $s(i)$ [Y2, 6.18], where $s(i) = \{\frac{i}{g}, \cdots, \frac{i}{g}, \frac{g-i}{g} \cdots, \frac{g-i}{g}\}$ (with multiplicity g). Let $\mathcal{N}_i^{(j)}$ be the Newton stratum with slope sequence $s(j)$ in $\mathcal{N}_i$. We know that each generic point of $\mathcal{N}_i^{(j)}$ has a-type $(i, \lceil j \rceil)$. By Lemma 6.19 of [Y2], the codimension of $\mathcal{N}_i^{(j)}$ in $\mathcal{N}_i$ is not less than $\lceil j \rceil - i$, hence that $\dim \mathcal{N}_i^{(j)} \le g - i - \lceil j \rceil$. ■

(4.7) Let $(A, \lambda, \iota, \eta) \in \mathcal{S}(k)$ that satisfies the Rapoport condition. Let M be the Dieudonné module of A and write $M = \oplus_{\mathfrak{p}|p} M_{\mathfrak{p}}$. By Proposition 4.4 of [Y2], we can choose a basis X, Y of $M_{\mathfrak{p}}$ such that

$$FX = \alpha X + Y, \quad FY = \pi^{e_{\mathfrak{p}}} X.$$

As A is supersingular, we have $\mathrm{ord}_\pi(\alpha) \ge e_{\mathfrak{p}}/2$ [Y2, 6.17]. We compute the Dieudonné modules in the canonical isogenies [Y1, Sect. 8]:
case 1: $e_{\mathfrak{p}}$ is even, write $e_{\mathfrak{p}} = 2c$.

$$\begin{aligned} M_0 := M_{\mathfrak{p}} &= W[\pi] < X, Y > \\ M_1 := (F, V) M_0 &= W[\pi] < Y, \pi^c X > \\ M_2 := (F, V) M_1 &= W[\pi] < \pi^c Y, \pi^{2c} X > \end{aligned} \tag{19}$$

One sees that $M_1/M_2 \simeq k[\pi]/\pi^c \oplus k[\pi]/\pi^c$, so M_1 is superspecial of type (c, c).
case 2: $e_{\mathfrak{p}}$ is odd, write $e_{\mathfrak{p}} = 2c+1$.

$$\begin{aligned} \frac{i}{g} M_0 := M_{\mathfrak{p}} &= W[\pi] < X, Y > \\ M_1 := (F, V) M_0 &= W[\pi] < Y, \pi^{c+1} X > \\ M_2 := (F, V) M_1 &= W[\pi] < \pi^{c+1} Y, \pi^{2c+1} X > \end{aligned} \tag{20}$$

One sees that $M_1/M_2 \simeq k[\pi]/\pi^c \oplus k[\pi]/\pi^{c+1}$, so M_1 is superspecial of type $(c, c+1)$.

(4.8) Let $x=(A_0,\lambda_0,\iota_0,\eta_0)$ be a *polarized* abelian $O_{\mathbf{F}}$-variety over k with a symplectic n-level structure for $(V_{\mathbb{Z}},\psi,i)$ such that each factor $M_{\mathfrak{p}}$ of the Dieudonné module M_0 of A_0 is isomorphic to M_1 above (i.e. (19) if $e_{\mathfrak{p}}$ is even and (20) if $e_{\mathfrak{p}}$ is odd). Let G_x be the automorphism group scheme of x (2.4) and Λ_x be the set of objects defined as in (2.4) together with a symplectic n-level structure.

