Taiwan lecture 1

Friday July 1 2005

1 Introduction

In these talks we consider mainly interpolation sequences for the analytic Besov-Sobolev spaces $B_p^\sigma(\mathbb{B}_n)$ on the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$, consisting of those holomorphic functions f on the ball such that

$$\|f\|_{B_p^\sigma(\mathbb{B}_n)} = \left(\int_{\mathbb{B}_n} \left| \left(1-|z|^2\right)^{m+\sigma} f^{(m)}(z) \right|^p d\lambda_n(z) + \sum_{k=0}^{m-1} \left| f^{(k)}(0) \right|^p \right)^{\frac{1}{p}} < \infty,$$

where $m+\sigma > \frac{n}{p}$, $d\lambda_n(z) = \left(1-|z|^2\right)^{-n-1} dz$ is invariant measure on the ball with dz Lebesgue measure on $\mathbb{C}^n$, and $f^{(m)}$ is the m^{th} order complex derivative of f. Thus $B_p^\sigma(\mathbb{B}_n)$ consists of those holomorphic functions on the ball having σ "invariant" derivatives in L^p with respect to invariant measure. This scale of spaces includes the Hardy space on the disc $H^2(\mathbb{D}) = B_2^{\frac{1}{2}}(\mathbb{D})$ with $\sigma = \frac{1}{2}$, the Dirichlet space $B_2^0(\mathbb{D})$ with $\sigma = 0$, and the various weighted Bergman and Dirichlet-type spaces. In fact, for $f(z) = \sum_{n=0}^\infty a_n z^n$, $z \in \mathbb{D}$, the orthogonality relations

$$\frac{1}{2\pi} \int_0^{2\pi} e^{i(n-m)\theta} d\theta = \begin{cases} 1 \text{ if } n = m \\ 0 \text{ if } n \neq m \end{cases}$$

yield

$$\begin{aligned} \sum_{n=0}^\infty |a_n|^2 &= \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{n=0}^\infty a_n \left(re^{i\theta}\right)^n \right|^2 d\theta \\ &= \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^2 d\theta \\ &\equiv \|f\|_{H^2(\mathbb{D})}^2, \end{aligned}$$

while the calculation

$$\int_0^1 \left(1-r^2\right) r^{2(n-1)} dr = \frac{1}{2n-1} - \frac{1}{2n+1} = \frac{2}{4n^2-1}$$

yields

$$\begin{aligned}
\|f\|^2_{B_2^{\frac{1}{2}}(\mathbb{D})} &= \int_{\mathbb{D}} \left| \left(1-|z|^2\right)^{1+\frac{1}{2}} f'(z) \right|^2 \frac{dz}{\left(1-|z|^2\right)^2} + |f(0)|^2 \\
&= \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \left| \sum_{n=1}^{\infty} n a_n \left(re^{i\theta}\right)^{n-1} \right|^2 \left(1-r^2\right) dr + |a_0|^2 \\
&= \sum_{n=1}^{\infty} |n a_n|^2 \int_0^1 \left(1-r^2\right) r^{2(n-1)} dr + |a_0|^2 \\
&= |a_0|^2 + \sum_{n=1}^{\infty} |a_n|^2 \frac{2n^2}{4n^2-1} \approx \sum_{n=0}^{\infty} |a_n|^2 .
\end{aligned}$$

Finally, the Dirichlet norm squared of f satisfies

$$\|f\|^2_{B_2^0(\mathbb{D})} = \int_{\mathbb{D}} |f'(z)|^2 dz + |f(0)|^2 ,$$

where

$$\int_{\mathbb{D}} |f'(z)|^2 dz = \int_{\mathbb{D}} \det \begin{pmatrix} u_x \, u_y \\ v_x \, v_y \end{pmatrix} dxdy = \int_{\mathbb{D}} J_f dxdy = \int_{f(\mathbb{D})} dudv$$

is the area of the image $f(\mathbb{D})$ of the disc under f by the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$ if $f = u + iv$.

The case $\sigma = \frac{1}{2}$ and $p = 2$ is the Drury-Arveson Hardy space $H_n^2 = B_2^{\frac{1}{2}}(\mathbb{B}_n)$ that can be identified with the symmetric Fock space over $\mathbb{C}^n$ (see [10] and [19]), and enjoys many universal operator-theoretic properties. An excellent survey of Hilbert space developments in this area up to now is the beautiful recent monograph of K. Seip [32].

1.1 Origins of interpolation in the Corona problem

The theory of Carleson measures and interpolating sequences has its roots in Lennart Carleson's 1958 paper [15], the first of his papers motivated by the corona problem for the Banach algebra $H^\infty(\mathbb{D})$ of bounded holomorphic functions in the unit disk $\mathbb{D}$: if $\{f_j\}_{j=1}^J$ is a finite set of functions in $H^\infty(\mathbb{D})$ satisfying

$$\sum_{j=1}^J |f_j(z)| \geq c > 0, \qquad z \in \mathbb{D},$$

are there are functions $\{g_j\}_{j=1}^J$ in $H^\infty(\mathbb{D})$ with

$$\sum_{j=1}^{J} f_j(z)\, g_j(z) = 1, \qquad z \in \mathbb{D},$$

i.e., is every multiplicative linear functional on $H^\infty(\mathbb{D})$ in the closure of the point evaluations, so that there is no "corona"? In [15], Carleson observed the following connection between the corona problem and interpolating sequences. A Blaschke product B_0 has the "baby corona" property,

$$\begin{aligned}&\text{For all } f_1 \in H^\infty(\mathbb{D}) \text{ satisfying } \inf_{z\in\mathbb{D}} \{|B_0(z)| + |f_1(z)|\} > 0,\\ &\text{there are } g_0, g_1 \in H^\infty(\mathbb{D}) \text{ with } B_0 g_0 + f_1 g_1 \equiv 1,\end{aligned} \tag{1}$$

if the zero set

$$Z_0 = \{z \in \mathbb{D} : B_0(z) = 0\} = \{z_j\}_{j=1}^\infty$$

of B_0 is an interpolating sequence for $H^\infty(\mathbb{D})$:

$$\text{The map } f \to \{f(z_j)\}_{j=1}^\infty \text{ takes } H^\infty(\mathbb{D}) \text{ boundedly into and onto } \ell^\infty(Z_0), \tag{2}$$

(if $g_1 \in H^\infty(\mathbb{D})$ satisfies $f_1(z_j)\, g_1(z_j) = 1$ for all j, then we can choose $g_0 = \frac{1-f_1 g_1}{B_0}$). Carleson solved this latter problem completely by showing that a sequence $Z = \{z_j\}_{j=1}^\infty$ is an interpolating sequence for $H^\infty(\mathbb{D})$ if and only if

$$\prod_{j:j\neq k} \left|\frac{z_j - z_k}{1 - \overline{z_k} z_j}\right| \geq c > 0, \qquad k = 1, 2, 3, \ldots \tag{3}$$

The necessity of (3) is easy. The open mapping theorem shows that given $\xi = \{\xi_j\}_{j=1}^\infty \in \ell^\infty$, there is an interpolating $f \in H^\infty(\mathbb{D})$ such that $\|f\|_{H^\infty(\mathbb{D})} \leq C\|\xi\|_\infty$. Let $B(z) = \prod_{k=1}^\infty \frac{|z_k|}{z_k} \frac{z_k - z}{1-\overline{z_k} z}$ be the Blaschke product with zeroes $\{z_k\}_{k=1}^\infty$ and $B_j(z) = \prod_{k\neq j} \frac{|z_k|}{z_k} \frac{z_k - z}{1-\overline{z_k} z}$. If f_j is such that $f_j(z_k) = \delta_{jk}$ and $\|f_j\|_{H^\infty(\mathbb{D})} \leq C$, then $\frac{1}{B_j(z_j)} = \frac{f_j}{B_j}(z_j) = \left\|\frac{f_j}{B_j}\right\|_{H^\infty(\mathbb{D})} \leq C$, which is (3). The rest of Carleson's proof made crucial use not only of Blaschke products, but also of duality. In the same paper he showed implicitly that the characterizing condition (3) can be rephrased in modern language as

$$\begin{aligned}&\left|\frac{z_j - z_k}{1 - \overline{z_k} z_j}\right| \geq c > 0 \text{ for } j \neq k, \text{ and}\\ &\mu = \sum_{j=1}^\infty \left(1 - |z_j|^2\right) \delta_{z_j} \text{ is a Carleson measure for } H^p(\mathbb{D}),\end{aligned}$$

where a positive Borel measure μ on the disk $\mathbb{D}$ is now said to be a Carleson measure for $H^p(\mathbb{D})$ if the embedding $H^p(\mathbb{D}) \subset L^p(d\mu)$ holds. Carleson later showed that μ is a Carleson measure if and only if

$$\mu(S(I)) \leq C|I|, \qquad \text{for all arcs } I \subset \mathbb{T},$$

where $S(I) = \left\{re^{i\theta} : \theta \in I \text{ and } 0 < 1 - r < |I|\right\}$, and solved the corona problem affirmatively in [16] by demonstrating the absence of a corona in the maximal ideal space of $H^\infty(\mathbb{D})$.

1.2 Peter Jones' proof in the upper half plane

We follow the excellent exposition in Seip [32]. First we show that the analogue of (3) in the upper half plane,

$$\prod_{j:j\neq k} \left|\frac{z_k - z_j}{z_k - \overline{z_j}}\right| \geq c > 0, \qquad k = 1, 2, 3, ... \tag{4}$$

implies the following separation condition on Z, and the following Carleson condition on the associated measure $\mu = \mu_Z = \sum_{j=1}^\infty y_j \delta_{z_j}$ where $z_j = x_j + iy_j$:

$$\begin{aligned} &\left|\frac{z_k - z_j}{z_k - \overline{z_j}}\right| \geq c > 0 \text{ for } j \neq k, \text{ and} \\ &\mu(T(z_k)) = \sum_{z_j \in T(z_k)} y_j \leq C y_k = C \times \text{ height of } T(z_k), \end{aligned} \tag{5}$$

where the *tent* $T(z_k)$ is the equilateral triangle with vertex z_k and opposite side on the x-axis. Clearly the separation condition in (5) is implied by (4). Fix k. If $B_k(z) = \prod_{j:j\neq k} \frac{z - z_j}{z - \overline{z_j}}$ denotes the Blaschke product with zeroes $Z \setminus \{z_k\}$, then from $-\ln t \geq 1 - t$ for $t > 0$ we have that

$$2\ln\frac{1}{c} \geq -\ln|B_k(z_k)|^2 = -\sum_{j\neq k} \ln\left|\frac{z_k - z_j}{z_k - \overline{z_j}}\right|^2 \geq \sum_{j\neq k}\left(1 - \left|\frac{z_k - z_j}{z_k - \overline{z_j}}\right|^2\right) = \sum_{j\neq k} \frac{4y_k y_j}{|z_k - \overline{z_j}|^2}.$$

Now if $z_j \in T(z_k)$, then $|z_k - \overline{z_j}|^2 \leq 4y_k^2$ and so,

$$\sum_{z_j \in T(z_k)} y_j \leq y_k \sum_{z_j \in T(z_k)} \frac{4y_k y_j}{|z_k - \overline{z_j}|^2} \leq \left(2\ln\frac{1}{c}\right) y_k.$$

