

## ABSTRACT

Let  $X$  be a based space, that is, a topological space with a base point  $*$ . Consider the smash product  $X \wedge X = \frac{X \times X}{X \times * \cup * \times X}$  with  $X \times * \cup * \times X = *$  as the base point. Elements of  $X \wedge X$  are denoted by  $x \wedge y$ . Let  $S^n$  be the standard unit sphere in  $\mathbb{R}^{n+1}$  ( $n \geq 0$ ). The ( $n^{\text{th}}$ ) quadratic construction  $D_n(X) = S^n \overline{\times}_{\mathbb{Z}_2} (X \wedge X)$  is the quotient space of  $S^n \times (X \wedge X)$  by the identifications  $(\lambda, x \wedge y) \sim (-\lambda, y \wedge x)$  and  $(\lambda, *) \sim (\lambda', *)$  for all  $\lambda, \lambda' \in S^n, x \wedge y \in X \wedge X$ . Elements of  $D_n(X)$  are denoted by  $[\lambda, x \wedge y]$  with  $[\lambda, *] = *$  as the base point. If  $Y$  is another based space then any base-point preserving map  $X \xrightarrow{f} Y$  induces a map  $D_n(X) \xrightarrow{D_n(f)} D_n(Y)$ , called the ( $n^{\text{th}}$ ) quadratic map, which is given by  $D_n(f)([\lambda, x \wedge y]) = [\lambda, f(x) \wedge f(y)]$ . We are going to study some basic properties of  $D_n(X)$  including the following:

- (1)  $D_0(X) = S^0 \overline{\times}_{\mathbb{Z}_2} (X \wedge X) \cong X \wedge X$ .
- (2)  $D_n(\Sigma^{n+1} X) \cong \Sigma^{2n+2} D_n(X)$  for  $n = 1, 3$  and  $7$  ( $\Sigma^m$  is the  $m$ -fold suspension).
- (3)  $D_n(S^m) \cong \Sigma^m P_m^{n+m}$  where  $P_l^k$  is the stunted real projective space  $P^k/P^{l-1}$  with  $P^t$  the  $t$ -dimensional real projective space.

Up to this point, only some knowledge of point set topology is required.

Next we will study the cup-1-construction  $\pi_{2k}(S^{2n}) \xrightarrow{Sq_1} \pi_{4k+1}(S^{4n})$  (for  $2k \geq 2n \geq 2$ ) via the quadratic construction  $D_1(S^{2l}) = \Sigma^{2l} P_{2l}^{2l+1}$  which has the homotopy type of  $S^{4l} \vee S^{4l+1}$ ; in fact, there is a canonical homotopy equivalence  $S^{4l} \vee S^{4l+1} \xrightarrow{\varphi_l} D_1(S^{2l}) = \Sigma^{2l} P_{2l}^{2l+1}$ . Given a homotopy class  $\alpha = [f] \in \pi_{2k}(S^{2n})$  represented by a (base-point preserving) map  $S^{2k} \xrightarrow{f} S^{2n}$ . Consider the composite

$$\widehat{f}: S^{4k} \vee S^{4k+1} \xrightarrow{\varphi_k} D_1(S^{2k}) \xrightarrow{D_1(f)} D_1(S^{2n}) \xrightarrow{\varphi_n^{-1}} S^{4n} \vee S^{4n+1}$$

where  $\varphi_n^{-1}$  is the homotopy inverse of  $\varphi_n$ . The restriction of  $\widehat{f}$  to  $S^{4k+1}$  followed by the projection  $S^{4n} \vee S^{4n+1} \xrightarrow{p} S^{4n}$  is a map  $S^{4k+1} \rightarrow S^{4n}$  which is denoted by  $Sq_1(f)$ . It can be shown (and this is easy) that the homotopy class  $[Sq_1(f)] \in \pi_{4k+1}(S^{4n})$  depends only on the homotopy class  $\alpha = [f] \in \pi_{2k}(S^{2n})$ . We write  $Sq_1(\alpha) = [Sq_1(f)]$ . This defines an operation

$$\pi_{2k}(S^{2n}) \xrightarrow{Sq_1} \pi_{4k+1}(S^{4n}), \text{ call the } \mathbf{cup-1-construction}.$$

We will show some of the following properties of the cup-1-construction  $Sq_1$ :

- (4)  $Sq_1(\Sigma^2 \alpha) = \Sigma^4 Sq_1(\alpha)$  for any  $\alpha \in \pi_{2k}(S^{2n})$ .
- (5)  $Sq_1(\iota) = 0$  where  $\iota = [1_{S^{2n}}] \in \pi_{2n}(S^{2n})$ .
- (6)  $Sq_1(2\iota) = \eta \in \pi_{4n+1}(S^{4n})$  where  $\eta$  is the first Hopf class which is the homotopy class represented by suspension  $S^{4n+1} \rightarrow S^{4n}$  of the attaching