It is clear that an element $g\in G_x(\hat{\mathbb{Z}})$ preserves η_0 if and only if $g\equiv 1 \mod n$. By Theorem 2.5, we have

$$\#\Lambda_x = G_x(\mathbb{Q})\backslash G_x(\mathbb{A}_f)/K_n = [G_x(\hat{\mathbb{Z}}):K_n]\cdot \mathrm{mass}(G_x(\hat{\mathbb{Z}})), \tag{21}$$

where K_n is the kernel of the map $G_x(\hat{\mathbb{Z}})\to G_x(\mathbb{Z}/n\mathbb{Z})$. We can choose an isomorphism $T_\ell(A_0)\simeq V_{\mathbb{Z}}\otimes\mathbb{Z}_\ell$ with the additional structure for each $\ell\neq p$, and obtain an isomorphism $G_x(\hat{\mathbb{Z}}^{(p)})\simeq G(\hat{\mathbb{Z}}^{(p)})$. It follows that

$$[G_x(\hat{\mathbb{Z}}):K_n] = [G_x(\hat{\mathbb{Z}}^{(p)}):K_n^p] = [G(\hat{\mathbb{Z}}^{(p)}):\widehat{\Gamma(n)}^p] = [G(\mathbb{Z}):\Gamma(n)], \tag{22}$$

where $\widehat{\Gamma(n)}$ is the closure of $\Gamma(n)$ in $G(\hat{\mathbb{Z}})$. As the corresponding Dieudonné module is minimal, we conclude that

$$\#\Lambda_x = [G(\mathbb{Z}):\Gamma(n)]\cdot\mathrm{mass}(G_x(\hat{\mathbb{Z}})) \tag{23}$$

where the formula $\mathrm{mass}(G_x(\hat{\mathbb{Z}}))$ is given in (18).

(4.9) Let $\xi\in\Lambda_x$, we consider the functor $\mathbf{X}_\xi$ which classifies the isomorphism classes of polarized $O_{\mathbf{F}}$-linear isogenies

$$\varphi:(A_1,\lambda_1,\iota_1,\eta_1)\to(A_0,\lambda_0,\iota_0,\eta_0)$$

over k-schemes S such that

(i) $(A_1,\lambda_1,\iota_1,\eta_1)\simeq\xi\times S$.

(ii) $\ker\varphi=\oplus_{\mathfrak{p}}(\ker\varphi)_{\mathfrak{p}}$ is an α-group and the α-sheaf $\mathrm{Lie}((\ker\varphi)_{\mathfrak{p}}^t)$ is a rank one locally free $O_S\otimes k[\pi]/\pi^{\lceil e_{\mathfrak{p}}/2\rceil}$-module.

Let $\alpha(\xi)$ denote the α-group of ξ. The isogeny φ gives a finite flat subgroup scheme $\ker\varphi$ of $\alpha(\xi)$ satisfying (ii). Conversely, given such finite flat subgroup scheme H, one has an $O_{\mathbf{F}}$-linear isogeny $\varphi:A_1\to A_1/H=:A_0$. The condition (ii) implies that H is isotropic for the Weil pairing of λ_1, hence the polarization descends to A_0. Since the isogeny has a p-power degree, the symplectic n-level structure identifies with that on A_0.

Let V_0 is the α-sheaf of the $\mathfrak{p}$-component of $\alpha(\xi)$. One has that

$$V_0\simeq\begin{cases} k[\pi]/\pi^c\oplus k[\pi]/\pi^{c+1}, & e_{\mathfrak{p}}=2c+1\\ k[\pi]/\pi^c\oplus k[\pi]/\pi^c, & e_{\mathfrak{p}}=2c.\end{cases}$$

Then $\mathbf{X}_\xi=\prod_{\mathfrak{p}|p}\mathbf{X}_{\mathfrak{p}}$, where $\mathbf{X}_{\mathfrak{p}}$ classifies the quotient bundles $\mathcal{F}$ of $V_0\otimes S$ such that $\mathcal{F}$ is a rank one locally free $O_S\otimes k[\pi]/\pi^{\lceil e_{\mathfrak{p}}/2\rceil}$-module. The freeness is an open condition, therefore $\mathbf{X}_{\mathfrak{p}}$ is representable by an quasi-projective scheme over k.