Finally, we remark that a covering lemma shows that we have the extended inequality,

$$\mu(T(z)) = \sum_{z_j \in T(z)} y_j \leq \left(4\ln\frac{1}{c}\right) y, \qquad \text{for all } z = x + iy. \tag{6}$$

To see this, fix z, let I_{z_j} be the base of the tent $T(z_j)$, and let $\{I_{z_k}\}_{k\in E}$ be a subcollection of the intervals $\left\{I_{z_j} : z_j \in T(z)\right\}$ having union $\cup_{z_j\in T(z)} I_{z_j}$, and finite overlap 2 (we may assume the sequence finite for this). Then

$$\begin{aligned}
\sum_{z_j\in T(z)} y_j &\le \sum_{k\in E} \sum_{z_j\in T(z_k)} y_j \le \sum_{k\in E} \left(2\ln\frac{1}{c}\right) y_k \\
&= 2\ln\frac{1}{c} \sum_{k\in E} \frac{\sqrt{3}}{2} |I_{z_k}| \le 4\ln\frac{1}{c}\frac{\sqrt{3}}{2} |I_z| \\
&= \left(4\ln\frac{1}{c}\right) y.
\end{aligned}$$

1.2.1 A linear interpolation operator

We wish to construct bounded analytic functions f_j in the upper half plane such that

$$\begin{aligned}
f_k(z_j) &= \delta_{j,k} \\
\sum_{k=1}^{\infty} |f_k(z)| &\le C < \infty, \qquad \text{for all } z.
\end{aligned} \tag{7}$$

With this done, the function $f = \sum_{k=1}^{\infty} a_k f_k(z)$ takes the value a_k at z_k and has $\|f\|_{H^\infty} \le C \left\| \{a_j\}_{j=1}^{\infty} \right\|_{\infty}$, showing that Z is an interpolating sequence for the upper half plane. A natural choice to try first is

$$f_k(z) = \frac{B_k(z)}{B_k(z_k)},$$

since $f_k \in H^\infty$ satisfies the first condition in (7), $f_k(z_j) = \delta_{j,k}$. However, the Blaschke products have modulus one on the boundary $\mathbb{R}$ (again it suffices to consider only finite sequences Z), and so typically $\sum_{k=1}^{\infty} |f_k(z)| \to \infty$ as $z \to \mathbb{R}$. To remedy the problem at infinity, we introduce the bounded holomorphic function $g(z) = \frac{-4}{(z+i)^2}$ that vanishes at ∞ (note $g(i) = 1$) and try the modification

$$f_k(z) = \frac{B_k(z)}{B_k(z_k)} g_k(z),$$

where $g_k(z) = g\left(\frac{z-x_k}{y_k}\right) = \frac{-4y_k^2}{(z-\overline{z_k})^2}$ is g rescaled and translated to take the value 1 at z_k. Of course the modified f_k's still fail to satisfy the second condition in (7). This could be fixed by further modifying the f_k by multiplying by an exponential factor of the form $e^{-\alpha \sum_{y_j \le y_k} \left| \frac{g_k(z)}{B_k(z_k)} \right|}$, but this has the defect that it fails to be holomorphic. Instead, we use a positive harmonic majorant u for $|g|$ and the corresponding exponential factor

$$e^{-\alpha \sum_{y_j \le y_k} \frac{u_j(z) + i v_j(z)}{|B_j(z_j)|}},$$

where v is the harmonic conjugate of u in the upper half plane. Here are the details.

Let $u(x, y) = \frac{4(1+y)}{x^2+(1+y)^2} = \frac{4(1+y)}{|z+i|^2}$ be 4 times the Poisson kernel for the half space $\mathbb{R} \times (-1, \infty)$ so that u is positive harmonic and coincides with $|g(x, y)| = \frac{4}{x^2+(1+y)^2}$ on the boundary $y = 0$. Obviously, $u \ge |g|$ (or more generally by the maximum principle). Define $u_k(z) = u\left(\frac{z - x_k}{y_k}\right) = \frac{4 y_k (y_k + y)}{|z - \overline{z_k}|^2}$ and

$$U_n(z) = \sum_{y_j \le y_n} \frac{u_j(z)}{|B_j(z_j)|}.$$

Note that $U_n(z)$ is finite, and so a positive harmonic function, by the extended Carleson inequality (6) with $z(\ell) = x + i 2^\ell y$:

$$\begin{aligned} U_n(z) &\le c^{-1} \sum_{\substack{z_j \in T(z(2)) \\ y_j \le y_n}} \frac{4 y_j (y_j + y)}{|z - \overline{z_j}|^2} + c^{-1} \sum_{\ell=2}^{\infty} \sum_{\substack{z_j \in T(z(\ell)) \setminus T(z(\ell-1)) \\ y_j \le y_n}} \frac{4 y_j (y_j + y)}{|z - \overline{z_j}|^2} \\ &\le C c^{-1} \frac{y_n + y}{y^2} \sum_{\substack{z_j \in T(z(2)) \\ y_j \le y_n}} y_j + C c^{-1} \sum_{\ell=2}^{\infty} \frac{y_n + y}{(2^\ell y)^2} \sum_{z_j \in T(z(\ell))} y_j \\ &\le C c^{-1} \sum_{\ell=1}^{\infty} \frac{y_n + y}{(2^\ell y)^2} \left(4 \ln \frac{1}{c}\right) 2^\ell y \\ &\le \frac{C}{c} \left(\ln \frac{1}{c}\right) \frac{y_n + y}{y} < \infty, \end{aligned}$$

but that this bound is *not* uniform in z and n. With V_n the harmonic conjugate of U_n in the upper half plane and $G_n = U_n + i V_n$, we set

$$f_n(z) = \frac{B_n(z)}{B_n(z_n)} g_n(z) e^{-\alpha \{G_n(z) - G_n(z_n)\}},$$

for a positive constant α to be chosen momentarily. Clearly, the f_n are holomorphic in the upper half plane and satisfy the first condition in (7). We estimate the second condition in terms of the quantity

$$\triangle(Z) = \sup_{n \ge 1} U_n(z_n) \le \frac{C}{c} \ln \frac{1}{c}$$

to obtain (**!!**)

$$\sum_{n=1}^{\infty}|f_n(z)| \leq \sum_{n=1}^{\infty}\left|\frac{g_n(z)}{B_n(z_n)}\right| e^{-\alpha\{U_n(z)-\triangle(Z)\}}$$
$$\leq \left(\frac{e^{\triangle(Z)}}{\alpha}\right)\sum_{n=1}^{\infty}\frac{\alpha u_n(z)}{|B_n(z_n)|}e^{-\sum_{y_j\leq y_n}\frac{\alpha u_j(z)}{|B_j(z_j)|}}$$
$$\leq \left(\frac{e^{\alpha\triangle(Z)}}{\alpha}\right)\int_0^{\infty}e^{-t}dt = \frac{e^{\alpha\triangle(Z)}}{\alpha}$$
$$\leq e\,\triangle(Z),$$

with $\alpha = \frac{1}{\triangle(Z)}$, since if the z_j are ordered with decreasing y_j and $t_n = \sum_{y_j\leq y_n}\frac{\alpha u_j(z)}{|B_j(z_j)|}$ for any fixed z, then

$$\int_0^{\infty}e^{-t}dt \geq \sum_{n=1}^{\infty}\int_{t_{n+1}}^{t_n}e^{-t}dt \geq \sum_{n=1}^{\infty}(t_n - t_{n+1})\,e^{-t_n}.$$

Thus we have obtained (7) with constant $C = e\,\triangle(Z)$ (see [32] for a discussion of sharpness).

Remark 1 *If B is the Blaschke product with zeroes Z and we write the above linear operator of interpolation as*

$$f(z) = \sum_{j=1}^{\infty}a_j f_j(z) = B(z)\sum_{j=1}^{\infty}a_j\frac{h_j(z)}{B'(z_j)(z-z_j)},$$

then a computation reveals that the function $u = \frac{f}{B}$ solves the equation

$$\frac{\partial u}{\partial\overline{z}} \equiv \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial x}\right)u = 2\pi i\sum_{j=1}^{\infty}\frac{a_j y_j}{B_j(z_j)}\delta_{z_j}.$$

By approximation, the equation

$$\frac{\partial u}{\partial\overline{z}} = \mu$$

can then be solved with $u \in L^{\infty}$ for any complex measure μ such that $|\mu|$ is a Carleson measure: $|\mu|(T(z)) \leq Cy$ for all z in the upper half plane. See Peter Jones [23] on this topic.

1.3 Origins of interpolation in control theory

Here we follow the excellent expository lecture by John M^{c}Carthy [25]. Let P denote an engineering plant that accepts an input $u = \{u_n\}_{n=0}^{\infty}$ and produces an output $y = \{y_n\}_{n=0}^{\infty}$. We assume

1. **C**ausality: $u_n = 0$ for $n \leq N$ implies $y_n = 0$ for $n \leq N$,

2. **T**ime invariance: input $\{0, u_1, u_2, ...\}$ has output $\{0, y_1, y_2, ...\}$,

3. **S**tability: energy $\sum_{n=0}^{\infty} |y_n|^2$ of output y is at most a constant times that of the input u,

4. **L**inearity: input $u^1 + \lambda u^2$ has output $y^1 + \lambda y^2$.

The setup can be transferred to the unit disk $\mathbb{D}$ by viewing u as the sequence of coefficients in a power series about the origin: $\widetilde{u}(z) = \sum_{n=0}^{\infty} u_n z^n$. The energy of u is then given by

$$\sum_{n=0}^{\infty} |u_n|^2 = \sup_{0<r<1} \sum_{n=0}^{\infty} |u_n|^2 r^{2n} = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left|\widetilde{u}\left(re^{i\theta}\right)\right|^2 d\theta \equiv \|\widetilde{u}\|_{H^2(\mathbb{D})}^2 .$$

Properties **S**tability and **L**inearity show that the plant P is a continuous (bounded) linear operator on $H^2(\mathbb{D})$. Properties **C**ausality, **T**ime invariance and **L**inearity show that P is the operator M_φ of multiplication by $\varphi = P1$: if $f = \sum_{n=0}^{N} a_n z^n$ is a polynomial, then

$$Pf(z) = \sum_{n=0}^{N} a_n P(z^n 1) = \sum_{n=0}^{N} a_n z^n P1(z) = f(z)\varphi(z) = M_\varphi f(z),$$

which by the density of polynomials in $H^2(\mathbb{D})$, extends to all $f \in H^2(\mathbb{D})$. Now the operator norm squared of M_φ satisfies

$$\|M_\varphi\|_{op}^2 = \sup_{\|f\|_{H^2(\mathbb{D})}^2 \leq 1} \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left|\varphi\left(re^{i\theta}\right)\right|^2 \left|f\left(re^{i\theta}\right)\right|^2 d\theta \leq \|\varphi\|_{H^\infty(\mathbb{D})}^2 ,$$

and with $f = 1$ and M_φ iterated n times, we have the reverse inequality

$$\left\{\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left|\varphi\left(re^{i\theta}\right)\right|^{2n} d\theta\right\}^{\frac{1}{2n}} = \left\{\left\|M_\varphi^n 1\right\|_{H^2(\mathbb{D})}^2\right\}^{\frac{1}{2n}} \leq \left\{\|M_\varphi\|_{op}^{2n}\right\}^{\frac{1}{2n}} = \|M_\varphi\|_{op} ,$$

which yields $\|\varphi\|_{H^\infty(\mathbb{D})} \leq \|M_\varphi\|_{op}$ upon letting $n \to \infty$.