- map  $S^3 \xrightarrow{\eta} S^2$  for the complex projective space  $\mathbb{C}P^2 = S^2 \cup_{\eta} e^4$ . It is known that  $2\eta = 0 \in \pi_{4n+1}(S^{4n})$ .
- (7) For  $\alpha, \beta \in \pi_{2k}(S^{2n})$ , there is the following relation:  
 $Sq_1(\alpha + \beta) = Sq_1(\alpha) + Sq_1(\beta) + (k + n + 1)\alpha\beta\eta$  in  $\pi_{4k+1}(S^{4n})$   
 where  $\alpha\beta\eta$  is the composite  $S^{4k+1} \xrightarrow{\eta} S^{4k} \xrightarrow{\Sigma^{2k}\alpha} S^{2k+2n} \xrightarrow{\Sigma^{2n}\beta} S^{4n}$ .
- (8) For  $\alpha = [f] \in \pi_{2k}(S^{2n})$ , consider the smash product  
 $X = (S^{2n} \cup_f e^{2k+1}) \wedge (S^{2n} \cup_f e^{2k+1})$  of the mapping cone  $S^{2n} \cup_f e^{2k+1}$ .  
 Let  $S^{2n+2k} \xrightarrow{f_1=\Sigma^{2n}f} S^{4n}$  and  $S^{4k+1} \xrightarrow{f_2=\Sigma^{2k+1}f} S^{2n+2k+1}$ . Then  $X$  has the homotopy type of the mapping cone of the map  
 $S^{4k+1} \xrightarrow{f^*} (S^{4n} \cup_{f_1} e^{2n+2k+1}) \vee S^{2n+2k+1}$  which, to the first factor, is the map  $S^{4k+1} \xrightarrow{Sq_1(f)} S^{4n} \hookrightarrow S^{4n} \cup_{f_1} e^{2k+n+1}$  and, to the second factor, is the map  $S^{4k+1} \xrightarrow{f_2} S^{2n+2k+1}$ .

Some of these properties will be needed when we go to discuss the Kervaire invariant problem (to be abbreviated as KIP) in terms of the quadratic construction.

An application of the quadratic construction is a simple proof of the following theorem of Barratt and Mahowald. Given a homotopy class  $\alpha \in \pi_l(S^m)$  with  $2\alpha = 0$  where  $l - m$  is even. One can then form the stable Toda bracket  $\langle \alpha, 2\iota, \alpha \rangle \subset \pi_{2l-2m+1}^S$  where  $\pi_k^S$  denotes the  $k$ -stem stable homotopy group of spheres.

**Theorem A.**  $0 \in \langle \alpha, 2\iota, \alpha \rangle$ .

This theorem actually is related to the KIP. This, however, will not be discussed in these lectures. Also, in case time is not allowed, we will skip this topic, that is, skip the proof of Theorem A.

Finally we sketch what we are going to say about the KIP in terms of the quadratic construction.

The product space  $S^n \times S^n$  ( $n \geq 1$ ) is well known to have a cell structure of the form  $(S^n \vee S^n) \cup_{\bar{w}} e^{2n}$  for some canonical map  $S^{2n-1} \xrightarrow{\bar{w}} S^n \vee S^n$ . Let  $S^n \vee S^n \xrightarrow{F} S^n$  be the folding map. The composite

$$\bar{w} : S^{2n-1} \xrightarrow{\bar{w}} S^n \vee S^n \xrightarrow{F} S^n$$

is called a **Whitehead square map** and the homotopy class  $w(n) = [\bar{w}]$  in  $\pi_{2n-1}(S^n)$  is called the  $n^{\text{th}}$  **Whitehead square**. We will be interested only in  $w_i = w(2^i - 1) \in \pi_{2^{i+1}-3}(S^{2^i-1})$ . A theorem of J. F. Adams says that  $w_i = 0$  for  $1 \leq i \leq 3$  and  $w_i \neq 0$  for  $i \geq 4$ . One version of the KIP is:

Is there a class  $\theta_i \in \pi_{2^{i+1}-1}(S^{2^i-1})$  with  $2\theta_i = w_i$  for each  $i \geq 4$ ?

Such a class  $\theta_i$ , if exists, is called a Kervaire invariant homotopy element. We remark that  $\theta_i$  here is  $\theta_{i-1}$  commonly used to denote such homotopy elements. So far it is known that  $\theta_i$  exists for  $4 \leq i \leq 6$  and for  $i \geq 7$  the answer is not known. One can describe  $\theta_i$  in terms of invariants in the Adams spectral sequence for spheres from which the original version of the KIP is expressed.

In what follows we assume  $i \geq 4$ . Let  $\rho(i) = 2^c + 8d$  if  $i = c + 4d$  with  $0 \leq c \leq 3$ . For example,  $\rho(4) = 9$ ,  $\rho(5) = 10$ ,  $\rho(6) = 12$ . A classical theorem (via Clifford algebras) says that the Whitehead square map  $S^{2^{i+1}-3} \xrightarrow{\bar{w}_i} S^{2^i-1}$ , that represents  $w_i$ , has a desuspension  $S^{2^{i+1}-2-\rho(i)} \xrightarrow{\hat{w}_i} S^{2^i-\rho(i)}$ , that is,  $\Sigma^{\rho(i)-1}\hat{w}_i \simeq \bar{w}_i$ . For  $1 \leq k \leq \rho(i)$  let  $\hat{w}_i(k)$  denote the suspension  $S^{2^{i+1}-2-k} \xrightarrow{\Sigma^{\rho(i)-k}\hat{w}_i} S^{2^i-k}$ . We are going to use property (8) of  $Sq_1$  to show the following in which  $k$  is assumed to be even so that  $Sq_1([\hat{w}_i(k)])$  is defined.

**Proposition B.**  $Sq_1([\hat{w}_i(k)]) = 0$  in  $\pi_{2^{i+2}-3-2k}(S^{2^{i+1}-2k})$  if  $2k \leq \rho(i)$ .

Is this result still true for  $2k > \rho(i)$ ? (we still assume  $k$  is even and recall that  $k \leq \rho(i)$ ) If one can show this positively then one can solve the KIP. Indeed, this is the main result of Barratt and Mahowald in their approach to the KIP from the quadratic construction viewpoint, and is stated as follows.

**Theorem C.** *If  $Sq_1([\hat{w}_i(k)]) = 0$  in  $\pi_{2^{i+2}-3-2k}(S^{2^{i+1}-2k})$  for some even  $k$  with  $2k > \rho(i)$  then  $\theta_i$  exists.*