(4.10) Lemma $\mathbf{X}_{\mathfrak{p}}$ *is a smooth irreducible quasi-projective scheme of dimension* $[e_{\mathfrak{p}}/2]$.

PROOF. Taking duality, we may identify $\mathbf{X}_{\mathfrak{p}}$ with the space of subspaces W of V_0 such that $W \simeq k[\pi]/\pi^{\lceil e_{\mathfrak{p}}/2\rceil}$. The algebraic group $H := \mathrm{Aut}_{k[\pi]}(V_0)$ acts naturally on the space $\mathbf{X}_{\mathfrak{p}}$. If $e_{\mathfrak{p}} = 2c$, then $H = \mathrm{GL}_2(k[\pi]/\pi^c)$, viewed as an algebraic group over k. If $e_{\mathfrak{p}} = 2c+1$, then

$$H = \left\{ \begin{pmatrix} a & b \\ \pi c & d \end{pmatrix} ; a, b \in (k[\pi]/\pi^c), c, d \in k[\pi]/\pi^{c+1}, a, d : \text{invertible} \right\}$$

with the natural multiplication. Namely, we lift g_1, g_2 with entries in $k[\pi]/\pi^{e_{\mathfrak{p}}}$, multiply them and project to quotient rings. It is clear that the group H is connected and the action is transitive. Then the space $\mathbf{X}_{\mathfrak{p}}$ is a homogeneous space of H, hence it is a smooth quasi-projective variety.

As $\mathbf{X}_{\mathfrak{p}}$ is smooth, the dimension can be computed by tangent spaces. Let $W_0 \in \mathbf{X}_{\mathfrak{p}}(k)$, then the tangent space at W_0 is given by $\mathrm{Hom}_{k[\pi]}(W_0, V_0/W_0)$, which has dimension c. ∎

Let pr denote the projection that sends the objects in $\mathbf{X}_\xi$ to their targets. The morphism $\mathrm{pr} : \mathbf{X}_\xi \to \mathcal{M}_{\mathrm{n,p}}$ factors through the supersingular locus and we have

$$\mathrm{pr} : \coprod_{\xi \in \Lambda_{\mathrm{x}}} \mathbf{X}_\xi \to \mathcal{S}. \tag{24}$$

Let $\mathcal{S}^R$ be the intersection of $\mathcal{S}$ with the Rapoport locus. For any $x \in \mathcal{S}^R(k)$, there is a unique $\xi \in \Lambda_x$ and $y \in \mathbf{X}_\xi(k)$ such that $\mathrm{pr(y) = x}$. By Proposition 4.6, the morphism pr is dominant. It follows from the irreducibility of $\mathbf{X}_\xi$ that $\#\Lambda_x = \mathrm{irrd}(\mathcal{S})$. By (3.10) and (23), we have

(4.11) Theorem *Assume that* $f_{\mathfrak{p}} = 1$ *for all primes* $\mathfrak{p}$ *of* $O_{\mathbf{F}}$ *over* p. *Then the number of the irreducible components of the supersingular locus* $\mathcal{S}$ *of* $\mathcal{M}_{n,p}$ *is*

$$[G(\mathbb{Z}) : \Gamma(n)] \cdot \left[\frac{-1}{2}\right]^g \cdot \zeta_{\mathbf{F}}(-1) \cdot \prod_{\mathfrak{p}|p,\ g_{\mathfrak{p}}:odd} (q_{\mathfrak{p}} - 1) \cdot \prod_{\mathfrak{p}|p} o_{\mathfrak{p}}, \tag{25}$$

where $q_{\mathfrak{p}} = N(\mathfrak{p})$ *and*

$$o_{\mathfrak{p}} = \begin{cases} q_{\mathfrak{p}}^{2e_{\mathfrak{p}}-1}(q_{\mathfrak{p}} - 1), & e_{\mathfrak{p}} \textit{ is even and } p = 2, \\ 1, & \textit{otherwise.} \end{cases} \tag{26}$$

For $e_{\mathfrak{p}} = 1$ and lower $f_{\mathfrak{p}}$, the description of the supersingular locus is given in [Y1]. The following theorem is a reformulation of the results in loc. cit. by the geometric mass formula in Sect. 3.