Thus the plant P is identified with an element φ of $H^\infty(\mathbb{D})$, and viewed as a multiplier operator M_φ on the Hilbert space $H^2(\mathbb{D})$.

Now we suppose there is another plant W that modulates a noise input e that is added to the output x of P to get y. In order to minimize the effects of the noise, the engineers create a feedback loop by subtracting the output y from the input u to get v and then modifying v by another plant C called a compensator, to get w which is then fed back into the plant P to get x. See the diagram. Thus we have

$$y = PC(y-u) + We, \text{ or solving for } y,$$
$$y = (I+PC)^{-1} PCu + (I+PC)^{-1} We.$$

The requirements that the internal signals x, v and w be stable (the plant C need not be assumed stable) leads, after a small calculation, to the requirement that $F = C(I+PC)^{-1}$ satisfy the **S**tability property. i.e.

$$C(I+PC)^{-1} \in H^\infty(\mathbb{D}).$$

The compensator C that minimizes the effects of noise is that which minimizes the operator norm

$$\left\|(I+PC)^{-1} W\right\|_{op} = \|(I-PF)W\|_{op},$$

since a short computation reveals $(I+PC)(I-PF) = I$. However, factoring P and W into their **i**nner and **o**uter factors P_iP_o and W_iW_o, we have (the following can be made rigorous by assuming P and W are rational)

$$\begin{aligned}
\inf_{F\in H^\infty(\mathbb{D})} \|(I-PF)W\|_{op} &= \inf_{F\in H^\infty(\mathbb{D})} \|(W_o - P_iP_oFW_o)W_i\|_{op} \\
&= \inf_{F\in H^\infty(\mathbb{D})} \|W_o - P_iP_oFW_o\|_{op} \\
&= \inf_{G\in H^\infty(\mathbb{D})} \|W_o - P_iG\|_{op} \\
&= \inf_{H\in H^\infty(\mathbb{D})} \left\{\|H\|_{op} : H(z_j) = W_o(z_j), 1 \le j < \infty\right\},
\end{aligned}$$

where $Z = \{z_j\}_{j=1}^\infty$ is the zero set of $P = \varphi$.

Thus the noise reduction problem is equivalent to the interpolation problem of finding the bounded analytic function of least norm that takes the values of W_o on the sequence Z of zeroes of P. For a given plant P, solving this problem for all stable plants W can be achieved by solving the interpolation problem (2) of Carleson.

1.4 Hilbert space methods

In 1961, H. Shapiro and A. Shields [35] demonstrated that the interpolation property (2) is equivalent to *weighted interpolation* for Hardy spaces $H^p(\mathbb{D})$,

The map $f \to \left\{\left(1-|z_j|^2\right)^{\frac{1}{p}} f(z_j)\right\}_{j=1}^\infty$ takes $H^p(\mathbb{D})$ boundedly into and onto $\ell^p(Z_f)$

The factor $\left(1-|z_j|^2\right)^{\frac{1}{p}}$ forces the map to be into $\ell^\infty(Z_f)$, since if $z_j = r_j e^{i\theta_j}$, then

$$\begin{aligned}|f(z_j)| &= \left|\frac{1}{2\pi}\int_0^{2\pi} P_{r_j}(\theta_j - t) f^*\left(e^{it}\right) dt\right| \\ &\le \left\|P_{r_j}\right\|_{L^{p'}(\mathbb{T})} \|f^*\|_{L^p(\mathbb{T})} \\ &\le C(1-r_j)^{-\frac{1}{p}} \|f\|_{H^p(\mathbb{D})};\end{aligned}$$

the Carleson measure condition ensures that it is into $\ell^p(Z_f)$.

We now recast the case $p = 2$ of this result in a way that will emphasize the analogy with what comes later. The Hardy space $H^2(\mathbb{D})$ is a Hilbert space with reproducing kernel, i.e. the point evaluations $f \to f(z)$ are continuous linear functionals. This means that for each $z \in \mathbb{D}$, there is $k_z \in H^2(\mathbb{D})$, the reproducing kernel for z, which is characterized by the fact that for any $f \in H^2(\mathbb{D})$ we have $f(z) = \langle f, k_z\rangle$. A sequence $Z = \{z_j\}_{j=1}^\infty$ is an interpolating sequence for $H^2(\mathbb{D})$ if one can freely assign the values of an $H^2(\mathbb{D})$ function on Z, subject only to the natural size restriction. More precisely, Hilbert space basics ensure that if $f \in H^2(\mathbb{D})$, then the function

$$z_i \to \frac{f(z_i)}{\|k_{z_i}\|} = \left(1-|z_i|^2\right)^{\frac{1}{2}} f(z_i)$$

is a bounded function on Z. The sequence Z is called an interpolating sequence for $H^2(\mathbb{D})$ if all the functions on Z which are obtained in this way are in $\ell^2(Z)$, and if furthermore, every function in $\ell^2(Z)$ can be obtained in this way. For any $z \in \mathbb{D}$, set $\widetilde{k_z} = \frac{k_z}{\|k_z\|}$, and note that by the Cauchy-Schwarz inequality,

$$\left|\left\langle \widetilde{k_z}, \widetilde{k_w}\right\rangle\right| \le 1, \qquad z, w \in \mathbb{D}.$$

If Z is an interpolating sequence, then it must be possible, given any i and j, to find $f \in H^2(\mathbb{D})$ so that

$$\left(1-|z_i|^2\right)^{\frac{1}{2}} f(z_i) = 0, \left(1-|z_j|^2\right)^{\frac{1}{2}} f(z_j) = 1;$$

and to do this with control on the size of $\|f\|$: say $\|f\| \le C$. This implies a weak separation condition on the points of Z which is necessary for Z to be an interpolating sequence: there is $\varepsilon > 0$ so that for all $i \neq j$, $\left|\left\langle \widetilde{k_{z_i}}, \widetilde{k_{z_j}}\right\rangle\right| \le 1 - \varepsilon$. Indeed, simply minimize over $\lambda \in \mathbb{C}$ the right side of

$$1 = \frac{f(z_j)^2}{\left\|k_{z_j}\right\|^2} = \frac{\left\langle f, k_{z_j} - \lambda k_{z_i}\right\rangle^2}{\left\|k_{z_j}\right\|^2} \le C^2 \frac{\left\|k_{z_j} - \lambda k_{z_i}\right\|^2}{\left\|k_{z_j}\right\|^2}.$$

This states that there is a uniform lower bound on the angle between reproducing kernels associated to the points in Z. An equivalent geometric statement is that there is a uniform lower bound on the hyperbolic distances $\beta(z_i, z_j)$, where the hyperbolic distance is obtained by transporting the Euclidean Riemannian metric at the origin to points of the disk via the automorphism group $z \to e^{i\theta}\frac{w-z}{1-\overline{w}z}$. The result of Shapiro and Shields can now be restated as saying that interpolating sequences for $H^2(\mathbb{D})$ are characterized by the following two conditions:

$$\text{There is } \varepsilon > 0 \text{ so that } \left|\left\langle \widetilde{k_{z_i}}, \widetilde{k_{z_j}} \right\rangle\right| \leq 1 - \varepsilon \text{ for all } i \neq j, \tag{8}$$

and

$$\sum_{j=1}^{\infty} \left\|k_{z_j}\right\|^{-2} \delta_{z_j} \text{ is a Carleson measure for } H^2(\mathbb{D}). \tag{9}$$

Interpolation problems, multiplier questions and Carleson measure characterizations have been studied by various authors in other classical function spaces on the disk, including certain of the spaces $B_p^\alpha(\mathbb{D})$ normed by

$$\sum_{k=0}^{m-1} \left|f^{(k)}(0)\right| + \left\{\int_{\mathbb{D}} \left|\left(1-|z|^2\right)^{m+\alpha} f^{(m)}(z)\right|^p d\lambda(z)\right\}^{\frac{1}{p}},$$

where $d\lambda(z) = \left(1-|z|^2\right)^{-2} dz$ is invariant measure on the disk, and for fixed α and p, the norms are equivalent for $(m+\alpha)p > 1$. This scale of spaces includes the Hardy space $H^2(\mathbb{D}) = B_2^{\frac{1}{2}}(\mathbb{D})$ with $\alpha = \frac{1}{2}$, the weighted Bergman spaces with $\alpha > \frac{1}{p}$, and the weighted Dirichlet-type spaces with $0 < \alpha < \frac{1}{p}$. See for example the recent book by K. Seip [32], which contains an in depth discussion of the history of interpolating sequences for Hilbert spaces of functions of a single variable. Interpolation proved more difficult for the family of analytic Besov spaces $B_p(\mathbb{D}) = B_p^0(\mathbb{D})$ on the disk, the prototypical Möbius invariant spaces, which do not admit *any* infinite Blaschke products - the Dirichlet norm $\|f\|_{B_2(\mathbb{D})}$ measures the square root of the area of the range of f counting multiplicities, and so is infinite for every infinite Blaschke product f. These Besov spaces are also distinguished by being the limit of those spaces $B_p^\alpha(\mathbb{D})$ with $\alpha < 0$ that are too smooth (they admit continuous extensions to the closed disk $\overline{\mathbb{D}}$) to contain any infinite interpolating sequences. Indeed, if $\zeta \in \mathbb{T}$ is an accumulation point of an interpolating sequence Z, say $z_{j_k} \to \zeta$ as $k \to \infty$, then the subsequence $\{f(z_{j_k})\}_{k=1}^{\infty}$ has limit $f(\zeta)$, and hence we cannot interpolate any bounded sequence $\{\xi_j\}_{j=1}^{\infty}$ for which the subsequence $\{\xi_{j_k}\}_{k=1}^{\infty}$ has no limit.

1.4.1 The Dirichlet space

In a revolutionary paper in 1994, D. Marshall and C. Sundberg [24] used Hilbert space methods (and independently C. Bishop [11] used different techniques) to characterize interpolating sequences for the Dirichlet space $B_2(\mathbb{D})$ and its multiplier space $M_{B_2(\mathbb{D})}$ (note the connection $H^\infty(\mathbb{D}) = M_{H^2(\mathbb{D})}$) by the condition

$$\begin{aligned}&\beta(z_i, 0) \leq C\beta(z_i, z_j) \text{ for } i \neq j \text{ and}\\&\sum_{j=1}^{\infty}\left(1+\log\frac{1}{1-|z_j|^2}\right)^{-1}\delta_{z_j} \text{ is a } B_2(\mathbb{D})\text{-Carleson measure,}\end{aligned}$$

where β is the Bergman metric, and a positive Borel measure μ is a $B_2(\mathbb{D})$-Carleson measure if the embedding $B_2(\mathbb{D}) \subset L^2(d\mu)$ holds:

$$\int |f(z)|^2 \, d\mu(z) \leq C \|f\|^2_{B_2(\mathbb{D})}.$$

A crucial part of their argument used the Nevanlinna-Pick property of $B_2(\mathbb{D})$: an important consequence of this property for any space X of analytic functions on the disk, is that X then has the same interpolating sequences as its multiplier algebra M_X - see e.g. [32]. The above two conditions can be rewritten in exactly the same form as (8) and (9) with only the natural changes; the $\widetilde{k}'s$ must now be normalized reproducing kernels for the Dirichlet space and the measure must be a Dirichlet space Carleson measure.

