Weights for maximal functions and singular integrals

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These notes are a guide for the course to be taught at the NCTS 2005 Summer School on Harmonic Analysis in Taiwan. They contain a description of results and sometimes short indications about the proofs. Moreover, each section contains a list of references. For an introduction to the subject it is better to go to the books mentioned in the bibliography at the end of the notes: each one of them includes at least one chapter on weighted inequalities. As the title suggests, the most complete of the books is the one by García-Cuerva and Rubio de Francia. Apart from the books we also give the references of two survey papers.

1 Introduction, motivation, examples

A weighted inequality for an operator is a boundedness result from some L^p space to some L^q space when at least one of those spaces is taken with respect to a measure different from Lebesgue measure.

Many times the measures are absolutely continuous with respect to Lebesgue measure and the densities are called weights; that is, if $d\mu(x) = w(x)dx$, w is the weight.

We will classify the weighted inequalities into three families.

1.1 Inequalities for an operator with one or two weights

We have an operator T and want to determine measures μ and ν such that T satisfies

$$\left(\int |Tf|^q d\nu\right)^{1/q} \leq C \left(\int |f|^p d\mu\right)^{1/p}.$$

Those inequalities in turn can be subdivided into two parts.

1.1.1 Inequalities for particular weights

There are many results for classical operators in which inequalities with respect to power weights (that is, weights of the form $|x|^\alpha$) are considered. Here are some examples.

- Hardy operator and related ones (G. H. Hardy, *Notes on some points in the integral calculus (LXIV)*, Messenger of Math. 57 (1928), 12-16).

- Hilbert transform (G. H. Hardy and J. E. Littlewood, *Some theorems on Fourier series and Fourier power series*, Duke Math. J. 2 (1936), 354-381, and K. I. Babenko, *On conjugate functions* (in Russian), Doklady-Akad.-Nauk-SSSR (N. S.) 62 (1948, 157-160).

- Singular integrals (E. M. Stein, *Note on singular integrals*, Proc. Amer. Math. Soc. 8 (1957), 250-254).

- Fractional integrals (G. H. Hardy and J. E. Littlewood, *Some properties of fractional integrals, I* Math. Zeit. 27 (1928), 565–606, and E. M. Stein and G. Weiss, *Fractional integrals on n-dimensional Euclidean space*, Jour. Math. Mech. 7 (1958), 503-514).

- Pitt's inequalities for the Fourier transform (H. R. Pitt, *Theorems on Fourier and power series*, Duke Math. J. 3 (1937), 747-755).

- The multiplier of the ball (I. Hirschman, *Multiplier transformations II*, Duke Math. J. 28 (1962), 45-56.

1.1.2 Description of *all* the weights for which the inequality holds

In 1962 H. Helson and G. Szëgo gave a necessary and sufficient condition on the measure μ for the boundedness of the Hilbert transform in $L^2(\mu)$ (*A problem in prediction theory*, Ann. Mat. Pura Appl. 51, 4 (1960), 107–138).

Nevertheless, the key stone in the theory of weights was the characterization by Muckenhoupt of the necessary and sufficient condition on w for the boundedness of the Hardy-Littlewood maximal operator on $L^p(w)$, $1 < p < \infty$. The classes of weights so obtained had a lot of structure and could be used for other operators.

1.2 One weight is the action of an operator on the other weight

Inequalities of the form

$$\int |Tf|^p u \leq C \int |f|^p Au,$$

where A is some operator acting on u.

1.2.1 An application

If T satisfies such an inequality for some p and A is bounded on L^q for some q, then T is bounded in L^r for $(r/p)' = q$ (that is, $r = pq'$). Use duality on $L^{r/p}$ to say that there exists $u \in L^{(r/p)'}$ such that

$$\left(\int |Tf|^r\right)^{p/r} = \int |Tf|^p u;$$

use then the hypothesis, Hölder inequality, and the boundedness of A.

If all the operators of a sequence $\{T_j\}$ satisfy uniform inequalities of that type, vector-valued inequalities can be deduced in a similar way.

- C. Fefferman and E. Stein (*Some maximal inequalities*, Amer. J. Math. 93 (1971), 107–115) proved that the Hardy-Littlewood maximal operator satisfies

$$\int |Mf|^p u \leq C \int |f|^p Mu, \quad 1 < p < \infty,$$

(here A is the operator M itself), and is of weak type $(1,1)$ with those weights. Actually, the usual proofs of the weak-type $(1,1)$ for the Hardy-Littlewood maximal operator provide the improvement to those weights. For $p > 1$ the result is deduced by interpolation.

- A. Córdoba and C. Fefferman (*A weighted norm inequality for singular integrals*, Studia Math. 57 (1976), 97-101) proved that the singular integrals with smooth kernel satisfy for all $s > 1$

$$\int |Tf|^p u \leq C_s \int |f|^p (Mu^s)^{1/s}, \quad 1 < p < \infty.$$

Here again M is the Hardy-Littlewood maximal operator. The characterization of A_1 weights included this inequality into that theory.

1.3 Weighted inequalities between two operators

These inequalities are of the type

$$\int |Tf|^p w \leq \int |Sf|^p w,$$

where both T and S are operators and w belongs to a large class of weights.

R. Coifman and C. Fefferman (*Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. 51 (1974), 241–250) proved an inequality like this for smooth singular integrals T with $S = M$ (the Hardy-Littlewood maximal operator), $0 < p < \infty$, and $w \in A_\infty$ (a class of weights to be described below).

1.4 Interpolation with change of measure

Apart from the usual interpolation theorems (Riesz-Thorin and Marcinkiewicz) in which the measure in the spaces is kept fixed, when dealing with weights another interpolation theorem is useful; this is the *interpolation theorem with change of measure* due to E. M. Stein and G. Weiss.

Theorem 1.1 *Let T be a linear operator such that*

$$\|Tf\|_{q_j,u_j} \leq A_j \|f\|_{p_j,v_j}, \qquad j = 0, 1,$$

where u_j, v_j are weights. Then

$$\|Tf\|_{q_t,u_t} \leq A_0^{1-t} A_1^t \|f\|_{p_t,v_t},$$

with $u_t^{1/q_t} = u_0^{(1-t)/q_0} u_1^{t/q_1}$ and $v_t^{1/p_t} = v_0^{(1-t)/p_0} v_1^{t/p_1}$, and

$$0 \leq t \leq 1, \ \frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \ \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

2 The Hardy-Littlewood maximal operator and A_p weights

Notation: M is the Hardy-Littlewood maximal function. If μ is a measure and E is a measurable set, $\mu(E)$ is the measure of E; if $d\mu(x) = w(x)dx$, then we write $w(E)$ instead of $\mu(E)$. χ_E is the characteristic function of E.

2.1 Necessary conditions

1. If the inequality

$$\int |Mf(x)|^p \, d\mu(x) \leq C \int |f(x)|^p \, d\mu(x)$$

holds for some positive measure μ, then μ is absolutely continuous with respect to Lebesgue measure.

2. Let w be nonnegative and locally integrable such that

$$\lambda^p w(\{x : Mf(x) > \lambda\}) \leq C \int |f(x)|^p w(x) \, dx.$$

Let g be nonnegative, integrable in the cube Q and let $f = g\chi_Q$. Then $Mf(x) \geq |Q|^{-1} g(Q)$ for $x \in Q$ and for $\lambda < |Q|^{-1} g(Q)$, we have $Q \subset \{x : Mf(x) > \lambda\}$ so that

$$\lambda^p w(Q) \leq C \int_Q g^p w.$$

Let $\lambda \to |Q|^{-1} g(Q)$ to obtain

$$w(Q) g(Q)^p |Q|^{-p} \leq C (g^p w)(Q).$$

Let $p > 1$. Take $g = w^{1-p'}$ (if it is not locally integrable take $g = (w + \epsilon)^{1-p'}$ and then $\epsilon \to 0$). We deduce

$$\Big(\frac{1}{|Q|} \int_Q w\Big)\Big(\frac{1}{|Q|} \int_Q w^{1-p'}\Big)^{p-1} \leq C$$

for all cube Q. This is the A_p *condition*. The set of weights for which it holds is the A_p *class*.

