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MULTICOLORED PARALLELISMS OF ISOMORPHIC SPANNING TREES*

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Abstract. A subgraph in an edge-colored graph is multicolored if all its edges receive distinct colors. In this paper, we prove that a complete graph on $2m$ ($m \neq 2$) vertices K_{2m} can be properly edge-colored with $2m-1$ colors in such a way that the edges of K_{2m} can be partitioned into m multicolored isomorphic spanning trees.

Key words. complete graph, multicolored tree, parallelism

AMS subject classifications. 05B15, 05C05, 05C15, 05C70

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A *spanning subgraph* of a graph G is a subgraph H with $V(H) = V(G)$. A *proper k-edge coloring* of a graph G is a mapping from $E(G)$ into a set of colors $\{1, \ldots, k\}$ such that incident edges of G receive distinct colors. An *h-total-coloring* of a graph G is a mapping from $V(G) \cup E(G)$ into a set of colors $\{1, \ldots, h\}$ such that (i) adjacent vertices in G receive distinct colors, (ii) incident edges in G receive distinct colors, and (iii) any vertex and its incident edges receive distinct colors. The *edge chromatic number* of a graph G is the minimum number k for which G has a proper k-edge coloring. Throughout this paper K_m and $K_{m,n}$ denote the complete graph of order m and the complete bipartite graph with partite sets of sizes m and n, respectively. It is well known that the edge chromatic number of K_m is m if m is odd, and $m-1$ if m is even [7, p. 15]. Assume that m is a natural number. For any integer i we denote the residue of i modulo m in the set $\{1, \ldots, m\}$ by $[i]_m$. The following result is known.

LEMMA 1 (see [7, p. 16]). *If m is an odd positive integer, then K_m has an m-total coloring.*

A *Latin square* of order m is an $m \times m$ array of m symbols in which every symbol occurs exactly once in each row and column of the array. A *Room square* of side $2m-1$ is a $(2m-1) \times (2m-1)$ array whose cells are empty or contain an unordered pair of distinct integers chosen from $R = \{1, \ldots, 2m\}$, such that the entries of a given row contain every member of R precisely once, and similarly for columns, and the array contains every unordered pair of members of R precisely once. Room squares have been found for all odd $2m-1 \geq 7$ [2, p. 239]. An example of a Room square of side 7 is shown in Table 1.

A subgraph in an edge-colored graph is said to be *multicolored* if no two edges have the same color. Using a Room square of side $2m-1$ one may obtain a proper

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TABLE 1

			35	17	28	46
	26	48			15	37
	13	57	68	24		
47		16		38		25
58		23	14		67	
12	78			56	34	
36	45		27			18

edge coloring of K_{2m} with $2m-1$ colors in which all edges can be partitioned into $2m-1$ multicolored perfect matchings. For example, using the rows of Table 1 we give a proper edge coloring of K_8 with 7 colors. We denote the vertices of K_8 by $1,\ldots,8$. In Table 1, if rs appears in the ith row, then we color the edge rs with color i. For instance, the edges $47, 16, 38, 25$ are colored with color 4. Each column in Table 1 corresponds to a multicolored perfect matching of K_8. In a recent paper [1] the existence of the multicolored matchings in an arbitrary edge-colored complete graph has been studied. A Latin square of order m corresponds to a proper edge coloring of $K_{m,m}$ with m colors. Indeed if $L=(L_{ij})$ is a Latin square of order m and $\{u_1,\ldots,u_m\}$ and $\{v_1,\ldots,v_m\}$ are two parts of $K_{m,m}$, then we color the edge u_iv_j with L_{ij}. Since L has m symbols, we have an m-edge coloring of $K_{m,m}$, and since every symbol occurs exactly once in each row and each column of L, the edge coloring is proper. Also the existence of two orthogonal Latin squares of order m corresponds to a proper edge coloring of $K_{m,m}$ with m colors for which all edges can be partitioned into m multicolored perfect matchings. For example, suppose that $L=(L_{ij})$ and $R=(R_{ij})$ are two orthogonal Latin squares of order m with symbols of the set $\{1,\ldots,m\}$, and $\{u_1,\ldots,u_m\}$ and $\{v_1,\ldots,v_m\}$ are two parts of $K_{m,m}$. As we saw before, the function c, where $c(u_iv_j)=L_{ij}$, is a proper m-edge coloring of $K_{m,m}$. For any r, $1\le r\le m$, let M_r be the set of all edges u_iv_j such that $R_{ij}=r$. Obviously $\{M_1,\ldots,M_n\}$ is an edge partition of $E(K_{m,m})$. Since the symbol r occurs exactly once in each row and each column of R, M_r is a perfect matching, and since L and R are orthogonal, if $R_{ij}=r$, then the symbols L_{ij} are distinct and we conclude that M_r is multicolored. There is a classic result which says that for any natural number m, $m\ne 2,6$, there exist two orthogonal Latin squares of order m; see [3].