(4.12) Theorem *Assume that* $e_{\mathfrak{p}} = 1$ *and* $f_{\mathfrak{p}} \le 4$ *for all primes* $\mathfrak{p}$ *of* $O_{\mathbf{F}}$ *over* p.

(1) The number irrd($\mathcal{S}$) *of the irreducible components of the supersingular locus* $\mathcal{S}$ *is*

$$\prod_{\mathfrak{p}|p} c(\mathfrak{p}) \cdot [G(\mathbb{Z}) : \Gamma(n)] \cdot \left[\frac{-1}{2}\right]^g \cdot \zeta_{\mathbf{F}}(-1) \cdot \prod_{\mathfrak{p}|p,\ g_{\mathfrak{p}}:odd} (q_{\mathfrak{p}} - 1) \cdot \prod_{\mathfrak{p}|p} o_{\mathfrak{p}}, \tag{27}$$

where $q_{\mathfrak{p}} = N(\mathfrak{p})$*,* $o_{\mathfrak{p}}$ *is given in (26), and*

$$c(\mathfrak{p}) = \begin{cases} 1, & f_{\mathfrak{p}} = 1 \\ 2, & f_{\mathfrak{p}} = 2 \\ 3, & f_{\mathfrak{p}} = 3 \\ 6, & f_{\mathfrak{p}} = 4. \end{cases}$$

(2) Each irreducible component of $\mathcal{S}$ *is isomorphic to* $\prod_{\mathfrak{p}|p} X_{\mathfrak{p}}$*, where*

$$X_{\mathfrak{p}} \simeq \begin{cases} \{point\}, & f_{\mathfrak{p}} = 1 \\ \mathbf{P}^1, & f_{\mathfrak{p}} = 2, 3 \\ \mathbf{P}^2, \ or\ \mathbf{P}(\mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(2)), & f_{\mathfrak{p}} = 4. \end{cases}$$

With Theorem 4.12 and 4.11, it is natural to expect the following

(4.13) Conjecture In the general case irrd($\mathcal{S}$) has the form (27), where $c(\mathfrak{p})$ only depends on $e_{\mathfrak{p}}$ and $f_{\mathfrak{p}}$.

(4.14) Remark (1) When $g = 2$ and $p \neq 2$, the theorems recover the main results of Bachmat and Goren in [BG]. The method in loc. cit. consists of several ingredients which all depends on $g = 2$: Zagier's explicit formula for quadratic zeta value $\zeta_{\mathbf{F}}(-1)$, the explicit description of $O_{\mathbf{F}}$ as $\mathbb{Z}[(d+\sqrt{d})/2]$, the fact that supersingular locus is codimension one, and the work of Katsura-Oort on moduli space of abelian surfaces. For details, see loc. cit. and references therein. Therefore, a different approach for the generalization of their theorem is required.

(2) It is not hard to see from the methods of the work [Y1] that irrd($\mathcal{S}$) is a sum of some class numbers. Indeed, we consider the type of canonical isogenies for generic points. The number of generic points with same type is the number of superspecial points appearing in the canonical isogenies, which is a class number [Y1, Thm. 10.5]. Therefore, irrd($\mathcal{S}$) is a sum of some class numbers. The point of the conjecture says that the superspecial points appearing in such canonical isogenies are minimal. If it is true, then the number $c(\mathfrak{p})$ should have an interesting group-theoretic meaning. A further computation (for $f_{\mathfrak{p}} \leq 8$) suggests that $c(\mathfrak{p}) = \binom{f_{\mathfrak{p}}}{[f_{\mathfrak{p}}/2]}$ when $e_{\mathfrak{p}} = 1$. This result and the analysis of the canonical isogenies will be published elsewhere.

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