Let $S \subset Q$ and $g = \chi_S$. Then

$$w(Q) \left(\frac{|S|}{|Q|}\right)^p \leq C w(S). \tag{2.1}$$

Let $p = 1$. For $x \in Q$ choose a sequence S_n of cubes contained in Q, containing x and such that their sides tend to 0; Lebesgue's differentiation theorem implies

$$\frac{w(Q)}{|Q|} \leq C w(x) \quad \text{a.e. } x \in Q$$

which is equivalent to

$$Mw(x) \leq C w(x) \quad \text{a.e.}$$

This is the A_1 *condition*, which defines the A_1 *class* of weights.

2.2 The doubling condition

A measure μ is $d$oubling if there exists $C > 0$ such that $\mu(2Q) \le C\mu(Q)$ for all cube Q of $\mathbf{R}^n$.

A measure $d\mu(x) = w(x)\,dx$ with $w \in A_p$ for some p is doubling.

Given a measure μ, define a maximal function associated to μ by

$$M_\mu f(x) = \sup_{x\in Q} \frac{1}{\mu(Q)} \int_Q |f(x)|\,d\mu(x)$$

Theorem 2.1 *If μ is doubling, then M_μ is bounded on $L^p(\mu)$ and is weak $(1,1)$ with respect to $L^1(\mu)$.*

The weak type $(1,1)$ comes from a Vitali's covering lemma and the strong (p,p) by interpolation.

2.3 Sufficient condition: weak case

Theorem 2.2 *Let $1 \le p < \infty$. The inequality*

$$w(\{x : Mf(x) > \lambda\}) \le \frac{C}{\lambda^p} \int |f(x)|^p w(x)\,dx$$

holds if and only if $w \in A_p$.

We need the sufficiency. Prove first that

$$Mf(x) \le C(M_w(f^p)(x))^{1/p}$$

for $w \in A_p$; deduce that

$$w(\{x : Mf(x) > \lambda\}) \le w(\{x : M_w(f^p)(x) > (C^{-1}\lambda)^p\})$$

and use the weak type $(1,1)$ for M_w.

Consequence. Using Marcinkiewicz interpolation theorem we deduce that M is bounded on $L^p(w)$ (strong) when $w \in A_q$ for some $q < p$.

2.4 Properties of A_p weights

Theorem 2.3

1. *$w \in A_p$ if and only if $w^{1-p'} \in A_{p'}$ $(1 < p < \infty)$.*
2. *If $w \in A_p$ and $q > p$, then $w \in A_q$.*
3. *If $w \in A_p$ and $0 \le \alpha \le 1$, then $w^\alpha \in A_p$.*
4. *If $w_0, w_1 \in A_p$ and $0 \le \alpha \le 1$, then $w_0^\alpha w_1^{1-\alpha} \in A_p$.*

5. *If* $w_0, w_1 \in A_1$, *then* $w_0 w_1^{1-p} \in A_p$.

These properties are easy to check using the definition and Hölder inequality. More difficult is the following key property of the A_p weights.

Theorem 2.4 (Reverse Hölder inequality) *Let* $w \in A_p$ *for some* p, $1 \le p < \infty$. *Then there exists* $\delta > 0$ *such that*

$$\left(\frac{1}{|Q|}\int_Q w^{1+\delta}\right)^{1/(1+\delta)} \le C\frac{1}{|Q|}\int_Q w$$

for all cube Q *of* $\mathbf{R}^n$. *(*C *depends on* w *but not on* Q.*)*

The converse inequality (with constant 1) is Hölder inequality; this is the reason for the name.

Corollary 2.5 *1. If* w *is in* A_p *for* $p > 1$, *there is some* $\epsilon > 0$ *such that* w *is in* $A_{p-\epsilon}$. *Then* $\bigcup_{q<p} A_q = A_p$.

2. If w *is in* A_p *for* $p \ge 1$, *there is some* $\epsilon > 0$ *such that* $w^{1+\epsilon}$ *is in* A_p.

2.5 Sufficient condition: strong case

Corollary 2.6 *Let* $1 < p < \infty$. M *is bounded on* $L^p(w)$ *if and only if* $w \in A_p$.

It is deduced from the consequence of Theorem 2.2 and the first part of the previous corollary.

2.6 Inequalities with two weights

Necessary and sufficient conditions on u and v such that the weak type inequality

$$u(\{x : Mf(x) > \lambda\}) \le \frac{C}{\lambda^p}\int |f(x)|^p v(x)\,dx$$

holds are similar to the one-weighted case, namely,

$$\sup_Q\Big(\frac{1}{|Q|}\int_Q u\Big)\Big(\frac{1}{|Q|}\int_Q v^{1-p'}\Big)^{p-1} \le C,$$

for $1 < p < \infty$, and $Mu(x) \le Cv(x)$ a.e. for $p = 1$.

The proof is like in the one-weighted case. The auxiliar maximal function to be used is

$$M_{u,v}f(x) = \sup_{r>0}\frac{1}{u(B(x,r))}\int_{B(x,r)}|f(y)|v(y)\,dy.$$

To prove that it is weak-type $(1,1)$ without knowing that u is doubling the Besicovitch covering lemma is used.

Unlike the one-weighted case the conditions are not sufficient for the strong inequalities to hold.

2.7 References and comments

The characterization of the A_p classes is due to B. Muckenhoupt (*Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. 165 (1972), 207-226). A proof of the strong case without using the reverse Hölder inequality is due to M. Christ and R. Fefferman (*A note on weighted norm inequalities for the Hardy-Littlewood maximal operator*, Proc. Amer. Math. Soc. 87 (1983), 447-448).

The characterization of strong inequalities with two weights is due to E. Sawyer (*A characterization of a two-weight norm inequality for maximal operators*, Studia Math. 75 (1982), 1-11): M is bounded from $L^p(v)$ to $L^p(u)$ $(1 < p < \infty)$ if and only if for all cube Q,

$$\int_Q (M(\chi_Q v^{1-p'})(x))^p u(x)\,dx \le C \int_Q v(x)^{1-p'}\,dx.$$

It is called S_p *condition* and we say $(u, v) \in S_p$. R. A. Hunt, D. S. Kurtz and C. J. Neugebauer gave a direct proof of the equivalence of A_p and S_p for equal weights (*A note on the equivalence of A_p and Sawyer's condition for equal weights* in *Conference in Harmonic Analysis in honor of A. Zygmund*, vol. 1, W. Beckner, A. P. Calderón, R. Fefferman and P. W. Jones ed., Wadsworth Inc., 1981, 156-158) .

The weak $(1, 1)$ result of C. Fefferman and E. Stein for M mentioned in 1.2.1 proves that M is weak $(1, 1)$ for A_1 weights. The strong L^p result deduced by interpolation shows that $(u, Mu) \in S_p$. This couple serves as a counterexample to show that the first property in Theorem 2.3 is false for S_p; indeed, $((Mu)^{1-p'}, u^{1-p'}) \in S_{p'}$ would lead to

$$\int_{\mathbf{R}^n} (Mf(x))^{p'} (Mu(x))^{1-p'}\,dx \le C \int_{\mathbf{R}^n} (f(x))^{p'} (u(x))^{1-p'}\,dx,$$

which cannot be true (take $u = f$).