We say that the complete graph K_{2m} admits a *multicolored tree parallelism* (MTP) if there exists a proper edge coloring of K_{2m} with $2m-1$ colors for which all edges can be partitioned into m isomorphic multicolored spanning trees. It is clear that the complete graph K_4 does not admit an MTP. We note here that such a partition of the edges of K_{2m} can be viewed as a parallelism as defined in [5] by Cameron, with an additional property due to edge colors. In fact, finding a partition as obtained above corresponds to an arrangement of the edges of K_{2m} into an array of $2m-1$ rows and m columns such that each row contains the edges with the same color which form a perfect matching and the edges in each column form a multicolored spanning tree of K_{2m}; moreover, all the m spanning trees are isomorphic. Therefore, the partition creates a double parallelism of K_{2m}, one from the rows of the perfect matchings and the other from the columns of the edge disjoint isomorphic spanning trees. The following result has been proven in [6].

THEOREM A (see [6]). *If $m\ne 1,3$ and K_{2m} admits an MTP, then for any $r\ge 1$, K_{2^rm} admits an MTP.*

There exist three interesting conjectures on the edge partitioning of the complete graphs into multicolored spanning trees.

TABLE 2

	T_1	T_2	T_3
c_1	35	46	12
c_2	24	15	36
c_3	25	34	16
c_4	26	13	45
c_5	14	23	56

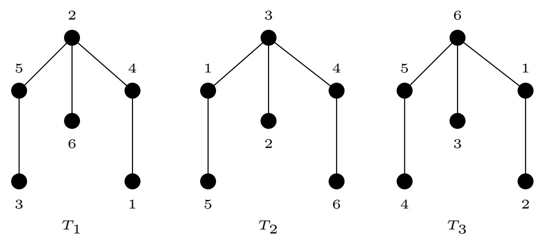

FIG. 1.

CONSTANTINE'S CONJECTURE (weak version; see [6]). *For any natural number m, $m > 2$, K_{2m} admits an MTP.*

BRUALDI–HOLLINGSWORTH CONJECTURE (see [4]). *If $m > 2$, then in any proper edge coloring of K_{2m} with $2m-1$ colors, all edges can be partitioned into m multicolored spanning trees.*

In [4] it was proved that in any proper edge coloring of K_{2m} ($m > 2$) with $2m-1$ colors there are at least two edge disjoint multicolored spanning trees.

CONSTANTINE'S CONJECTURE (strong version; see [6]). *If $m > 2$, then in any proper edge coloring of K_{2m} with $2m-1$ colors, all edges can be partitioned into m isomorphic multicolored spanning trees.*

The main goal of this paper is to prove the first conjecture.

Example 1. The complete graph K_6 admits an MTP. To see this consider the complete graph K_6 with the vertex set $\{1,\dots,6\}$. Table 2 gives a proper edge coloring of K_6 with colors $c_1,\dots,c_5$ as well as an MTP for it. The ith row of this table is the set of all edges with color c_i. Each column denotes the edges of a multicolored spanning tree. Figure 1 shows that the spanning trees T_1, T_2, T_3 are isomorphic.

In [6] it has been shown that K_8 admits an MTP.

Using the software Gap, Peter Cameron found a decomposition of $K_{6,6}$ into six isomorphic multicolored graphs $K_{1,3}\cup 3K_2\cup 2K_1$. In the next lemma, using Cameron's decomposition we find an MTP for K_{12}.

