On the solutions of an ergodic type Bellman equation

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Abstract:

We consider an ergodic type of Bellman equation in R^d,

$$\frac{1}{2}\sum_{ij} a_{ij}(x)D_{ij}W(x) + b(x)\cdot\nabla W(x) + \frac{1}{2}\hat{a}_{ij}(x)D_iW(x)D_jW(x) + V(x) = \Lambda.$$

We assume $a_{ij}(x), \hat{a}_{ij}(x), b_i(x), V(x)$ are smooth and $(a_{ij}(x)), (\hat{a}_{ij}(x))$ are positive definite for all x. A solution is given by a pair $(\Lambda, W(x))$, $W(x)$ is smooth. In particular, we are interested in the set of Λ such that there is W satisfying the above equation. Under very general condition, there is Λ^* such that there exists a W satisfying above equation if and only if $\Lambda \geq \Lambda^*$.

This type of Bellman equations arise from some ergodic control problems or risk senstive control problems in infinite time horizon. Λ^* has natural interpretation as the optimal rate. In the particular case that $\hat{a}_{ij}(x) = a_{ij}(x)$, this equation can be rewritten as

$$L\phi + V\phi = \Lambda\phi,$$

with $\phi = \exp(W)$,

$$Lf(x) = \frac{1}{2}\sum_{ij} a_{ij}(x)D_{ij}f(x) + b(x)\cdot\nabla f(x).$$

Then Λ^* relates to the principal eigenvalue of the operator $L + V$. It also relates to the infinite time growth rate of the expectation

$$E[\exp(\int_0^T V(X(t))dt)],$$

$X(t)$ is the diffusion process generated by L.

The equation with $a_{ij} = 0$ will be also discussed. This can be considered as the limit of the small noise problem (replace $a_{ij}(x)$ by $\epsilon a_{ij}(x)$ and ϵ is small). The solution is not smooth in general, the concept of viscosity solution will be needed. Some parallel results can be obtained.