B. Jawerth extended the results to maximal operators defined through a basis $\mathcal{B}$, which is a collection of open sets in $\mathbf{R}^n$ (*Weighted norm inequalities: linearization, localization and factorization*, Amer. J. Math. 108 (1986), 361-414). Given a weight w define

$$M_{\mathcal{B},w} f(x) = \sup_{x \in B \in \mathcal{B}} \frac{1}{w(B)} \int_B |f(y)| w(y)\,dy \quad \text{if } x \in \bigcup_{B \in \mathcal{B}} B,$$

and 0 otherwise. For $w \equiv 1$ we write $M_{\mathcal{B}}$. $w \in A_{p,\mathcal{B}}$ is defined in a similar way to A_p with the elements of $\mathcal{B}$ instead of the cubes. Jawerth's theorem is the following: *Let $\mathcal{B}$ a basis, w a weight, $1 < p < \infty$, and write $\sigma = w^{1-p'}$. Then $M_{\mathcal{B}}$ is bounded on $L^p(w)$ and on $L^{p'}(\sigma)$ if and only if $w \in A_{p,\mathcal{B}}$, $M_{\mathcal{B},\sigma}$ is bounded on $L^p(\sigma)$ and $M_{\mathcal{B},w}$ on $L^p(w)$.* The result can be applied to the strong maximal function and many other operators.

3 Structure of A_p classes: factorization and extrapolation

3.1 Factorization

The last property in Theorem 2.3 says that if $w_0, w_1 \in A_1$, then $w_0 w_1^{1-p} \in A_p$. The converse is true.

Theorem 3.1 (Factorization of A_p weights) *Let $w \in A_p$. There exist $w_0, w_1 \in A_1$ such that $w = w_0 w_1^{1-p}$.*

Notation: Let T be a nonnegative operator, that is, $Tf \geq 0$ for $f \geq 0$. We denote $W_p(T) = \{w : T \text{ is bounded on } L^p(w)\}$ for $1 < p < \infty$ and $W_1(T) = \{w : Tw(x) \leq Cw(x) \text{ a.e.}\}$.

Let S be a linear nonnegative operator and let S^* be its adjoint. It is easy to check that S is bounded on $L^1(w)$ if and only if $w \in W_1(S^*)$.

Let $w_0 \in W_1(S^*)$ and $w_1 \in W_1(S)$. Define T as $Tf = w_1^{-1} S(w_1 f)$; then T is bounded on $L^1(w_0 w_1)$ and on $L^\infty(w_0 w_1)$, so that T is also bounded on $L^p(w_0 w_1)$. Then $w_0 w_1^{1-p} \in W_p(S)$. As for A_p weights, the converse is true.

Both factorization theorems for M and for the linear operator S can be proved as particular cases of an abstract setting.

Definition: A sublinear operator T is *admissible* if $Tf \geq 0$ and

$$T\left(\sum_{j=0}^{\infty} f_j\right) \leq \sum_{j=0}^{\infty} T f_j$$

for f_j in the domain of T.

If T is bounded on $L^q(\mu)$, the condition holds for f_j and $f = \sum_j f_j$ in that space.

The following key lemma is sometimes called *Rubio de Francia algorithm*.

Lemma 3.2 *Let T be an admissible operator, bounded on $L^q(\mu)$ and $u \geq 0$ a function of $L^q(\mu)$. Then there exists $v \in L^q(\mu)$ such that*

1. *$u(x) \leq v(x)$ μ-a.e. x;*
2. *$\|v\|_{q,\mu} \leq C\|u\|_{q,\mu}$;*
3. *$Tv(x) \leq Cv(x)$ μ-a.e.*

If A is the operator norm of T on $L^q(\mu)$, define

$$v = \sum_{k=0}^{\infty} \frac{1}{(2A)^k} T^k u$$

and check the inequalities.

Remark. If $T_1, T_2, \ldots, T_m$ are admissible and bounded on $L^q(\mu)$, take $T = T_1 + T_2 + \cdots + T_m$ and define v as before. Then it satisfies the third property of the lemma for each T_j.

Theorem 3.3 *Let M_1 and M_2 be two operators and w a weight in $W_p(M_1)$ such that $w^{1-p'} \in W_{p'}(M_2)$. If $T_1 u = (M_1 u^{p'-1})^{p-1}$ and $T_2 u = w^{-1} M_2(wu)$ are admissible, then there exist $w_0 \in W_1(M_2)$ and $w_1 \in W_1(M_1)$ such that $w = w_0 w_1^{1-p}$.*

To prove the theorem write $u = w_1^{p-1}$, so that $w_1 = u^{p'-1}$ and $w_0 = wu$. We need u such that $M_2(wu) \leq Cwu$ and $M_1(u^{p'-1}) \leq Cu^{p'-1}$ a.e. This means $T_1 u \leq Cu$ and $T_2 u \leq Cu$ a.e. It is enough to check that T_1 and T_2 are bounded on $L^{p'}(w)$ and apply the Rubio de Francia algorithm.

Consequences. If M_1 and M_2 are both the Hardy-Littlewood maximal function, the conclusion is Theorem 3.1. The factorization theorem for the weights associated to a linear nonnegative operator are deduced with $M_1 = S$ and $M_2 = S^*$.

3.2 A_1 weights

A_1 weights are interesting to build A_p weights. Given $u \in L^q$ for some $q > 1$, Rubio de Francia algorithm gives a way to find an A_1 weight majorizing u with L^q-norm controlled by the norm of u. This can be done in a different way using the following result.

Theorem 3.4 *Let μ a finite measure such that $M\mu(x) < \infty$ a.e. and $0 \leq \delta < 1$. Then $(M\mu)^\delta \in A_1$ with constant depending on δ, but not on μ.*

An application. Let μ be the Dirac mass at the origin. Then $M\mu(x) = c|x|^{-n}$ and the theorem implies $|x|^\alpha \in A_1$ for $-n < \alpha \leq 0$. This range is sharp. Using the factorization theorem we deduce that $|x|^\alpha \in A_p$ for $-n < \alpha < n(p-1)$ when $1 < p < \infty$. This range is also sharp (outside either w or $w^{1-p'}$ are not locally integrable).

A slight modification of Theorem 3.4 gives a characterization of A_1 weights.

Theorem 3.5 *$w \in A_1$ if and only if there exist $f \in L^1_{loc}(\mathbf{R}^n)$, $k \in L^\infty(\mathbf{R}^n)$ and $0 \leq \delta < 1$ such that $k^{-1} \in L^\infty$ and $w(x) = k(x)(Mf(x))^\delta$ a.e.*

3.3 Extrapolation

Using factorization and interpolation with change of measure the following result is deduced: *Let T be bounded on $L^{p_0}(w)$ for all $w \in A_{p_0}$ and on $L^{p_1}(w)$ for all $w \in A_{p_1}$. Then it is bounded on $L^p(w)$ for all $w \in A_p$ and $p_0 \leq p \leq p_1$.*

It is quite surprising that actually the hypotheses on one side (either p_0 or p_1) lead to the same conclusion. This is the extrapolation theorem.

Theorem 3.6 *Let T be bounded on $L^{p_0}(w)$ for all $w \in A_{p_0}$. Then T is bounded on $L^p(w)$ for all $w \in A_p$ and $1 < p < \infty$.*

Step 1. We show first that if $1 < q < p_0$ and $w \in A_1$, then T is bounded on $L^q(w)$. The function $(Mf)^{(p_0-q)/(p_0-1)}$ is in A_1 since $p_0 - q < p_0 - 1$ (Theorem 3.4), so that $w(Mf)^{q-p_0}$

is in A_{p_0} (Theorem 2.3). Therefore,

$$\begin{aligned}\int_{\mathbf{R}^n}|Tf|^q w &= \int_{\mathbf{R}^n}|Tf|^q (Mf)^{-(p_0-q)q/p_0}(Mf)^{(p_0-q)q/p_0} w \\ &\leq \left(\int_{\mathbf{R}^n}|Tf|^{p_0} w (Mf)^{q-p_0}\right)^{q/p_0}\left(\int_{\mathbf{R}^n}(Mf)^q w\right)^{(p_0-q)/p_0} \\ &\leq C\left(\int_{\mathbf{R}^n}|f|^{p_0} w (Mf)^{q-p_0}\right)^{q/p_0}\left(\int_{\mathbf{R}^n}|f|^q w\right)^{(p_0-q)/p_0} \leq C\int_{\mathbf{R}^n}|f|^q w.\end{aligned}$$

(We use $Mf(x)^{q-p_0} \leq |f(x)|^{q-p_0}$ a.e., which comes from $|f(x)| \leq Mf(x)$ a.e. and $q-r<0$.)