1.5 Interpolation in Besov spaces

More recently, in 2002 in [12], B. Böe has extended the above theorem to all $1 < p < \infty$ by a long and clever construction involving Carleson measures, that was in turn based on an earlier construction in [24] (see also the analogous construction on trees in Section 6 of [7]), together with, in Böe's words, a "curious lemma" on unconditional basic sequences $\{f_j\}_{j=1}^{\infty}$ of *positive* functions in a Lebesgue space $L^q(d\mu)$:

$$\left\|\sum_{j=1}^{\infty}|a_j f_j|\right\|_{L^q(d\mu)} \approx C_q \left\|\sup_{j\geq 1}|a_j f_j|\right\|_{L^q(d\mu)}.$$

The sequence $\{f_j\}_{j=1}^{\infty}$ is an unconditional basic sequence if $\left\|\sum_{j=1}^{\infty} b_j f_j\right\| \leq C \left\|\sum_{j=1}^{\infty} a_j f_j\right\|$ whenever $|b_j| \leq |a_j|$.

1.5.1 Higher dimensions

In Arcozzi, Rochberg and Sawyer [8], Böe's results were extended to the analytic Besov spaces $B_p(\mathbb{B}_n)$ on the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$ for $n > 1$. We note that the corresponding questions for the Hardy spaces on the ball remain open in higher dimensions, due in part to the lack of Blaschke products, but also since the relevant separation condition fails to be sparse enough to accommodate the "hands-on" type of construction used by Böe. The Nevanlinna-Pick property fails as well.

At least two difficulties arise immediately in higher dimensions. Böe makes use of Stegenga's 1980 characterization [37] of $B_2(\mathbb{D})$-Carleson measures by a capacity condition, as well as later extensions to $p>1$:

$$\mu(T(E)) \leq C\, cap_p(E),$$

for all compact subsets E (or equivalently finite unions of arcs) of the circle $\mathbb{T}$, and where $T(E)$ denotes the Carleson tent associated to E, and

$$cap_p(E) = \inf\left\{\int_{-\pi}^{\pi} f\left(e^{i\theta}\right)^p d\theta : f \geq 0 \text{ and } \int_{-\pi}^{\pi} f\left(e^{i(\phi-\theta)}\right)|\theta|^{-\frac{1}{2}} d\theta \geq \chi_E(\phi)\right\}.$$

This characterization is not yet available in higher dimensions, and as indicated in [12], seems difficult to check even in certain one-dimensional situations. Instead, the characterization in [7] involving the discrete Bergman tree condition,

$$\sum_{\beta\in\mathcal{T}:\beta\geq\alpha}\left(\sum_{\gamma\in\mathcal{T}:\gamma\geq\beta}\mu(\gamma)\right)^{p'} \leq C^{p'}\sum_{\beta\in\mathcal{T}:\beta\geq\alpha}\mu(\alpha) < \infty, \qquad \alpha\in\mathcal{T}, \tag{10}$$

is extended to higher dimensions where it plays a crucial role both as a substitute for a capacity condition, and in generalizing the clever Carleson measure construction of Böe in [12].

The second difficulty runs deeper. It is connected to the fact that the reproducing kernel $k_w(z) = \log\frac{1}{1-\overline{w}z}$ for $B_p(\mathbb{D})$ has derivative $\overline{w}\frac{1}{1-\overline{w}z}$ where $\frac{1}{1-\overline{w}z}$ has positive real part, and that this positivity played a crucial role in part of Böe's argument when $p<2$. In particular his "curious lemma", which deals with positive functions, is applied to those real parts. This property persists in dimension n only for $1<p<1+\frac{1}{n-1}$, where the analogous derivative $\mathcal{R}^{\alpha}_{\frac{n+1+\alpha}{p'}}$ of the reproducing kernel $k_w^{\alpha,p}(z)$ is

$$\mathcal{R}^{\alpha}_{\frac{n+1+\alpha}{p'}}k_w^{\alpha,p}(z) = (1-\overline{w}\cdot z)^{-\frac{n+1+\alpha}{p'}}, \qquad \alpha > -1,$$

which has positive real part only when $\frac{n+1+\alpha}{p'}\leq 1$ for some $\alpha>-1$, i.e. $p<1+\frac{1}{n-1}$.

As a consequence, the aforementioned "curious lemma" of Böe only generalizes to prove the necessity of the discrete tree condition for $M_{B_p(\mathbb{B}_n)}$ interpolation in the thin range $1<p<1+\frac{1}{n-1}$ (where reproducing kernels for $B_p(\mathbb{B}_n)$ have the requisite positivity property). To combat the failure of this positivity property for larger p, we introduce "holomorphic" Besov spaces $HB_p(\mathcal{T}_n)$ on Bergman trees $\mathcal{T}_n$ whose reproducing kernels *do* enjoy a suitable positivity property, and such that the restriction map from $B_p(\mathbb{B}_n)$ to $HB_p(\mathbb{B}_n\mathcal{T}_n)$, as well as the restriction map between their multiplier spaces, is bounded. This requires a great deal of effort and is accomplished in the latter half of the paper [8]. Another consequence is that our one-dimensional proof of the characterization of Carleson measures by the discrete tree condition extends to dimension n only in the thin range of p given by $1 < p < 1 + \frac{1}{n-1}$. A TT^* argument lifts the proof to the larger range $1 < p < 2 + \frac{1}{n-1}$, beyond which we are unable to proceed at this time.

1.5.2 The Drury-Arveson space

In [9], Carleson measures are characterized in particular for the Besov-Sobolev spaces $B_2^\sigma(\mathbb{B}_n)$, $0 \le \sigma < \frac{1}{2}$, by the tree condition

$$\sum_{\beta \in \mathcal{T}_n : \beta \ge \alpha} 2^{2\sigma d(\beta)} \left(\sum_{\gamma \in \mathcal{T}_n : \gamma \ge \beta} \mu(\gamma) \right)^2 \le C \sum_{\beta \in \mathcal{T}_n : \beta \ge \alpha} \mu(\alpha) < \infty, \qquad \alpha \in \mathcal{T}_n.$$

Combined with recent work of Böe [12] and Agler and M^{c}Carthy [1], the above Carleson measure characterization yields that a sequence $Z = \{z_j\}_{j=1}^\infty$ in the ball $\mathbb{B}_n$ is an interpolating sequence for $B_2^\sigma(\mathbb{B}_n)$ *if and only if* it is an interpolating sequence for the multiplier algebra $M_{B_2^\sigma(\mathbb{B}_n)}$ *if and only if* Z is separated in the sense that $\inf_{i \ne j} \beta(z_i, z_j) > 0$ and the measure $\mu = \sum_{j=1}^\infty \left(1 - |z_j|^2\right)^{2\sigma} \delta_{z_j}$ satisfies the above tree condition. This characterization of Carleson measures fails for the Drury-Arveson Hardy space $B_2^{\frac{1}{2}}(\mathbb{B}_n)$ (the "endpoint" case), and is instead replaced by the simple condition $2^{d(\alpha)} I^* \mu(\alpha) \le C$, $\alpha \in \mathcal{T}_n$, together with the "split" tree condition

$$\sum_{k \ge 0} \sum_{\gamma \ge \alpha} 2^{d(\gamma) - 2k} \sum_{(\delta, \delta') \in \mathcal{G}^{(k)}(\gamma)} I^* \mu(\delta) I^* \mu(\delta') \le C I^* \mu(\alpha), \qquad \alpha \in \mathcal{T}_n.$$

The restriction $(\delta, \delta') \in \mathcal{G}^{(k)}(\gamma)$ in the sum above means that we sum over all pairs (δ, δ') of grand^k-children of γ that have γ as their minimum in $\mathcal{T}_n$, and do not lie in a common ring in the quotient tree $\mathcal{R}_n$, but whose immediate predecessors do.

von Neumann's inequality We can now give a sharp estimate for the generalization of von Neumann's celebrated inequality [26] to the complex ball by Drury [21]. Let $A = (A_1, ..., A_n)$ be an n-contraction on a complex Hilbert space $\mathcal{H}$, i.e. an n-tuple of linear operators on $\mathcal{H}$ satisfying

$$A_j A_k = A_k A_j \text{ for all } 1 \le j, k \le n, \text{ and } \sum_{j=1}^n \|A_j h\|^2 \le \|h\|^2 \text{ for all } h \in \mathcal{H}.$$

Equivalently, the A_j commute and the row operator $A = (A_1, ..., A_n)$ is bounded with norm one from $\bigoplus_{j=1}^n \mathcal{H}$ to $\mathcal{H}$: $\left\| \sum_{j=1}^n A_j h_j \right\|^2 \le \sum_{j=1}^n \|h_j\|^2$. Drury showed in [21] that if f is a complex polynomial on $\mathbb{C}^n$, then

$$\sup_{A \text{ an } n\text{-contraction}} \|f(A)\| = \|f\|_{M_{\mathcal{K}(\mathbb{B}_n)}}, \tag{11}$$

where $\|f(A)\|$ is the operator norm of $f(A)$ on $\mathcal{H}$, and $\|f\|_{M_{\mathcal{K}(\mathbb{B}_n)}}$ denotes the multiplier norm of the polynomial f on Drury's Hardy space of holomorphic functions

$$\mathcal{K}(\mathbb{B}_n)=\left\{\sum_k a_k z^k,\ z\in\mathbb{B}_n:\sum_k |a_k|^2\frac{k!}{|k|!}<\infty\right\},$$

denoted by H_n^2 in Arveson [10] (who also proves (11) in Theorem 8.1). The original inequality of von Neumann in dimension $n=1$ is

$$\sup_{A\text{ a contraction}}\|f(A)\|=\|f\|_{M_{H^2(\mathbb{D})}}=\|f\|_{H^\infty(\mathbb{D})}.$$

Chen [19] has identified the Drury-Arveson Hardy space $\mathcal{K}(\mathbb{B}_n)=H_n^2$ as the Besov-Sobolev space $B_2^{\frac{1}{2}}(\mathbb{B}_n)$ consisting of those holomorphic functions $\sum_k a_k z^k$ in the ball with coefficients a_k satisfying

$$\sum_k |a_k|^2\frac{|k|^{n-1}(n-1)!k!}{(n-1+|k|)!}<\infty.$$

Indeed, the coefficient multipliers in the definitions of $\mathcal{K}(\mathbb{B}_n)$ and $B_2^{\frac{1}{2}}(\mathbb{B}_n)$ are easily seen to be comparable. It now follows that the multiplier norms are equivalent:

$$\|f\|_{M_{\mathcal{K}(\mathbb{B}_n)}}\approx\|f\|_{M_{B_2^{\frac{1}{2}}(\mathbb{B}_n)}}.$$

We note in passing that a number of important operator-theoretic properties of the Hilbert space H_n^2 are developed by Arveson in [10] that establish its central position in multivariable operator theory.

