THE ASYMPTOTIC RESULTS OF A SARS EPIDEMIC MODEL WITHOUT QUARANTINE

SZE-BI HSU AND LIH-ING W. ROEGER

Abstract. In this short article, we present the continuing work on a SARS model without quarantine by Hsu and Hsieh [SIAM J. Appl. Math., 66 (2006), 627–647]. We find the relation between the initial susceptible population S_0 and the asymptotic susceptible population S_∞.

1. Introduction

The following is a differential equation model for severe acute respiratory syndrome (SARS) without quarantine proposed by Hsu and Hsieh [1].

$$
\begin{aligned}
S' &= \frac{-\beta I S}{E+I+S}\,\frac{c}{1+a(P+R+D)}, \\
E' &= \frac{\beta I S}{E+I+S}\,\frac{c}{1+a(P+R+D)} - \mu E, \\
I' &= \mu E - (\sigma_1+\rho_1+\gamma_3) I, \\
P' &= \gamma_3 I - (\sigma_2+\rho_2) P, \\
R' &= \sigma_1 I + \sigma_2 P, \\
D' &= \rho_1 I + \rho_2 P.
\end{aligned} \tag{1}
$$

The initial conditions are $S(0) = S_0 > 0$, $I(0) = I_0 > 0$, $E(0) = P(0) = R(0) = D(0) = 0$, and $S(t) + E(t) + I(t) + P(t) + R(t) + D(t) \equiv N = I_0 + S_0$. The variables $S(t)$, $E(t)$, $I(t)$, and $P(t)$ are the number of susceptible individuals, infected asymptomatic individuals, infected individuals with onset of symptoms, and isolated probable SARS cases at time t. $R(t)$ is the cumulative number of discharged SARS patients and $D(t)$, the cumulative number of SARS deaths, at time t. System (1) is an SARS epidemic model without quarantine. To see the details of the modeling process and the related model with quarantine, please refer to [1].

Hsu and Hsieh [1] had done some analysis for this model. We will state their results. The disease-free equilibrium (DFE) for the system in (S, E, I, P, R, D) is

$(S^*, 0, 0, 0, R^*, D^*)$ with $S^*+R^*+D^* = N$; the endemic equilibrium is $(0, 0, 0, 0, R^\#, D^\#)$ with $R^\# + D^\# = N$. The basic reproduction number $\mathcal{R}_0$ for this model is

$$\mathcal{R}_0 = \frac{1}{\sigma_1 + \rho_1 + \gamma_3} \, \frac{\beta c}{1 + a(N - S^*)}.$$

If $\mathcal{R}_0 < 1$, the disease-free equilibrium is locally asymptotically stable and if $\mathcal{R}_0 > 1$, unstable. Hsu and Hsieh [1] had obtained the following results.

THEOREM 1. *For the SARS model with quarantine (1), the solutions have the following asymptotic properties:* $S(t) \to S_\infty \geq 0$, $R(t) \to R_\infty > 0$, $D(t) \to D_\infty > 0$, $I(t) \to 0$, $E(t) \to 0$, *and* $P(t) \to 0$ *as* $t \to \infty$.

THEOREM 2. *Consider system (1). Let* $\tilde{\beta} = \frac{\beta c}{1+aN}$ *and* $q = \sigma_1 + \rho_1 + \gamma_3$.

(i) *If* $q > \tilde{\beta}$, *then* $S(t) \to S_\infty > 0$ *as* $t \to \infty$.
(ii) *If* $q < \tilde{\beta}$, *then* $S(t) \to 0$ *as* $t \to \infty$.

The above theorems state that the asymptotic dynamics are actually global so that we can write the disease-free equilibrium or the endemic equilibrium to be $(S_\infty, 0, 0, 0, R_\infty, D_\infty)$ and $(0, 0, 0, 0, R_\infty, D_\infty)$ depending on $\mathcal{R}_0$. In the following section, we will give details of finding the relation between S_0 and S_∞.

We note that in the classical Kermack-Mckendric SIR model, the asymptotic state S_∞ satisfies a transcendental equation [2, 3], so does S_∞ obtained in this model (1).

2. Main Results

We can integrate the equations of P, R, and D in (1) from $t = 0$ to $t = \infty$. Since $I(\infty) = 0 = P(\infty)$, we have

$$\gamma_3 \int_0^\infty I(t)dt = (\sigma_2 + \rho_2) \int_0^\infty P(t)dt, \tag{2}$$

$$R_\infty = \sigma_1 \int_0^\infty I(t)dt + \sigma_2 \int_0^\infty P(t)dt, \tag{3}$$

and

$$D_\infty = \rho_1 \int_0^\infty I(t)dt + \rho_2 \int_0^\infty P(t)dt. \tag{4}$$

By substituting (2) into (3) and (4), we obtain

$$R_\infty = (\sigma_1 + \sigma_2 \frac{\gamma_3}{(\sigma_2 + \rho_2)}) \int_0^\infty I(t)dt,$$

and

$$D_\infty = (\rho_1 + \rho_2 \frac{\gamma_3}{(\sigma_2 + \rho_2)}) \int_0^\infty I(t)dt.$$

Let

(5) $$r = \frac{\sigma_1 + \sigma_2 \frac{\gamma_3}{(\sigma_2+\rho_2)}}{\rho_1 + \rho_2 \frac{\gamma_3}{(\sigma_2+\rho_2)}}.$$