Step 2. Next we show that given any p, $1<p<\infty$, and q, $1<q<\min(p,p_0)$, T is bounded on $L^p(w)$ for $w \in A_{p/q}$. This implies the theorem.

Let $w \in A_{p/q}$. Then by duality there exists $u \in L^{(p/q)'}(w)$ with norm 1 such that

$$\left(\int_{\mathbf{R}^n}|Tf|^p w\right)^{q/p} = \int_{\mathbf{R}^n}|Tf|^q wu.$$

For any $s>1$, $wu \leq M((wu)^s)^{1/s}$ and $M((wu)^s)^{1/s} \in A_1$. Therefore, by the first part of the proof,

$$\begin{aligned}\int_{\mathbf{R}^n}|Tf|^q wu &\leq \int_{\mathbf{R}^n}|Tf|^q M((wu)^s)^{1/s} \leq C\int_{\mathbf{R}^n}|f|^q M((wu)^s)^{1/s} \\ &\leq C\left(\int_{\mathbf{R}^n}|f|^p w\right)^{q/p}\left(\int_{\mathbf{R}^n}M((wu)^s)^{(p/q)'/s} w^{1-(p/q)'}\right)^{1/(p/q)'}.\end{aligned}$$

Since $w \in A_{p/q}$, then $w^{1-(p/q)'} \in A_{(p/q)'}$ by Theorem 2.3. For some s close to 1 we have $w^{1-(p/q)'} \in A_{(p/q)'/s}$, and this completes the proof.

Remarks. 1. It is possible to start with weak $(1,1)$ inequalities with respect to all A_1 weights and deduce strong $L^p(w)$ inequalities with $w \in A_p$, but it is not possible to deduce weak $(1,1)$ inequalities in Theorem 3.6. Indeed, there exist operators bounded on $L^p(w)$ for all $w \in A_p$, $1<p<\infty$, which are not weak $(1,1)$ (even unweighted).

2. The same proof works for the following: *Let T be bounded on $L^{p_0}(w)$ for all $w \in A_s$ with $s<p_0$. Then T is bounded on $L^p(w)$ for all $w \in A_{p_0/s}$ and $p>p_0/s$.*

3.4 The class A_∞

A_∞ is the name given to the class of all A_p weights, that is, $A_\infty = \bigcup_{1\leq p<\infty} A_p$. The following theorem characterizes this class.

Theorem 3.7 *The following are equivalent:*

1. *$w \in A_p$ for some $p \geq 1$.*

2. *w satisfies a reverse Hölder inequality.*

3. There exist $\delta > 0$ *and* $C > 0$ *such that for all cube* Q *and* $S \subset Q$,

$$\frac{w(S)}{w(Q)} \leq C \left(\frac{|S|}{|Q|} \right)^{\delta}. \tag{3.1}$$

4. For all cube Q,

$$\frac{1}{|Q|} \int_Q w \leq C \exp \left(\frac{1}{|Q|} \int_Q \log w \right).$$

Sometimes (3.1) is called the A_∞ *condition*. The last characterization is the limit when p goes to infinity of the A_p condition.

3.5 A_p weights and the space BMO

A locally integrable function is of *bounded mean oscillation* if

$$\sup_Q \frac{1}{|Q|} \int_Q |f - f_Q| < \infty$$

where the supremum is taken over all cubes of $\mathbf{R}^n$ and f_Q is the average of f on Q. BMO is the space of functions of bounded mean oscillation. If we define the *sharp maximal function* as

$$M^{\#} f(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f - f_Q|,$$

then f is in BMO if and only if $M^{\#} f \in L^\infty$. To define a norm on BMO we identify functions whose difference is a constant and put $\|M^{\#} f\|_\infty$ as the norm.

The relation between A_p and BMO is contained in the following theorem.

Theorem 3.8 *1. If* $w \in A_p, 1 \leq p < \infty$, *then* $\log w \in BMO$.

2. If $f \in BMO$ *and is real, and* $1 < p < \infty$, *then there exists* $a > 0$ *such that* $e^{af} \in A_p$.

As a consequence, if μ is a measure such that $M\mu(x) < \infty$ a.e., then $\log M\mu \in BMO$. In particular, $\log |x|$ is in BMO.

3.6 References and comments

The factorization theorem, conjectured by Muckenhoupt, was proved by P. Jones (*Factorization of A_p weights*, Ann. of Math. 111 (1980), 511–530). The proof was simplified by R. Coifman, P. Jones and J. L. Rubio de Francia (*Constructive decomposition of BMO functions and factorization of A_p weights*, Proc. Amer. Math. Soc. 87 (1983), 675–676) following a more general approach by Rubio de Francia. See also the paper of B. Jawerth mentioned in Section 2.7.

The characterization of A_1 weights in Theorem 3.4 is due to R. Coifman and appeared in a paper with R. Rochberg (*Another characterization of BMO*, Proc. Amer. Math. Soc. 79 (1980), 249–254). For other maximal operators a similar result needs not be true; for instance, if M_S is the strong maximal function (defined with rectangles instead of cubes) then $(M_S\mu)^\delta$ for $\delta < 1$ can fail to be in $W_1(M_S)$ (F. Soria, *A remark on A_1-weights for the strong maximal function*, Proc. Amer. Math. Soc. 100 (1987), 46-48).

The extrapolation theorem is due to J. L. Rubio de Francia (*Factorization theory and A_p weights*, Amer. J. Math. 106 (1984), 533–547), who proved it using the connection between weighted norm inequalities and vector-valued inequalities; a direct proof without using vector-valued inequalities is due to J. García-Cuerva (*An extrapolation theorem in the theory of A_p-weights*, Proc. Amer. Math. Soc. **87** (1983), 422–426).

The class A_∞ and its properties were studied independently by B. Muckenhoupt (*The equivalence of two conditions for weight functions*, Studia Math. 49 (1974), 101-106) and R. Coifman and C. Fefferman (*Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. 51 (1974), 241–250). The last characterization in Theorem 3.7 is due to S. V. Hruščev (*A description of weights satisfying the A_∞ condition of Muckenhoupt*, Proc. Amer. Math. Soc. 90 (1984), 253-257) and independently to García-Cuerva and Rubio de Francia in the book [2].

4 Weights for smooth singular integrals

We consider convolution operators $Tf = \text{p.v.}\,K * f$ where the distribution p.v. K coincides outside the origin with a locally integrable function (which is also denoted by K). For $f \in \mathcal{C}_0^\infty(\mathbf{R}^n)$, the operator is

$$Tf(x) = \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} K(x-y)f(y)\,dy := \lim_{\epsilon \to 0} T_\epsilon f(x).$$

We also consider the associated maximal operator

$$T^* f(x) = \sup_{\epsilon > 0} |T_\epsilon f(x)|.$$

For the smooth singular integrals in this section we make the following assumptions:

1. Size: $|K(x)| \le C|x|^{-n}$.
2. Regularity: $|K(x-y) - K(x)| \le C \dfrac{|y|}{|x|^{n+1}} \quad \text{for} \quad |x| > 2|y|$.
3. The Fourier transform of the distribution p.v. K is a bounded function.

The second condition holds in particular if $|\nabla K(x)| \le C|x|^{-(n+1)}$ for $x \neq 0$. The third condition is equivalent to saying that T is bounded on L^2.

4.1 An integral inequality

The following integral inequality will be enough to deduce weighted inequalities for singular integrals from those of M.