Ortega and Fabrega [28] have shown that f is a pointwise multiplier on $B_2^{\frac{1}{2}}(\mathbb{B}_n)$ if and only if f is a bounded holomorphic function and the measure

$$d\mu_f(z)=\left|R^{\left(\frac{n+1}{2}\right)}f(z)\right|^2\left(1-|z|^2\right)dz$$

is a Carleson measure for the Drury-Arveson Hardy space $B_2^{\frac{1}{2}}(\mathbb{B}_n)$. In fact, we can replace $d\mu_f$ by any of the measures

$$d\mu_f^m(z)=\left|f^{(m)}(z)\right|^2\left(1-|z|^2\right)^{2m-n}dz,\qquad m>\frac{n-1}{2}.$$

Using this we obtain the following estimate.

Theorem 1 *For any $m>\frac{n-1}{2}$,*

$$\sup_{A\text{ an }n\text{-contraction}}\|f(A)\|\approx\|f\|_\infty+\sup_{\alpha\in\mathcal{T}_n}\sqrt{2^{d(\alpha)}I^*\mu_f^m(\alpha)}\tag{12}$$
$$+\sup_{\alpha\in\mathcal{T}_n}\sqrt{\frac{1}{I^*\mu_f^m(\alpha)}\sum_{k\geq 0}\sum_{\gamma\geq\alpha}2^{d(\gamma)-k}\sum_{\delta,\delta'\in\mathcal{G}^{(k)}(\gamma)}I^*\mu_f^m(\delta)\,I^*\mu_f^m(\delta')},$$

for all polynomials f on $\mathbb{C}^n$.

The right side of (12) can of course be transported onto the ball using that $\cup_{\beta\geq\alpha}K_\beta$ is an appropriate nonisotropic tent in $\mathbb{B}_n$, and that $2^{-d(\alpha)} \approx \left(1-|z|^2\right)$ for $z \in K_\alpha$.

Complete Nevanlinna-Pick kernels The *universal* complete Nevanlinna-Pick property of the Drury-Arveson space $H_n^2 = B_2^{\frac{1}{2}}(\mathbb{B}_n)$ provides another application of Carleson measures for H_n^2. We recall the theory of Hilbert spaces with a complete Nevanlinna-Pick kernel $k(x,y)$ in Agler and M^{c}Carthy [1], keeping in mind the classical model of the Szego kernel $k(x,y) = \frac{1}{1-\overline{x}y}$ on the unit disk $\mathbb{D}$. Let X be an infinite set and $k(x,y)$ be a positive definite kernel function on X, i.e. for all finite subsets $\{x_i\}_{i=1}^m$of X,

$$\sum_{i,j=1}^m a_i\overline{a_j}k(x_i,x_j) \geq 0 \text{ with equality } \Leftrightarrow \text{ all } a_i = 0.$$

Denote by $\mathcal{H}_k$ the Hilbert space obtained by completing the space of finite linear combinations of k_{x_i}'s, where $k_x(y) = k(x,y)$, with respect to the inner product

$$\left\langle \sum_{i=1}^m a_i k_{x_i}, \sum_{j=1}^m b_j k_{y_j} \right\rangle = \sum_{i,j=1}^m a_i \overline{b_j} k(x_i,y_j).$$

The kernel k is called a *complete Nevanlinna-Pick kernel* if the solvability of the matrix-valued Nevanlinna-Pick problem is characterized by the contractivity of a certain family of adjoint operators $R_{x,\Lambda}$ (we refer to [1] for an explanation of this generalization of the classical Pick condition).

Let $a_n(x,y) = \frac{1}{1-\langle y,x\rangle}$ for $x,y \in \mathbb{B}_n$, the unit ball in n-dimensional Hilbert space ℓ_n^2 of cardinality n, and denote the Hilbert space $\mathcal{H}_{a_n}$ by H_n^2 (so that $H_n^2 = B_2^{\frac{1}{2}}(\mathbb{B}_n)$ when n is finite). Theorem 4.2 of [1] shows that if k is an irreducible kernel on X, and if for any fixed point $x_0 \in X$, the Hermitian form

$$F(x,y) = 1 - \frac{k(x,x_0)\,k(x_0,y)}{k(x,y)\,k(x_0,y_0)}$$

has rank n, then k is a complete Nevanlinna-Pick kernel if and only if there is an injective function $f : X \to \mathbb{B}_n$ and a nowhere vanishing function δ on X such that

$$k(x,y) = \overline{\delta(x)}\delta(y)\,a_n(f(x),f(y)) = \frac{\overline{\delta(x)}\delta(y)}{1-\langle f(x),f(y)\rangle}.$$

Moreover, if this happens, then the map $k_x \to \overline{\delta(x)}\,(a_n)_{f(x)}$ extends to an isometric linear embedding T of $\mathcal{H}_k$ into H_n^2. If in addition there is a topology on X so that k is continuous on $X \times X$, then the map f will be a continuous embedding of X into $\mathbb{B}_n$.

As a result, the Carleson embedding norm $\|\mu\|_{Carleson}$ of $H_n^2 \subset L^2(\mu)$ can be used to give a necessary and sufficient condition for Carleson measures on any Hilbert space $\mathcal{H}_k$ with a complete continuous irreducible Nevanlinna-Pick kernel k. To see this, consider first the case where the Hermitian form F above has finite rank (F is positive semi-definite if k is a complete Nevanlinna-Pick kernel by Theorem 2.1 in [1]). Denote by $f_*\nu$ the pushforward of a Borel measure ν on X under the continuous map f. If μ is a positive Borel measure on X, then μ is $\mathcal{H}_k$-Carleson, i.e.

$$\int_X |h(x)|^2 \, d\mu(x) \leq C \|h\|_{\mathcal{H}_k}^2, \qquad h \in \mathcal{H}_k, \tag{13}$$

if and only if the measure $\mu^\natural = f_*\left(|\delta|^2 \mu\right)$ is $B_2^{\frac{1}{2}}(\mathbb{B}_n)$-Carleson, i.e.

$$\int_{\mathbb{B}_n} |G|^2 \, d\mu^\natural \leq C \|G\|_{B_2^{\frac{1}{2}}(\mathbb{B}_n)}^2, \qquad G \in B_2^{\frac{1}{2}}(\mathbb{B}_n). \tag{14}$$

Indeed, the functions $h = \sum_{i=1}^m c_i k_{x_i}$ are dense in $\mathcal{H}_k$ and have norm squared

$$\left\langle \sum_{i=1}^m c_i k_{x_i}, \sum_{i=1}^m c_i k_{x_i} \right\rangle = \sum_{i,j=1}^m c_i \overline{c_j} k(x_i, x_j) = \sum_{i,j=1}^m c_i \overline{c_j} \overline{\delta(x_i)} a_n(f(x_i), f(x_j)) \delta(x_j),$$

which coincides with the norm squared in H_n^2 of $H = Th = \sum_{i=1}^m c_i \overline{\delta(x_i)} (a_n)_{f(x_i)}$:

$$\left\langle \sum_{i=1}^m c_i \overline{\delta(x_i)} (a_n)_{f(x_i)}, \sum_{i=1}^m c_i \overline{\delta(x_i)} (a_n)_{f(x_i)} \right\rangle = \sum_{i,j=1}^m c_i \overline{\delta(x_i)} \overline{c_j} \delta(x_j) a_n(f(x_i), f(x_j)).$$

The change of variable f yields

$$\begin{aligned}
\int_X |h(y)|^2 \, d\mu(y) &= \int_X \left| \sum_{i=1}^m c_i k(x_i, y) \right|^2 d\mu(y) \\
&= \int_X \left| \sum_{i=1}^m c_i \overline{\delta(x_i)} a_n(f(x_i), f(y)) \right|^2 |\delta(y)|^2 \, d\mu(y) \\
&= \int_{f(X)} |H|^2 \, d\mu^\natural = \int_{\mathbb{B}_n} |H|^2 \, d\mu^\natural,
\end{aligned}$$

and it follows immediately that (14) implies (13).

For the converse, we observe that if $G \in H_n^2 = B_2^{\frac{1}{2}}(\mathbb{B}_n)$, then we can write $G = H + J$ where $H \in T(\mathcal{H}_k)$ and J is orthogonal to the closed subspace $T(\mathcal{H}_k)$.

Now since J is orthogonal to all functions $\overline{\delta(x)}(a_n)_{f(x)}$ with $x \in X$, and since δ is nonvanishing on X, we obtain that J vanishes on the subset $f(X)$ of the ball $\mathbb{B}_n$. Since $\mu^\natural$ is carried by $f(X)$ and orthogonal projections have norm 1, we then have with $H = Th$,

$$\int_{\mathbb{B}_n} |G|^2 \, d\mu^\natural = \int_{\mathbb{B}_n} |H|^2 \, d\mu^\natural = \int_X |h|^2 \, d\mu,$$

and

$$\|h\|_{\mathcal{H}_k} = \|H\|_{H_n^2} \leq \|G\|_{H_n^2}.$$

It follows immediately that (13) implies (14).

We can extend the above characterization to the case of infinite rank n, by characterizing Carleson measures on H_∞^2 (where ∞ denotes any infinite cardinal) as follows. Given a finite dimensional subspace L of $\mathbb{C}^\infty$, let P_L denote orthogonal projection onto L and set $\mathbb{B}_L = \mathbb{B}_\infty \cap L$, which we identify with the complex ball $\mathbb{B}_n$, $n = \dim L$. We say that a positive measure ν on $\mathbb{B}_L$ is $H_n^2(\mathbb{B}_L)$-Carleson if, when viewed as a measure on $\mathbb{B}_n$, $n = \dim L$, it is $H_n^2(\mathbb{B}_n)$-Carleson.

Lemma 2 *A positive Borel measure ν on $\mathbb{B}_\infty$ is H_∞^2-Carleson if and only if $(P_L)_* \nu$ is uniformly $H_n^2(\mathbb{B}_L)$-Carleson, $n = \dim L$, for all finite-dimensional subspaces L of $\mathbb{C}^\infty$.*

Proof. Suppose that $(P_L)_* \nu$ is uniformly $H_n^2(\mathbb{B}_L)$-Carleson for all finite-dimensional subspaces L of $\mathbb{C}^\infty$, $n = \dim L$. Let

$$f(z) = \sum_{i=1}^{m} c_i a_\infty (w_i, z) = \sum_{i=1}^{m} c_i \frac{1}{1 - \langle z, w_i \rangle} \tag{15}$$

for a finite sequence $\{w_i\}_{i=1}^m \subset \mathbb{B}_\infty$ (such functions are dense in H_∞^2). If we let L be the linear span of $\{w_i\}_{i=1}^m$ in $\mathbb{C}^\infty$, then since $f(P_L z) = f(z)$, we can view f as a function on both $\mathbb{B}_\infty$ and $\mathbb{B}_L$, and from our hypothesis we have

$$\int_{\mathbb{B}_\infty} |f|^2 \, d\nu = \int_{\mathbb{B}_L} |f|^2 \, d(P_L)_* \nu \leq C \|f\|_{H_n^2(\mathbb{B}_L)}^2 = C \|f\|_{H_\infty^2}^2, \tag{16}$$

with a constant C independent of f. Since such functions f are dense in H_∞^2, we conclude that ν is H_∞^2-Carleson. Conversely, given a subspace L and a measure ν that is H_∞^2-Carleson, functions of the form (15) with $\{w_i\}_{i=1}^m \subset \mathbb{B}_L$ are dense in $H_n^2(\mathbb{B}_L)$ and so (16) shows that $(P_L)_* \nu$ is a $H_n^2(\mathbb{B}_L)$-Carleson measure on $\mathbb{B}_L$ with constant C independent of L, $n = \dim L$.