LEMMA 2. *The complete graph K_{12} admits an MTP.*

Proof. Consider the complete graph K_{12} with the vertex set $\{u_1,\dots,u_6,v_1,\dots,v_6\}$. Table 3 gives a proper edge coloring of K_{12} with colors $c_1,\dots,c_{11}$ as well as an MTP for it. The ith row of this table is the set of all edges with color c_i. Each column denotes the edges of a multicolored spanning tree. Note that the first six rows of the table determine a decomposition of $K_{6,6}$ into six multicolored subgraphs isomorphic to $K_{1,3}\cup 3K_2\cup 2K_1$. □

Now, we are ready to prove our main result.

THEOREM. *For $m \neq 2$, K_{2m} admits an MTP.*

Proof. First suppose that m is an odd integer. Consider the complete graph K_{2m} defined on the set $A\cup B$ where $A = \{a_1,\dots,a_m\}$ and $B = \{b_1,\dots,b_m\}$. For

TABLE 3

	T_1	T_2	T_3	T_4	T_5	T_6
c_1	u_2v_5	u_1v_6	u_6v_1	u_3v_2	u_4v_3	u_5v_4
c_2	u_2v_3	u_5v_2	u_6v_6	u_4v_5	u_3v_4	u_1v_1
c_3	u_4v_1	u_3v_3	u_6v_4	u_1v_2	u_5v_5	u_2v_6
c_4	u_1v_4	u_3v_5	u_5v_3	u_6v_2	u_2v_1	u_4v_6
c_5	u_2v_2	u_4v_4	u_1v_5	u_5v_1	u_6v_3	u_3v_6
c_6	u_5v_6	u_3v_1	u_4v_2	u_2v_4	u_1v_3	u_6v_5
c_7	u_3u_5	u_4u_6	u_1u_2	v_3v_5	v_4v_6	v_1v_2
c_8	u_2u_4	u_1u_5	u_3u_6	v_2v_4	v_1v_5	v_3v_6
c_9	u_2u_5	u_3u_4	u_1u_6	v_2v_5	v_3v_4	v_1v_6
c_{10}	u_2u_6	u_1u_3	u_4u_5	v_2v_6	v_1v_3	v_4v_5
c_{11}	u_1u_4	u_2u_3	u_5u_6	v_1v_4	v_2v_3	v_5v_6

convenience, let G and H be the complete graphs on the sets A and B, respectively. Since m is odd, G has a total coloring π which uses m colors, $1, \dots, m$. Now, define an edge-coloring c of K_{2m} as follows:

(a) For each edge $a_ja_k \in E(G)$, let $c(a_ja_k) = \pi(a_ja_k)$.
(b) For each edge $b_jb_k \in E(H)$, let $c(b_jb_k) = \pi(a_ja_k)$.
(c) For each edge a_ib_i, $1 \le i \le m$, let $c(a_ib_i) = \pi(a_i)$.
(d) For each edge a_jb_k, $j \ne k$, let $c(a_jb_k) = [k-j]_m + m$.

Clearly, c is a proper $(2m-1)$-edge-coloring of K_{2m}. It is left to decompose K_{2m} into m multicolored isomorphic spanning trees. First, for each $i \in \{1, \dots, m\}$, let T_i be defined on the set $A \cup B$ and $E(T_i) = \{a_ia_{[i+2t]_m}, b_ib_{[i+2t-1]_m}, b_ia_{[i+2t-1]_m}, a_{[i+1]_m}b_{[i+2t]_m} \,|\, t = 1, 2, \dots, \frac{m-1}{2}\} \cup \{a_ib_i\}$. It is easy to check that each T_i is a multicolored spanning tree, and all the T_i's are isomorphic.

Now, if m is not an odd integer, then $2m = 2^tm'$ where $t \ge 2$ and m' is odd. In the case where $m' = 1$, t must be at least 3. Then it is a direct consequence of Theorem A. Assume $m' \ge 3$. Thus $K_{2^tm'}$ admits an MTP by Theorem A except when $m' = 3$ and $t = 2$. Since this case can be handled by Lemma 2, we conclude the proof. □

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