Theorem 4.1 *Let T be a smooth singular integral, $f \in \mathcal{C}_0^\infty(\mathbf{R}^n)$, $w \in A_\infty$, and $0 < p < \infty$. Then*

$$\int_{\mathbf{R}^n} |Tf(x)|^p w(x)\,dx \le C_p \int_{\mathbf{R}^n} |Mf(x)|^p w(x)\,dx;$$

and

$$\sup_\lambda \lambda w(\{x : |Tf(x)| > \lambda\}) \le C_w \sup_\lambda \lambda w(\{x : |Mf(x)| > \lambda\}).$$

The same inequalities hold for T^*.

The proof of Theorem 4.1 is based on several previous results.

Lemma 4.2 (Good lambda inequality) *Let $w \in A_\infty$, $0 < \gamma < 1$ and $\lambda > 0$. Then*

$$w(\{x : Mf(x) > 2\lambda, M^\# f(x) \le \gamma\lambda\}) \le A\gamma^\delta w(\{x : Mf(x) > \lambda\})$$

with A independent of γ and λ and where δ is the exponent of the A_∞ condition of w.

(Here $M^\#$ is the sharp maximal function of Section 3.5.)

The proof is easier with the dyadic maximal function instead of M, which is enough for the sequel.

The lemma leads to an integral inequality.

Theorem 4.3 *Let $w \in A_\infty$. (i) If $0 < p < \infty$ and f is such that $Mf \in L^p(w)$, then*

$$\int_{\mathbf{R}^n} |Mf(x)|^p w(x)\,dx \le C_w \int_{\mathbf{R}^n} |M^\# f(x)|^p w(x)\,dx;$$

(ii) if $\varphi : (0,\infty) \longrightarrow (0,\infty)$ is increasing and satisfies $\varphi(2t) \le C\varphi(t)$, then

$$\sup_\lambda \varphi(\lambda) w(\{x : Mf(x) > \lambda\}) \le C_w \sup_\lambda \varphi(\lambda) w(\{x : M^\# f(x) > \lambda\})$$

whenever the left-hand side is finite.

To use the inequalities for singular integrals, we need a pointwise inequality which appears in the following lemma.

Lemma 4.4 *Let T be a smooth singular integral. Then for all $s > 1$, there exists C_s such that for all $x \in \mathbf{R}^n$,*

$$M^\#(Tf)(x) \le C_s (Mf^s(x))^{1/s},$$
$$T^* f(x) \le C_s (Mf^s(x))^{1/s} + cM(Tf)(x).$$

4.2 A_p weights for smooth singular integrals

Sufficiency. An immediate consequence of Theorem 4.1 and the results for the Hardy-Littlewood maximal function is the following.

Corollary 4.5 *Let T be a smooth singular integral.*

1. *If $1 < p < \infty$ and $w \in A_p$, then*

$$\int_{\mathbf{R}^n} |Tf(x)|^p w(x)\,dx \le C \int_{\mathbf{R}^n} |f(x)|^p w(x)\,dx;$$

2. *if $w \in A_1$, then*

$$w(\{x : Tf(x) > \lambda\}) \le \frac{C}{\lambda} \int_{\mathbf{R}^n} |f(x)| w(x)\,dx.$$

The same results hold for T^*.

Necessity. The simplest smooth singular integrals in $\mathbf{R}^n$ are the Riesz transforms defined as (a multiple of)

$$R_j f(x) = p.v. \int_{\mathbf{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\,dy.$$

Theorem 4.6 *If the Riesz transforms are of weak type on $L^p(w)$, then $w \in A_p$.*

4.3 Weighted inequalities for a Littlewood-Paley decomposition

Let Ψ be a function in the Schwartz class of $\mathbf{R}^n$ such that $\hat{\Psi}(0) = 0$ and $\Psi_j(x) = 2^{jn}\Psi(2^j x)$. Construct the square function

$$g(f)(x) = \left(\sum_{j=-\infty}^{\infty} |\Psi_j * f(x)|^2\right)^{1/2}.$$

This operator can be viewed as a vector-valued convolution operator $Tf = \{\Psi_j * f(x)\}_{j=-\infty}^{\infty}$. The norm of g in a space B is the norm of $\|Tf\|_{l^2}$ in B. A theory for vector-valued singular integrals can be developped and the size and smoothness conditions are in this case

$$\|\Psi_j(x)\|_{l^2} \le C|x|^{-n},$$
$$\|\Psi_j(x-y) - \Psi_j(x)\|_{l^2} \le C|y||x|^{-n-1}.$$

Those inequalities are easy to check for Ψ in the Schwartz class.

The weighted theory can be repeated in the vector-valued case and the conclusion is that *g is bounded on $L^p(w)$ for all $w \in A_p$.* If moreover we assume that $\sum_j |\hat{\Psi}_j(\xi)|^2 = C$ independent of ξ, then f and $g(f)$ have equivalent norms in $L^p(w)$. By duality, the operator $\{f_j\} \mapsto \sum_j \Psi_j * f_j$ is bounded from $L^p(l^2, w)$ to $L^p(w)$ for all $w \in A_p$.

4.4 References and comments

The boundedness of the Hilbert transform on $L^p(w)$ with A_p weights was proved by R. Hunt, B. Muckenhoupt and R. Wheeden (*Weighted norm inequalities for the conjugate function and the Hilbert transform*, Trans. Amer. Math. Soc. 176 (1973), 227–251) but their method does not extend to higher dimensions. For smooth singular integrals the result is due to R. Coifman and C. Fefferman (paper cited in Section 3.6) who proved Theorem 4.1. They used good lambda inequalities.

Several years before the A_p theory H. Helson and G. Szegö characterized the weights for the Hilbert transform when $p = 2$ (*A problem in prediction theory*, Ann. Mat. Pura Appl. 51 (4) (1960) 107-138). Their theorem is as follows: *the Hilbert transform is bounded on $L^2(\mu)$ if and only if μ is absolutely continuous with respect to Lebesgue measure, $d\mu = w(x)dx$, and w is of the form $\log w = u + Hv$ with $u, v \in L^\infty$ and $||v||_\infty < \pi/2$.* This condition must be equivalent to A_2 but there is not a direct proof of the equivalence. Coifman, Jones and Rubio de Francia (paper cited in Section 3.6) used the factorization of w with weights satisfying $|Hw_j(x)| \le Cw_j(x)$ a.e. to write w in a way similar to Helson-Szegö's condition but with $\|v\|_\infty < \pi$.

A. Córdoba and C. Fefferman proved (*A weighted norm inequality for singular integrals*, Studia Math. 57 (1976), 97–101) the inequality of Lemma 4.4 and using it

$$\int |Tf(x)|^p u(x)\,dx \le C_{p,s} \int |f(x)|^p (Mu^s(x))^{1/s}\,dx. \tag{4.1}$$

This inequality is now a consequence of Corollary 4.5 and the fact that $(Mu^s)^{1/s}$ is an A_1-weight majoring u. Córdoba and Fefferman proved first that $(Mu^s)^{1/s} \in A_\infty$.

Instead of Lemma 4.4 the following variant can be used.

Lemma 4.7 *Let T be a smooth singular integral, $0 < t < 1$, and $M_t^{\#}g(x) = M^{\#}(|g|^t)(x)^{1/t}$. Then $M_t^{\#}(Tf)(x) \le C_t Mf(x)$ for all x in $\mathbf{R}^n$, and the same is true for T^*.*

This variant can be seen in the paper of J. Alvarez and C. Pérez, *Estimates with A_∞ weights for various singular integral operators*, Boll. Un. Mat. Ital. A (7) 8 (1994), 123-133.