The above lemma now yields the following characterization of Carleson measures on any Hilbert space $\mathcal{H}_k$ with a complete continuous irreducible Nevanlinna-Pick kernel k. Note that the irreducibility assumption on k can be removed using Lemma 1.1 of [1].

Theorem 3 *Let k be a complete continuous irreducible Nevanlinna-Pick kernel on a set X. With notation as above, and rank $(F) = m$, a positive measure μ on X is $\mathcal{H}_k$-Carleson if and only if there is a positive constant C such that*

$$\left\| (P_L)_* f_* \left(|\delta|^2 \mu \right) \right\|_{Carleson} \leq C,$$

for all finite-dimensional subspaces L of $\mathbb{C}^m$, and where $\|\nu\|_{Carleson}$ denotes the norm of the embedding $H_n^2(\mathbb{B}_L) \subset L^2(\mu)$ with $n = \dim L$. Note that $(P_L)_* f_* = (P_L \circ f)_*$.

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Taiwan lecture 2

Monday July 4 2005

1 Plan of the attack on interpolation in Besov spaces in higher dimensions

We will introduce a tree structure for the unit ball $\mathbb{B}_n$ by choosing a set $\mathcal{T}_n$ of points in the ball at roughly a fixed distance apart in the Bergman metric, and declaring a point $\beta \in \mathcal{T}_n$ to be a child of another point $\alpha \in \mathcal{T}_n$ if the Bergman ball around β lies just "beyond" the Bergman ball around α. This simple construction suffices for dealing with Carleson measures and sufficient conditions for interpolation. The construction must be significantly refined in order to deal with the holomorphic Besov spaces on trees. The refinement allows us to develop an effective discrete version of passing from spaces defined by a single derivative to spaces of functions defined using higher derivatives. We postpone the rigorous construction of $\mathcal{T}_n$ for now, since we only need it for the sake of completeness in stating our interpolation theorem below.

We use this tree structure $\mathcal{T}_n$ to characterize Carleson embeddings for Besov spaces $B_p(\mathbb{B}_n)$ on the ball,

$$\int_{\mathbb{B}_n} |f(z)|^p \, d\mu(z) \leq C \|f(z)\|^p_{B_p(\mathbb{B}_n)},$$

in terms of a discrete condition on the Bergman tree,

$$\sum_{\beta \in \mathcal{T}_n : \beta \geq \alpha} I^*\mu(\beta)^{p'} \leq C^{p'} I^*\mu(\alpha) < \infty, \qquad \alpha \in \mathcal{T}_n, \tag{1}$$

where $I^*\mu(\beta) = \sum_{\gamma \in \mathcal{T}_n : \gamma \geq \beta} \mu(\gamma)$ - we are however unable to obtain the necessity of the tree condition when $2 + \frac{1}{n-1} \leq p < \infty$. It turns out that the one-dimensional methods in [7], using a positivity property of the reproducing kernels, generalize to obtain the characterization in the thin range $1 < p < 1 + \frac{1}{n-1}$. A standard TT^* argument can be used to obtain the case $p = 2$ since the kernel $K(z, w)$ of TT^*

turns out to have appropriate "derivative" $\log\left(\frac{1}{1-\overline{w}\cdot z}\right)$, whose real part is positive. We combine the two techniques to obtain the larger range $1 < p < 2 + \frac{1}{n-1}$.

Interpolating sequences for both Besov spaces $B_p(\mathbb{B}_n)$ and their multiplier spaces $M_{B_p(\mathbb{B}_n)}$ are the treated by following, for the most part, the development in Böe [12]. In Theorem 5, weighted $B_p(\mathbb{B}_n)$ interpolation is characterized by separation and Carleson embedding conditions for all $1 < p < \infty$:

$$\beta(z_i, 0) \leq C\beta(z_i, z_j), i \neq j \text{ and}$$
$$\sum_{j=1}^{\infty} \left\| k_{z_j}^{\alpha,p} \right\|_{B_{p'}}^{-p} \delta_{z_j} \text{ is a } B_p\text{-Carleson measure,}$$

where $k_w^{\alpha,p}(z)$ is a reproducing kernel for B_p. In Theorem 6, the separation and tree conditions,

$$d(\alpha_i, o) \leq Cd(\alpha_i, \alpha_j), i \neq j \text{ and}$$
$$\sum_{j=1}^{\infty} \left(1 + \log \frac{1}{1 - \left|c_{\alpha_j}\right|^2}\right)^{1-p} \delta_{c_{\alpha_j}} \text{ satisfies the tree condition (1),}$$

are proved sufficient for $M_{B_p(\mathbb{B}_n)}$ interpolation, and the separation and Carleson embedding conditions above are proved necessary, for all $1 < p < \infty$. As well, in the range $p > 2n$, we prove that the separation and Carleson embedding conditions are sufficient. The necessity of the Carleson embedding condition is proved first for p in the two ranges $\left(1, 1 + \frac{1}{n-1}\right)$ and $[2, \infty)$. The first range exploits the positivity of a reproducing kernel on the ball, and the second range exploits the embedding of ℓ^q spaces in connection with Khinchine's inequality.

Necessity of the Carleson embedding condition for $M_{B_p(\mathbb{B}_n)}$ interpolation in the remaining range $1 + \frac{1}{n-1} \leq p < 2$ is much more difficult, and is the subject of later talks.

2 Interpolating sequences

Let $\{z_j\}_{j=1}^{\infty}$ be a sequence of points in the unit ball $\mathbb{B}_n$, and $1 < p < \infty$. Here we will prove that weighted ℓ^p interpolation for Besov spaces $B_p(\mathbb{B}_n)$ holds on the sequence $\{z_j\}_{j=1}^{\infty}$ if and only if the following separation condition and Carleson embedding hold;

$$\beta(z_i, 0) \leq C\beta(z_i, z_j) \text{ and} \tag{2}$$
$$\sum_{j=1}^{\infty} \left(1 + \log \frac{1}{1 - |z_j|^2}\right)^{1-p} \delta_{z_j} \text{ is a } B_p(\mathbb{B}_n)\text{-Carleson measure.}$$

We may assume without loss of generality that the points z_j occur as the centers of Bergman kubes for a corresponding sequence $\{\alpha_j\}_{j=1}^{\infty}$ in the Bergman tree $\mathcal{T}_n$

(this requires only a much weaker notion of separation, $\beta(z_i, z_j) \geq c > 0$). Note that

$$d(\alpha_i, o) \approx \beta(c_{\alpha_i}, 0) \approx \log \frac{1}{1 - |c_{\alpha_i}|^2},$$

where d denotes distance in the Bergman tree $\mathcal{T}_n$. Furthermore, the separation condition $\beta(z_i, 0) \leq C\beta(z_i, z_j)$ on the ball implies the tree separation condition $d(\alpha_i, o) \leq Cd(\alpha_i, \alpha_j)$, but not conversely. We then show that the analogue of condition (2) on the Bergman tree $\mathcal{T}_n$,

$$\begin{aligned} &\beta(z_i, 0) \leq C\beta(z_i, z_j) \text{ and} \\ &\sum_{j=1}^{\infty} (1 + d(\alpha_j, o))^{1-p} \delta_{\alpha_j} \text{ is a } B_p(\mathcal{T}_n)\text{-Carleson measure,} \end{aligned} \tag{3}$$

is sufficient for ℓ^∞ interpolation of the multiplier spaces $M_{B_p(\mathbb{B}_n)}$ on $\{z_j\}_{j=1}^\infty$ for all $1 < p < \infty$, and necessary provided $1 < p < 2 + \frac{1}{n-1}$. More precisely, for the sufficiency, we need (3) taken over all unitary rotations of the Bergman tree $\mathcal{T}_n$, since on average over the unitary group $\mathcal{U}_n$, tree distance is comparable to Bergman distance.

We are however able to show that (2) is sufficient for ℓ^∞ interpolation of the multiplier spaces $M_{B_p(\mathbb{B}_n)}$ for $p > 2n$, and that (2) is necessary for ℓ^∞ interpolation of the multiplier spaces $M_{B_p(\mathbb{B}_n)}$ for all $1 < p < \infty$. Since a measure μ is a $B_p(\mathcal{T}_n)$-Carleson measure if and only if it satisfies the tree condition (1), we see that one obstacle to obtaining a characterization of ℓ^∞ interpolation of the multiplier spaces $M_{B_p(\mathbb{B}_n)}$ in the exceptional range $\left[2 + \frac{1}{n-1}, 2n\right]$ is our failure to find a characterization of Carleson measures for $B_p(\mathbb{B}_n)$ when $p \geq 2 + \frac{1}{n-1}$. We consider mostly Besov spaces $B_p(\mathbb{B}_n)$ on the unit ball, and for convenience in notation, we will suppress the dependence on the ball by writing simply B_p for $B_p(\mathbb{B}_n)$.