Then we have

(6) $$\frac{R_\infty}{D_\infty} = r.$$

Let $V = S + E + I$. Then $P + R + D = N - V$, and system (1) can be simplified so that the two equations for S and V are

$$\begin{aligned} S' &= \frac{-\beta IS}{V} \frac{c}{1 + a(N - V)}, \\ V' &= -qI. \end{aligned}$$

Then we have

$$\frac{dS}{dV} = \frac{\beta c}{q} \frac{S}{V(1 + a(N - V))}.$$

Applying the method of separation of variables and integrating both sides of the equation leads to

$$\ln\left(\frac{S}{S_0}\right) = \frac{\beta c}{q(1 + aN)} \ln\left(\frac{V}{V_0(1 + a(N - V))}\right).$$

Since $V_0 = S_0 + E_0 + I_0 = N$, and $V_\infty = S_\infty$, S_∞ satisfies the following

(7) $$\left(\frac{S_\infty}{S_0}\right)^{\frac{q(1+aN)}{\beta c}} = \frac{S_\infty}{N(1 + a(N - S_\infty))}.$$

We can show that the two equations

(8) $$y = \left(\frac{x}{S_0}\right)^{\frac{q(1+aN)}{\beta c}} \text{ and } y = \frac{x}{N(1 + a(N - x))}$$

only intersect at $x = 0$ and $0 < x = S_\infty < S_0$ as seen in Figure 1. We obtain the results in Figure 1 by choosing the parameters $\beta = 1$, $c = 0.4$, $a = 0.0013$, $\mu = 0.14$, $\sigma_1 = \sigma_2 = 0.2$, $\rho_1 = \rho_2 = 0.1$, $\gamma_3 = 0.4$, and the initial conditions $S_0 = 10$, $I_0 = 1$, and $E_0 = P_0 = R_0 = D_0 = 0$. We found that $S_\infty = 8.8105$ which also satisfies (7).

In case (i) of Theorem 2, since $S_\infty + R_\infty + D_\infty = N$, and R_∞ and D_∞ satisfied (6), we can find all three values, S_∞, R_∞, and D_∞. In case (ii) of Theorem 2, all solutions approach the endemic equilibrium $(0, 0, 0, 0, R_\infty, D_\infty)$, that is to say that

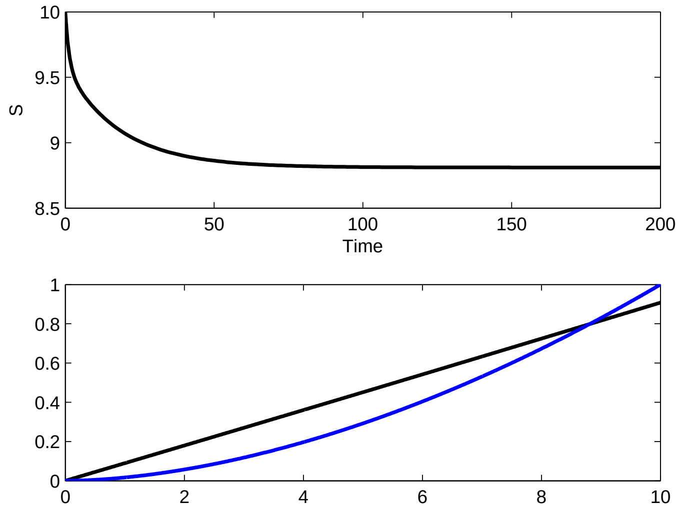

FIGURE 1. The top figure shows $S_\infty = 8.8105$. The number is also the x value of the intersection of the equations (8) in the bottom figure.

eventually everyone gets infected and recovered or died. Since $R_\infty + D_\infty = N$, it is also easy to find R_∞ and D_∞. Therefore, we have the following results.

THEOREM 3. *Let $\tilde{\beta}$ and q are defined as in Theorem 2 and r as in (5).*

(i) *If $q > \tilde{\beta}$, then $S_\infty > 0$ and satisfies (7), $R_\infty = \frac{r(N-S_\infty)}{1+r}$, and $D_\infty = \frac{N-S_\infty}{a+r}$.*
(ii) *If $q < \tilde{\beta}$, then $S_\infty = 0$, $R_\infty = \frac{rN}{1+r}$, and $D_\infty = \frac{N}{a+r}$.*

REFERENCES

[1] Sze-Bi Hsu and Ying-Hen Hsieh. *Modeling Intervention Measures and Severity-dependent Public Response During Severe Acute Respiratory Syndrome Outbreak*, SIAM J. Appl. Math., 66 (2006), 627–647. Add references:

[2] J. D. Murray. *Mathematical Biology.* Springer-Verlag, New York, 1989.

[3] P. E. Waltman. *Deterministic Threshold Models in the Theory of Epidemics.* Lecture Notes in Biomathematics, Vol 1, Springer-Verlag, New York, 1974.

Sze-Bi Hsu, Department of Mathematics, National Tsing Hua University, Hsinchu, Taiwan

E-mail address: sbhsu@math.nthu.edu.tw

Lih-Ing W. Roeger, Department of Mathematics and Statistics, Box 41042, Texas Tech University, Lubbock, TX 79409, Tel: 806-742-2580, Fax: 806-742-1112

E-mail address: lih-ing.roeger@ttu.edu