The pointwise inequality $M^{\#}(Tf)(x) \le CMf(x)$ or (4.1) with $s = 1$ are not true, in general. (A counterexample to the last one for the Hilbert transform is obtained with $f(x) = \dfrac{1}{\log x}\chi_{(e,e^n)}(x)$ and $u(x) = \chi_{(0,1)}(x)$.) Nevertheless (4.1) can be improved to

$$\int |Tf(x)|^p u(x)\,dx \le C_p \int |f(x)|^p M^{[p]+1}u(x)\,dx$$

where M^k is obtained iterating k times the operator M. The result is sharp and does not hold with $M^{[p]}$. It is due to J. M. Wilson (*Weighted norm inequalities for the continuous square function*, Trans. Amer. Math. Soc. 314 (1989), 661-692) for $p \le 2$ and to C. Pérez (*Weighted norm inequalities for singular integral operators*, J. London Math. Soc. 49 (1994), 296-308) for all p.

The results obtained for smooth singular integrals of convolution type can be extended to generalized Calderón-Zygmund operators in the sense of Coifman and Meyer. Let T be bounded on L^2 and

$$Tf(x) = \int K(x,y)f(y)\,dy$$

whenever $f \in \mathcal{C}_0^\infty(\mathbf{R}^n)$ and x is not in the support of f. K is a standard kernel if it is a function defined outside the diagonal of $\mathbf{R}^n \times \mathbf{R}^n$ such that $|K(x,y)| \le C|x-y|^{-n}$ (size condition) and for some $\delta > 0$ and $|x-y| > 2|y-y'|$ satisfies

$$|K(x,y) - K(x,y')| + |K(y,x) - K(y',x)| \le C\frac{|y-y'|^\delta}{|x-y|^{n+\delta}}$$

(regularity condition). Operators with standard kernel can be treated as smooth singular integrals to show that they are bounded on $L^p(w)$ for $w \in A_p$.

When working with singular integrals the regularity condition can be relaxed to

$$\sup_{y\neq 0}\int_{|x|>2|y|} |K(x-y) - K(x)|\,dx < \infty,$$

(Hörmander's condition) and still get that T and T_* are bounded on $L^p(\mathbf{R}^n), 1 < p < \infty$, and are of weak type $(1,1)$. Nevertheless, this condition is not sufficient to get the weighted estimates.

There are intermediate conditions on the kernel for which weighted results are possible but not for the whole A_p classes. Let K be the convolution kernel of the singular integral T such that

$$\left(\int_{S_k(|y|)} |K(x-y) - K(x)|^r\,dx\right)^{1/r} \le a_k |S_k(|y|)|^{1/r'}, \quad k \ge 1,$$

where $S_k(|y|) = \{x : 2^k|y| < |x| \le 2^{k+1}|y|\}$ and $\sum_k a_k < +\infty$. (For $r = 1$ this is Hörmander's condition and when $r \to \infty$ we get the regularity condition on K mentioned before Section 4.1.) When $1 < r < \infty$ it is possible to see that $M^{\#}(Tf)(x) \le C_r(Mf^{r'}(x))^{1/r'}$ and deduce from it that T is bounded on $L^p(w)$ if (i) $w \in A_{p/r'}$ and $p \ge r'$; (ii) $w^{1-p'} \in A_{p'/r'}$ and $1 < p \le r$; (iii) $w^{r'} \in A_p$ y $1 < p < \infty$. About this type of conditions see the paper by D. Kurtz and R. Wheeden (*Results on weighted norm inequalities for multipliers*, Trans. Amer. Math. Soc. 255 (1979), 343-362) and for vector valued singular integrals the paper by J. L. Rubio de Francia, F. J. Ruiz and J. L. Torrea (*Calderón-Zygmund theory for operator-valued kernels*, Adv. Math. 62 (1986), 7-48).

5 Weights for some rough singular integrals

Let Ω be an integrable function on S^{n-1} with integral zero and consider the singular integral operator T_Ω with convolution kernel p.v. $\Omega(x')|x|^{-n}$, where $x' = x/|x|$. The method of rotations of Calderón and Zygmund applies to prove that if $\Omega \in L^q(S^{n-1})$ for some $q > 1$, T_Ω is bounded on L^p for $1 < p < \infty$. Nevertheless, the method is not well adapted to prove weighted norm inequalities.

An alternative approach for the unweighted case was introduced by J. Duoandikoetxea and J. L. Rubio de Francia. It applies to the weighted case.

5.1 An abstract setting

Theorem 5.1 *Let $T_j f = \sigma_j * f$ where σ_j is a finite Borel measure such that*

$$|\hat{\sigma}_j(\xi)| \leq C \min(|2^j\xi|, |2^j\xi|^{-1})^\alpha \text{ for some } \alpha > 0.$$

Assume that for some $w \in A_2$ the inequality

$$\int ||\sigma_j| * f|^2 w \leq C \int |f|^2 w$$

holds with C independent of j. Then $Tf(x) = \sum_j \sigma_j * f(x)$ is bounded on $L^2(w^\theta)$ for $0 \leq \theta < 1$.

Let Ψ be a radial function in $\mathcal{S}(\mathbf{R}^n)$ such that supp $\hat{\Psi} \subset \{\xi : 1/2 < |\xi| \leq 2\}$ and $\sum_{k\in Z} |\hat{\Psi}(2^k\xi)|^2 = 1$, for all $\xi \neq 0$. Let $\Psi_j(x) = 2^{-jn}\Psi(2^{-j}x)$ and S_j the convolution with Ψ_j. Write

$$\tilde{T}_k = \sum_j T_j S_{j+k}.$$

The assumption on $\hat{\sigma}_j$ implies

$$\|\tilde{T}_k f\|_2 \leq C 2^{-\alpha|k|}\|f\|_2.$$

Using the weighted inequalities for the Littlewood-Paley square function and the assumption of the theorem we have

$$\int |\tilde{T}_k f|^2 w \leq C \int |\tilde{f}|^2 w,$$

with constant independent of k. Interpolating with change of measure and adding in k we get the result.

5.2 An application

The rough singular integral T_Ω can be written as a sum $\sum_j \sigma_j * f$ with

$$\sigma_j = \Omega(x')|x|^{-n}\chi_{2^j<|x|<2^{j+1}}.$$

For $\Omega \in L^q$ with $q > 1$ the condition on the Fourier transform of σ_j holds. (It is enough to check it for $j = 0$ and use the dilation property of the Fourier transform.)

If Ω is bounded, the second condition in the theorem holds with $w \in A_2$ and the conclusion is that T_Ω is bounded on $L^2(w)$ for all $w \in A_2$. Using the extrapolation theorem it is bounded on $L^p(w)$ for all $w \in A_p$.

If $\Omega \in L^q$ and $q > 2$, then the second condition holds with weights $w \in A_{2/q'}$ and the extrapolation theorem applies to give the boundedness of T_Ω on $L^p(w)$ for all $w \in A_{p/q'}$ when $p > q'$. This result can be proved also for all $q > 1$ without using L^2 as the space for the basic estimate. Some more weights are obtained by duality and interpolation. Nevertheless, those classes of weights are worse than expected and, for instance, they do not give all the power weights $|x|^\alpha$. This can be corrected introducing appropriate classes of weights defined for the maximal operator

$$M_\Omega f(x) = \sup_{R>0} R^{-n} \int_{|y|<R} |\Omega(y') f(x-y)|\, dy.$$

5.3 Weights for the dyadic spherical maximal operator

Define

$$S_t f(x) = \int_{S^{n-1}} f(x - ty)\, d\sigma(y)$$

where $d\sigma$ is the normalized Lebesgue measure over the unit sphere S^{n-1}. The dyadic spherical maximal operator is

$$\mathcal{M}^d f(x) = \sup_{k \in Z} |S_{2^k} f(x)|.$$

This operator is known to be bounded on L^p for $p > 1$ while the general operator with the supremum on all radii is bounded only for $p > n/(n-1)$.