2.1 Invariant metrics, measures and derivatives

We recall some basic definitions and properties from W. Rudin's book [29], K. Zhu's book [38] and our paper [8]. For $a \in \mathbb{B}_n$ let P_a denote orthogonal projection onto the one-dimensional complex subspace $\mathbb{C}a$ generated by a, i.e.

$$P_a z = \frac{z \cdot \overline{a}}{|a|^2} a, \tag{4}$$

and let $Q_a = I - P_a$ denote orthogonal projection onto the orthogonal complement of $\mathbb{C}a$. Define an involutive automorphism of the ball $\mathbb{B}_n$ by ([29], page 25)

$$\begin{aligned} \varphi_a(z) &= \frac{a - P_a z - \left(1 - |a|^2\right)^{\frac{1}{2}} Q_a z}{1 - z \cdot \overline{a}}, \\ &= \frac{a - \frac{z \cdot \overline{a}}{|a|^2} a - \left(1 - |a|^2\right)^{\frac{1}{2}} \left(z - \frac{z \cdot \overline{a}}{|a|^2} a\right)}{1 - z \cdot \overline{a}}, \end{aligned} \tag{5}$$

for $z \in \mathbb{B}_n$. Then $Aut(\mathbb{B}_n)$, the group of automorphisms of $\mathbb{B}_n$, consists of all maps $U\varphi_a$ where U is a unitary transformation and $a \in \mathbb{B}_n$.We have $\varphi_a(0) = a$, $\varphi_a(a) = 0$ and $\varphi_a \circ \varphi_a = I$. We also have the following identities ([29], Theorem 2.2.2),

$$\begin{aligned} \varphi_a'(0) &= -\left(1-|a|^2\right) P_a - \left(1-|a|^2\right)^{\frac{1}{2}} Q_a, \\ \varphi_a'(a) &= -\left(1-|a|^2\right)^{-1} P_a - \left(1-|a|^2\right)^{-\frac{1}{2}} Q_a, \\ 1-\overline{\varphi_a(w)}\cdot\varphi_a(z) &= \frac{(1-\overline{a}\cdot a)(1-\overline{w}\cdot z)}{(1-\overline{w}\cdot a)(1-\overline{a}\cdot z)}, \\ 1-|\varphi_a(z)|^2 &= \frac{\left(1-|a|^2\right)\left(1-|z|^2\right)}{|1-\overline{a}\cdot z|^2}, \end{aligned} \tag{6}$$

and ([29], Theorem 2.2.6)

$$J\varphi_a(z) = |\det \varphi_a'(z)|^2 = \left(\frac{1-|a|^2}{|1-\overline{a}\cdot z|^2}\right)^{n+1},$$

where $J\varphi_a(z)$ denotes the real Jacobian of φ_a at z. For example, in dimension $n=1$, $\varphi_a(z) = \frac{a-z}{1-\overline{a}z}$, and we have

$$\begin{aligned} 1-|\varphi_a(z)|^2 &= \frac{|1-\overline{a}z|^2 - |z-a|^2}{|1-\overline{a}z|^2} \\ &= \frac{\left\{1-2\operatorname{Re}(\overline{a}z)+|a|^2|z|^2\right\} - \left\{|z|^2 - 2\operatorname{Re}(\overline{a}z)+|a|^2\right\}}{|1-\overline{a}z|^2} \\ &= \frac{\left(1-|a|^2\right)\left(1-|z|^2\right)}{|1-\overline{a}\cdot z|^2}. \end{aligned}$$

An invariant measure on $\mathbb{B}_n$ is given by ([29], Theorem 2.2.6)

$$d\lambda_n(z) = \left(1-|z|^2\right)^{-n-1} dz.$$

The invariance of $d\lambda_n$ follows from the above Jacobian formula and the last identity in (6).

An invariant metric on $\mathbb{B}_n$ is the Bergman metric $\beta(z,w)$ given by ([38], Proposition 1.21)

$$\beta(z,w) = \frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}, \qquad z,w \in \mathbb{B}_n. \tag{7}$$

By invariance, the Bergman metric balls $B_\beta(a,r)$ of radius r at the point $a \in \mathbb{B}_n$ satisfy

$$B_\beta(a,r) = \varphi_a(B_\beta(0,r)),$$

and if $t > 0$ is such that $B_\beta(0,r) = B(0,t)$ (note from (7) that Bergman metric balls centered at the origin are Euclidean balls), then the β-balls are the ellipsoids ([29], page 29)

$$B_\beta(a,r) = \left\{ z \in \mathbb{B}_n : \frac{|P_a z - c_a|^2}{t^2 \rho_a^2} + \frac{|Q_a z|^2}{t^2 \rho_a} < 1 \right\},$$

where

$$c_a = \frac{(1-t^2)\,a}{1 - t^2 |a|^2}, \ \rho_a = \frac{1 - |a|^2}{1 - t^2 |a|^2}.$$

We have the reproducing formula of Bergman ([29], Theorem 3.1.3),

$$f(z) = \frac{n!}{\pi^n} \int_{\mathbb{B}_n} \frac{f(w)}{(1 - \overline{w} \cdot z)^{n+1}} dw, \qquad f \in L^1(d\lambda_n) \cap H(\mathbb{B}_n), \tag{8}$$

and the following variants ([29], Theorem 7.1.2)

$$f(z) = \frac{n!}{\pi^n} \binom{n+s}{n} \int_{\mathbb{B}_n} \frac{\left(1 - |w|^2\right)^s}{(1 - \overline{w} \cdot z)^{s+n+1}} f(w)\, dw, \qquad \mathrm{Re}\, s > -1, \tag{9}$$

valid for all $f \in H(\mathbb{B}_n)$ for which the integrand is in L^1.

We now recall the invertible "radial" operators $R^{\gamma,t} : H(\mathbb{B}_n) \to H(\mathbb{B}_n)$ given in [38] by

$$R^{\gamma,t} f(z) = \sum_{k=0}^{\infty} \frac{\Gamma(n+1+\gamma)\,\Gamma(n+1+k+\gamma+t)}{\Gamma(n+1+\gamma+t)\,\Gamma(n+1+k+\gamma)} f_k(z),$$

provided neither $n+\gamma$ nor $n+\gamma+t$ is a negative integer, and where $f(z) = \sum_{k=0}^{\infty} f_k(z)$ is the homogeneous expansion of f. Note that

$$\frac{\Gamma(n+1+\gamma)\,\Gamma(n+1+k+\gamma+t)}{\Gamma(n+1+\gamma+t)\,\Gamma(n+1+k+\gamma)} \approx (1+k)^t.$$

If the inverse of $R^{\gamma,t}$ is denoted $R_{\gamma,t}$, then Proposition 1.14 of [38] yields

$$\begin{aligned} R^{\gamma,t} \left(\frac{1}{(1 - \overline{w} \cdot z)^{n+1+\gamma}} \right) &= \frac{1}{(1 - \overline{w} \cdot z)^{n+1+\gamma+t}}, \\ R_{\gamma,t} \left(\frac{1}{(1 - \overline{w} \cdot z)^{n+1+\gamma+t}} \right) &= \frac{1}{(1 - \overline{w} \cdot z)^{n+1+\gamma}}, \end{aligned} \tag{10}$$

for all $w \in \mathbb{B}_n$. Thus for any γ, $R^{\gamma,t}$ is approximately differentiation of order t. From Theorem 6.1 and Theorem 6.4 of [38] we have that the derivatives $R^{\gamma,m} f(z)$ are "L^p norm equivalent" to $\sum_{k=0}^{m-1} \left|\nabla^k f(0)\right| + \nabla^m f(z)$ for m large enough.

Proposition 1 *(Theorem 6.1 and Theorem 6.4 of [38]) Suppose that* $0 < p < \infty$, $n+\gamma$ *is not a negative integer, and* $f \in H(\mathbb{B}_n)$. *Then the following four conditions are equivalent:*

$$\begin{aligned}
\left(1-|z|^2\right)^m \nabla^m f(z) &\in L^p(d\lambda_n) \text{ for some } m > \frac{n}{p}, m \in \mathbb{N}, \\
\left(1-|z|^2\right)^m \nabla^m f(z) &\in L^p(d\lambda_n) \text{ for all } m > \frac{n}{p}, m \in \mathbb{N}, \\
\left(1-|z|^2\right)^m R^{\gamma,m} f(z) &\in L^p(d\lambda_n) \text{ for some } m > \frac{n}{p}, m+n+\gamma \notin -\mathbb{N}, \\
\left(1-|z|^2\right)^m R^{\gamma,m} f(z) &\in L^p(d\lambda_n) \text{ for all } m > \frac{n}{p}, m+n+\gamma \notin -\mathbb{N}.
\end{aligned}$$

Moreover, with $\sigma(z) = 1-|z|^2$, *we have for* $1 < p < \infty$,

$$\begin{aligned}
& C^{-1}\left\|\sigma^{m_1} R^{\gamma,m_1} f\right\|_{L^p(d\lambda_n)} \\
\leq & \sum_{k=0}^{m_2-1}\left|\nabla^k f(0)\right| + \left(\int_{\mathbb{B}_n}\left|\left(1-|z|^2\right)^{m_2} \nabla^{m_2} f(z)\right|^p d\lambda_n(z)\right)^{\frac{1}{p}} \\
\leq & C\left\|\sigma^{m_1} R^{\gamma,m_1} f\right\|_{L^p(d\lambda_n)}
\end{aligned} \tag{11}$$

for all $m_1, m_2 > \frac{n}{p}$, $m_1+n+\gamma \notin -\mathbb{N}$, $m_2 \in \mathbb{N}$, *and where the constant* C *depends only on* m_1, m_2, n, γ *and* p.

Definition 2 *We define the analytic Besov spaces* $B_p(\mathbb{B}_n)$ *on the ball* $\mathbb{B}_n$ *by taking* $\gamma = 0$ *and* $m = \left[\frac{n}{p}\right]+1$ *and setting*

$$B_p = B_p(\mathbb{B}_n) = \left\{f \in H(\mathbb{B}_n) : \left\|\sigma^m R^{0,m} f\right\|_{L^p(d\lambda_n)} < \infty\right\}. \tag{12}$$

We will indulge in the usual abuse of notation by using $\|f\|_{B_p(\mathbb{B}_n)}$ to denote any of the norms appearing in (11).

2.1.1 Duality and reproducing kernels

For $\alpha > -1$, let $\langle \cdot, \cdot \rangle_\alpha$ denote the inner product for the weighted Bergman space A^2_α:

$$\langle f, g \rangle_\alpha = \int_{\mathbb{B}_n} f(z)\overline{g(z)} d\nu_\alpha(z), \quad f, g \in A^2_\alpha,$$

where $d\nu_\alpha(z) = \left(1-|z|^2\right)^\alpha dz$. Recall that $K^\alpha_w(z) = K^\alpha(z,w) = (1-\overline{w}\cdot z)^{-n-1-\alpha}$ is the reproducing kernel for A^2_α (Theorem 2.7 in [38]):

$$f(w) = \langle f, K^\alpha_w \rangle_\alpha = \int_{\mathbb{B}_n} f(z)\overline{K^\alpha_w(z)} d\nu_\alpha(z), \quad f \in A^2_\alpha.$$

This formula continues to hold as well for $f \in A^p_\alpha$, $1 < p < \infty$, since the polynomials are dense in A^p_α.

Corollary 6.5 of [38] states that $R^{\gamma,\frac{n+1+\alpha}{p}}$ is a bounded invertible operator from B_p onto A^p_α, provided that neither $n+\gamma$ nor $n+\gamma+\frac{n+1+\alpha}{p}$ is a negative integer. It turns out to be convenient to take $\gamma = \alpha - \frac{n+1+\alpha}{p}$ here (with this choice we can explicitly compute certain derivatives and $B_{p'}$ norms of our reproducing kernels - see (15) and (22) below), and thus we single out the special operators

$$\mathcal{R}^\alpha_t = R^{\alpha-t,t}.$$

Note that the operators $\mathcal{R}^\alpha_t$ and their inverses $\left(\mathcal{R}^\alpha_t\right)^{-1} = R_{\alpha-t,t}$ are self-adjoint with respect to $\langle \cdot,\cdot \rangle_\alpha$ since the monomials are orthogonal with respect to $\langle \cdot,\cdot \rangle_\alpha$ (see (1.21) and (1.23) in [38]), and the operators act on the homogeneous expansion of f by multiplying the homogeneous coefficients of f by certain positive constants. The next definition is motivated by the fact that $\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}}$ is a bounded invertible operator from B_p onto A^p_α, and that $\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p'}}$ is a bounded invertible operator from $B_{p'}$ onto $A^{p'}_\alpha$, provided that neither $n+\alpha$, $n+\alpha-\frac{n+1+\alpha}{p}$ nor $n+\alpha-\frac{n+1+\alpha}{p'}$ is a negative integer. Note that this proviso holds in particular for $\alpha > -1$.