Using the notation introduced in Section 3.1 and putting $AB^t = \{u : u = u_0 u_1^t, u_0 \in A, u_1 \in B\}$, we can describe weighted inequalities for $\mathcal{M}^d$ saying that

$$\bigcup_{s<1} [(W_1^d)(W_1^d)^{1-p}]^s \subset W_p^d \subset \bigcap_{s>1} [(W_1^d)(W_1^d)^{1-p}]^s.$$

5.4 References and comments

The method and the case Ω bounded appear in the paper *Maximal and singular integral operators via Fourier transform estimates* by J. Duoandikoetxea and J. L. Rubio de Francia (Invent. Math. 84 (1986), 541–561).

For $\Omega \in L^q$ see the papers by D. K. Watson (*Weighted estimates for singular integrals via Fourier transform estimates*, Duke Math. J. 60 (1990), 389–399) and J. Duoandikoetxea (*Weighted norm inequalities for homogeneous singular integrals*, Trans. Amer. Math. Soc. 336 (1993), 869–880). For the factorization and extrapolation of weights coming from rough operators see also works by the same authors (D. K. Watson, *Vector-valued inequalities, factorization, and extrapolation for a family of rough operators*, J. Funct. Anal. 121 (1994), 389–415, and J. Duoandikoetxea, *Almost-orthogonality and weighted inequalities* in Harmonic analysis and operator theory (Caracas, 1994), 213–226, Contemp. Math., 189, Amer. Math. Soc., Providence, RI, 1995). For weights involving the operator M_Ω there is a previous work by S. Hofmann (*Weighted norm inequalities and vector valued inequalities for certain rough operators*, Indiana Univ. Math. J. 42 (1993), 1–14).

The weights for the spherical dyadic maximal operator are in a paper by J. Duoandikoetxea and L. Vega (*Spherical means and weighted inequalities*, J. London Math. Soc. 53 (1996), 343–353).

6 Some applications

6.1 Vector valued inequalities

Given a sequence $\{T_j\}$ of operators and a sequence $\{f_j\}$ of functions we consider inequalities of the type

$$\left\| \left(\sum_j |T_j f_j|^q \right)^{1/q} \right\|_p \leq C_{p,q} \left\| \left(\sum_j |f_j|^q \right)^{1/q} \right\|_p . \tag{6.1}$$

Instead of the L^p-norm we can also consider an $L^p(w)$-norm for some weight w. When $p = q$ the inequality is clearly equivalent to the L^p boundedness of all the T_j with constants uniform in j.

The following theorem holds.

Theorem 6.1 *Assume that the operators T_j are bounded on $L^p(w)$ for all $w \in A_1$, uniformly in j. Then (6.1) holds for $1 \leq q \leq p$.*

For some $u \in L^{(p/q)'}(\mathbf{R}^n)$ with norm 1 we have

$$\left\| \left(\sum_j |T_j f_j|^q \right)^{1/q} \right\|_p = \int \sum_j |T_j f_j|^q u.$$

Majorize u by the A_1 weight $(Mu^s)^{1/s}$ with $1 < s < (p/q)'$, use the hypotheses and Hölder inequality.

When all the T_j coincide with the unique operator T the hypothesis is just the $L^p(w)$-boundedness of T for A_1 weights. Then the theorem can be applied to the Hardy-Littlewood maximal operator and the singular integrals. But in those cases the theorem can be improved.

Corollary 6.2 *Let T be either the Hardy-Littlewood maximal operator, a smooth singular integral or a rough singular integral like those in Section 5.2 with $\Omega \in L^\infty$. Then (6.1) holds for $1 < p, q < \infty$ ($q = \infty$ is valid for the maximal operator).*

The case $q \leq p$ is contained in the theorem; we only need the case $p < q$. For the maximal operator if $q = \infty$ we have $|Mf_j(x)| \leq M(\sup_k |f_k|)(x)$ and from here the case $q = \infty$ follows. By interpolation the range is completed.

For the singular integral use that the adjoint operator satisfies the inequality for $q \leq p$ and dualize.

An interesting case with different operators in the sequence is the following.

Corollary 6.3 *Let $\{I_j\}$ be a sequence of intervals of the real line (possibly of infinite length) and T_j the operator defined as $(T_j f)\hat{}(\xi) = \chi_{I_j}(\xi)\hat{f}(\xi)$. Then (6.1) holds for $1 < p, q < \infty$.*

If $I_j = (a_j, b_j)$ then we have the formula

$$S_j f_j = \frac{i}{2}\left(M_{a_j} H M_{-a_j} f_j - M_{b_j} H M_{-b_j} f_j\right),$$

where $M_a f(x) = e^{2\pi i a x} f(x)$ and H is the Hilbert transform, and with the obvious modifications if the interval is unbounded. In all cases we get operators bounded on $L^p(w)$ for all $w \in A_p$ with constant independent of the interval. Moreover they are (almost) self-adjoint. Then the proof goes as in the previous corollary.

6.2 Hörmander multiplier theorem

Given a function m, the multiplier operator associated to it is defined through the Fourier transform as $(T_m f)\hat{} = m\hat{f}$. We say that m is an L^p-multiplier if T_m is bounded on L^p. The Hörmander multiplier theorem gives sufficient conditions on m such that it is an L^p-multiplier for $1 < p < \infty$.

Define the Sobolev space $L^2_a(\mathbf{R}^n)$ as the set of functions g such that $(1+|\xi|^2)^{a/2}\hat{g}(\xi) \in L^2$ and the norm of this function is the norm of g in L^2_a.

If $a > n/2$ and $m \in L^2_a(\mathbf{R}^n)$ then $\hat{m} \in L^1$ (in particular, m is continuous and bounded). It follows that if $m \in L^2_a$ with $a > n/2$ then m is a multiplier on L^p, $1 \le p \le \infty$. In fact, $Tf = K * f$ with $K \in L^1$. Hörmander's theorem shows that m is a multiplier on L^p under much weaker hypotheses. A proof is possible using a Littlewood-Paley decomposition and weighted inequalities.

Let $\psi \in C^\infty$ be a radial function supported on $1/2 \le |\xi| \le 2$ and such that

$$\sum_{j=-\infty}^{\infty} |\psi(2^{-j}\xi)|^2 = 1, \quad \xi \neq 0.$$

Theorem 6.4 (Hörmander) *Let m be such that for some $a > n/2$,*

$$\sup_j \|m(2^j\cdot)\psi\|_{L^2_a} < \infty.$$

Then the operator T associated with the multiplier m is bounded on $L^p(\mathbf{R}^n)$, $1 < p < \infty$.

The weighted inequality is contained in the following lemma.

Lemma 6.5 *Let $m \in L^2_a$, $a > n/2$, and let $\lambda > 0$. Define the operator T_λ by $(T_\lambda f)\hat{}(\xi) = m(\lambda\xi)\hat{f}(\xi)$. Then*

$$\int_{\mathbf{R}}^{n} |T_\lambda f(x)|^2 u(x)\,dx \le C \int_{\mathbf{R}}^{n} |f(x)|^2 Mu(x)\,dx,$$

where the constant C is independent of u and λ, and M is the Hardy-Littlewood maximal operator.

The usual statement of Hörmander's theorem is somewhat different. It is a corollary to the previous theorem.

Corollary 6.6 *If for $k = [n/2] + 1$, $m \in C^k$ away from the origin, and if for $|\beta| \le k$*

$$\sup_R R^{|\beta|} \left(\frac{1}{R^n} \int_{R<|\xi|<2R} |D^\beta m(\xi)|^2 \, d\xi \right)^{1/2} < \infty, \tag{6.2}$$

then m is a multiplier on L^p, $1 < p < \infty$. In particular, m is a multiplier if

$$|D^\beta m(\xi)| \le C|\xi|^{-|\beta|}, \quad |\beta| \le k.$$

6.3 Boundedness on Morrey spaces

Given $0 \le \alpha \le n$ and $p \ge 1$ we define the Morrey space $L^{p,\alpha}$ as the set of functions $f \in L^p_{\text{loc}}$ such that

$$\sup_B \frac{1}{|B|^{1-\alpha/n}} \int_B |f(x)|^p \, dx := \|f\|^p_{L^{p,\alpha}} < +\infty$$

where the supremum is taken over all the balls B of $\mathbf{R}^n$. It is a Banach space; for $\alpha = n$ coincides with L^p and for $\alpha = 0$ with L^∞.

Theorem 6.7 *Assume that T is bounded on $L^p(w)$ for all $w \in A_1$. Then T is bounded on $L^{p,\alpha}(\mathbf{R}^n), 0 < \alpha \le n$.*

Using the characterization of A_1 weights we can write

$$\int_B |Tf(x)|^p \, dx = \int_{\mathbf{R}^n} |Tf(x)|^p \chi_B(x) \, dx \le C_s \int_{\mathbf{R}^n} |f(x)|^p (M\chi_B(x))^{1/s} \, dx.$$

Use that

$$M\chi_B(x) \sim \chi_B(x) + \sum_{k=0}^\infty 2^{-kn} \chi_{2^{k+1}B \setminus 2^k B}(x)$$

to majorize the right-hand side by the appropriate norm of f on Morrey spaces.

The theorem applies to the Hardy-Littlewood maximal function and the smooth singular integrals.

6.4 Conmutators and BMO

Given b define the operator M_b as $M_b f(x) = b(x)f(x)$; it is bounded on L^p if and only if $b \in L^\infty$. Given the operator T, the conmutator of T and b is defined as $[b, T] = M_b T - T M_b$. When T is bounded on L^p and $b \in L^\infty$, $[b, T]$ is also bounded on L^p. Nevertheless, when T is a smooth singular integral, $b \in BMO$ is enough for the conclusion. A proof of this result can be obtained using weighted inequalities for T.

Theorem 6.8 *Let T be a linear operator and $b \in BMO$. Assume that T is bounded on $L^p(w)$ for all $w \in A_p$ and $1 < p < \infty$. Then $[b, T]$ is bounded on L^p.*

6.5 Bochner-Riesz operators

A classical problem in Harmonic Analysis is the inversion of the Fourier transform. Bochner-Riesz operators appear as a scale of summability methods to recover a function from its Fourier transform. More precisely, given $f \in L^p(\mathbf{R}^n)$, we want to know whether

$$f(x) = \lim_{R\to\infty} \int_{|\xi|<R} \hat{f}(\xi)\left(1 - \frac{|\xi|^2}{R^2}\right)^{\lambda} e^{2\pi i x.\xi}\, d\xi$$

for appropriate λ either in L^p or pointwise. In one dimension the result is valid with $\lambda \geq 0$ (the pointwise version with $\lambda = 0$ is the celebrated theorem of Carleson-Hunt). In higher dimensions a necessary condition is

$$\frac{2n}{n+1+2\lambda} < p < \frac{2n}{n-1-2\lambda} \quad \text{if } 0 < \lambda < \frac{n-1}{2}.$$

For $\lambda \geq (n-1)/2$, convergence holds in $1 < p < \infty$. To study pointwise convergence, it is customary to consider the associated maximal operator

$$T_*^{\lambda} f(x) = \sup_{R>0} \left| \int_{|\xi|<R} \hat{f}(\xi)\left(1 - \frac{|\xi|^2}{R^2}\right)^{\lambda} e^{2\pi i x.\xi}\, d\xi \right|.$$

It is well-known that if T_*^{λ} is weak (p,p), there is pointwise convergence for $f \in L^p$. The converse is true for $1 < p \leq 2$, but not neccessarily for $p > 2$. A. Carbery, J. L. Rubio de Francia and L. Vega proved the following weighted inequality.

Theorem 6.9 *Let $\lambda > 0$ and $0 < \alpha < 1 + 2\lambda \leq n$. Then*

$$\int |T_*^{\lambda} f(x)|^2 |x|^{-\alpha}\, dx \leq C_{\alpha,\lambda} \int |f(x)|^2 |x|^{-\alpha}\, dx.$$

From this theorem one can deduce pointwise convergence for $f \in L^2(|x|^{\alpha})$. When $2 \leq p < (2n)/(n-1-2\lambda)$ the embedding $L^p \subset L^2 + L^2(|x|^{\alpha})$ holds with α in the previous range. Then pointwise convergence holds in the expected range of values of p. Nevertheless, we do not know whether the Bochner-Riesz operators are bounded in that range of values of p.

6.6 References and comments

The proof of Theorem 6.1 is similar to part of the extrapolation theorem and actually one can prove a more general result: *Let $\{T_j\}$ be a sequence of bounded operators on $L^{p_0}(w)$ for all $w \in A_{p_0}$, uniformly in j. Then* (6.1) *holds on $L^p(w)$ for all $w \in A_p$ and $1 < p, q < \infty$.*

We proved vector valued inequalities from weighted inequalities but it is possible to do it conversely and get weighted inequalities from vector valued inequalities. The equivalence is part of the interesting work of J. L. Rubio de Francia and can be seen in [2], for instance. A sample of those results is the following:

Theorem 6.10 *Let $\{T_j\}$ be a sequence of sublinear operators uniformly bounded on $\mathbf{R}^n$. The vector valued inequality*

$$\left\| \left(\sum_j |T_j f_j|^q \right)^{1/q} \right\|_p \le A \left\| \left(\sum_j |f_j|^q \right)^{1/q} \right\|_p$$

is equivalent to:

1. *if $p > q$ and $\alpha = p/q$, for all nonnegative u in $L^{\alpha'}(\mathbf{R}^n)$ there exists $v \in L^{\alpha'}(\mathbf{R}^n)$ such that $\|v\|_{\alpha'} \le C\|u\|_{\alpha'}$ and*
$$\int_{\mathbf{R}^n} |T_j f|^q u \le A^q \int_{\mathbf{R}^n} |f|^q v, \quad \textit{for all } j;$$
2. *if $p < q$ and $\alpha = q/p$, for all nonnegative u in $L^{\alpha'/\alpha}(\mathbf{R}^n)$ there exists $v \in L^{\alpha'/\alpha}(\mathbf{R}^n)$ such that $\|v\|_{\alpha'/\alpha} \le C\|u\|_{\alpha'/\alpha}$ and*
$$\int_{\mathbf{R}^n} |T_j f|^q v^{-1} \le A^q \int_{\mathbf{R}^n} |f|^q u^{-1}, \quad \textit{for all } j.$$

J. L. Rubio de Francia used these results to answer in some cases the following question about the two-weight problem: *Given an operator T, find conditions on the weight u (resp. v), such that there exists some v (resp. u) for which T is bounded from $L^p(v)$ to $L^p(u)$.* When T is a singular integral or a fractional integral, these results are in the paper *Weighted norm inequalities and vector-valued inequalities* by J. L. Rubio de Francia (in *Harmonic Analysis* (Proceedings, Minneapolis 1981), F. Ricci and G. Weiss ed., Lecture Notes in Math. 908, Springer Verlag, Berlin, 1981, 86–101).

The result in Corollary 6.2 for M was proved by C. Fefferman and E. M. Stein; they proved the inequality mentioned in Subsection 1.2.1 to deduce it (see the reference given there). The details of the proof of Hörmander multiplier theorem following the approach mentioned here are in the book [1]. Theorem 6.7 for Morrey spaces in Section 6.3 is due to F. Chiarenza and M. Frasca (*Morrey spaces and Hardy-Littlewood maximal function*, Rend. Mat. Appl. (7), 7 (1987), 273–279). Theorem 6.8 for commutators is due to R. Rochberg and uses the relation between A_p weights and BMO mentioned in Section 3.5; it was published in the paper *Factorization theorems for Hardy spaces in several variables* by R. R. Coifman, R. Rochberg and G. Weiss (Ann. of Math. 103 (1976), 611-635). Theorem 6.9 was obtained by A. Carbery, J. L. Rubio de Francia and L. Vega (*Almost everywhere summability of Fourier integrals*, J. London Math. Soc. 38 (1988), 513–524).

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