Definition 3 *For $\alpha > -1$ and $1 < p < \infty$, we define a pairing $\langle \cdot,\cdot \rangle_{\alpha,p}$ for B_p and $B_{p'}$ using $\langle \cdot,\cdot \rangle_\alpha$ as follows:*

$$\begin{aligned}\langle f,g\rangle_{\alpha,p} &= \left\langle \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}} f, \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p'}} g \right\rangle_\alpha = \int_{\mathbb{B}_n} \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}} f(z)\, \overline{\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p'}} g(z)}\, d\nu_\alpha(z) \\ &= \int_{\mathbb{B}_n} \left\{ \left(1-|z|^2\right)^{\frac{n+1+\alpha}{p}} \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}} f(z) \right\} \overline{\left\{ \left(1-|z|^2\right)^{\frac{n+1+\alpha}{p'}} \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p'}} g(z) \right\}}\, d\lambda_n(z).\end{aligned}$$

With $K^\alpha_w(z)$ the reproducing kernel for A^2_α, we have that the kernel

$$k^{\alpha,p}_w(z) = \left(\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p'}}\right)^{-1} \left(\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}}\right)^{-1} K^\alpha_w(z) \tag{13}$$

satisfies the following reproducing formula for B_p:

$$f(w) = \langle f, k^{\alpha,p}_w \rangle_{\alpha,p} = \int_{\mathbb{B}_n} \mathcal{R}^\alpha_{\frac{n+1+\alpha}{p}} f(z)\, \overline{\mathcal{R}^\alpha_{\frac{n+1+\alpha}{p'}} k^{\alpha,p}_w(z)}\, d\nu_\alpha(z), \qquad f \in B_p. \tag{14}$$

Thus we have the following theorem.

Theorem 4 *Let $1 < p < \infty$ and $\alpha > -1$. Then the dual space of B_p can be identified with $B_{p'}$ under the pairing $\langle \cdot,\cdot \rangle_{\alpha,p}$, and the reproducing kernel $k^{\alpha,p}_w$ for this pairing is given by (13).*

From (13) and (10) we have

$$\begin{aligned}\mathcal{R}^{\alpha}_{\frac{n+1+\alpha}{p'}} k^{\alpha,p}_w(z) &= \left(\mathcal{R}^{\alpha}_{\frac{n+1+\alpha}{p}}\right)^{-1} K^{\alpha}_w(z) \\ &= R_{\alpha-\frac{n+1+\alpha}{p},\frac{n+1+\alpha}{p}}\left((1-\overline{w}\cdot z)^{-(n+1+\alpha)}\right) \\ &= (1-\overline{w}\cdot z)^{-\frac{n+1+\alpha}{p'}}.\end{aligned} \tag{15}$$

Using this formula we will show in (22) below that the $B_{p'}$ norm of the reproducing kernel $k^{\alpha,p}_w$ is comparable to $\left(1+\log\frac{1}{1-|w|^2}\right)^{\frac{1}{p'}}$.

We now state our analogue of Böe's interpolation theorem in two separate statements.

Theorem 5 *Let $1<p<\infty$, $\alpha>-1$ and $k^{\alpha,p}_w(z)$ be the reproducing kernel for B_p relative to the pairing $\langle\cdot,\cdot\rangle_{\alpha,p}$ given in Theorem 4 above. Let $\{z_j\}_{j=1}^{\infty}$ be a sequence in the unit ball $\mathbb{B}_n$. Then the following conditions are equivalent.*

1. *$\{z_j\}_{j=1}^{\infty}$ interpolates B_p:*

$$\text{The map } f \to \left\{\frac{f(z_j)}{\|k^{\alpha,p}_{z_j}\|_{B_{p'}}}\right\}_{j=1}^{\infty} \text{ takes } B_p \text{ boundedly into and onto } \ell^p. \tag{16}$$

2. *The following norm equivalence holds:*

$$\left\|\sum_{j=1}^{\infty} a_j \frac{k^{\alpha,p}_{z_j}}{\|k^{\alpha,p}_{z_j}\|_{B_{p'}}}\right\|_{B_{p'}} \approx \left(\sum_{j=1}^{\infty}|a_j|^{p'}\right)^{\frac{1}{p'}}. \tag{17}$$

3. *The following separation condition and Carleson embedding hold:*

$$\begin{aligned}&\beta(z_i,0)\le C\beta(z_i,z_j), i\neq j \text{ and} \\ &\sum_{j=1}^{\infty}\left\|k^{\alpha,p}_{z_j}\right\|^{-p}_{B_{p'}}\delta_{z_j} \text{ is a } B_p\text{-Carleson measure.}\end{aligned} \tag{18}$$

Theorem 6 *Let $1<p<\infty$, $\alpha>-1$ and $k^{\alpha,p}_w(z)$ be the reproducing kernel for B_p relative to the pairing $\langle\cdot,\cdot\rangle_{\alpha,p}$ given in Theorem 4 above. Let $\{z_j\}_{j=1}^{\infty}$ be a sequence in the unit ball $\mathbb{B}_n$. If $p\in\left(1,2+\frac{1}{n-1}\right)$, then each of conditions (19) and (21) below is equivalent to the three conditions in Theorem 5. In general, for $1<p<\infty$, (21) implies (19) implies (20). For $p>2n$ (18) implies (19). If $p\in\left(1,1+\frac{1}{n-1}\right)\cup[2,\infty)$, we also have that (20) implies (18):*

1. $\{z_j\}_{j=1}^{\infty}$ *interpolates* M_{B_p}:

$$\text{The map } f \to \{f(z_j)\}_{j=1}^{\infty} \text{ takes } M_{B_p} \text{ boundedly into and onto } \ell^{\infty}. \tag{19}$$

2. $\left\{k_{z_j}^{\alpha,p}\right\}_{j=1}^{n}$ *is an unconditional basic sequence in* $B_{p'}$:

$$\left\| \sum_{j=1}^{\infty} b_j k_{z_j}^{\alpha,p} \right\|_{B_{p'}} \leq C \left\| \sum_{j=1}^{\infty} a_j k_{z_j}^{\alpha,p} \right\|_{B_{p'}}, \quad \text{whenever } |b_j| \leq |a_j|. \tag{20}$$

3. $\{z_j\}_{j=1}^{\infty} = \{c_{\alpha_j}\}_{j=1}^{\infty}$ *where* $\{\alpha_j\}_{j=1}^{\infty}$ *is a sequence in a Bergman tree* $\mathcal{T}_n$ *satisfying*

$$\beta(z_i, 0) \leq C\beta(z_i, z_j), i \neq j \text{ and} \tag{21}$$
$$\sum_{j=1}^{\infty} (1 + d(\alpha_j, o))^{1-p} \delta_{\alpha_j} \text{ satisfies the tree condition (1).}$$

Note in particular that for $p \in \left(1, 2 + \frac{1}{n-1}\right) \cup (2n, \infty)$, multiplier interpolation (19) is characterized by the separation condition and Carleson embedding in (18).

The parameter $\alpha > -1$ appearing in condition (18) is not essential, as evidenced by the following calculation.

Lemma 7 *For* $\alpha > -1$ *and* $1 < p < \infty$, *we have*

$$\|k_w^{\alpha,p}\|_{B_{p'}} \approx \left(1 + \log \frac{1}{1 - |w|^2}\right)^{\frac{1}{p'}} \approx (1 + \beta(0, w))^{\frac{1}{p'}}. \tag{22}$$

Proof. Using (15) and $m = \frac{n+1+\alpha}{p'} > \frac{n}{p'}$, we compute that

$$\begin{aligned}
\|k_w^{\alpha,p}\|_{B_{p'}} &= \left(\int_{\mathbb{B}_n} \left| \left(1 - |z|^2\right)^{\frac{n+1+\alpha}{p'}} \mathcal{R}^{\alpha}_{\frac{n+1+\alpha}{p'}} k_w^{\alpha,p}(z) \right|^{p'} d\lambda_n(z) \right)^{\frac{1}{p'}} \\
&= \left(\int_{\mathbb{B}_n} \left| \frac{1 - |z|^2}{1 - \overline{w} \cdot z} \right|^{n+1+\alpha} d\lambda_n(z) \right)^{\frac{1}{p'}} \\
&= \left(\int_{\mathbb{B}_n} \frac{\left(1 - |z|^2\right)^{\alpha}}{|1 - \overline{w} \cdot z|^{n+1+\alpha}} dz \right)^{\frac{1}{p'}} \\
&\approx \left(1 + \log \frac{1}{1 - |w|^2}\right)^{\frac{1}{p'}}
\end{aligned}$$

by Theorem 1.12 of [38]. Note also that $1+\log\frac{1}{1-|w|^2}\approx d(\alpha)$ if $w\in K_\alpha$.

Thus we can restate condition (18) in the equivalent form,

$$\begin{aligned}&\beta(z_i,0)\leq C\beta(z_i,z_j)\ \text{ and}\\&\sum_{j=1}^{\infty}\left(1+\log\frac{1}{1-|z_j|^2}\right)^{1-p}\delta_{z_j}\ \text{ is a } B_p\text{-Carleson measure.}\end{aligned}\tag{23}$$

Remark 1 *Our proofs show that the interpolations in (19) and (16) can be taken to be linear, i.e. there are bounded linear maps* $R:\ell^\infty\to M_{B_p}$ *and* $S:\ell^p\to B_p$ *that yield right inverses to the restriction maps in (19) and (16) respectively. In dimension* $n=1$ *Böe has shown [12] the stronger result that there are functions* $f_k\in M_{B_p}$ *such that* $\|f_k\|_{M_{B_p}}\leq C$, $f_k(z_j)=\delta_k^j$ *and* $\sum_k|f_k(z)|\leq C$ *for all* $z\in\mathbb{D}$ *(compare Theorem 2.1 in chapter 7 of [22]). It seems likely that this extends to* $1<p<2+\frac{1}{n-1}$ *for* $n>1$*, but we will not pursue this here.*

Remark 2 *We do not know if (18) is sufficient for (21) when* $p\in\left[2+\frac{1}{n-1},\infty\right)$*. Note that (3) and (21) are equivalent.*

Proof. Here we only prove the following "soft" implications: that (19) implies (20), that (16) and (17) are equivalent, that (16) implies (18), and that if $p\in\left(1,1+\frac{1}{n-1}\right)\cup[2,\infty)$, then (20) implies (18).

2.2 Multiplier space necessity

We begin with the straightforward necessity implications; (19) implies (20), (16) implies (17), and (17) implies (18). For the most part, we follow Böe [12], who in turn generalized the Hilbert space arguments in Marshall and Sundberg [23]. First, we have that condition (20) follows from (19) using that the reproducing kernels are eigenfunctions of adjoints of multiplier operators: $M_\varphi^*\left(k_{z_j}^{\alpha,p}\right)=\overline{\varphi(z_j)}k_{z_j}^{\alpha,p}$. Indeed, for all $f\in B_p$,

$$\left\langle f,M_\varphi^*\left(k_{z_j}^{\alpha,p}\right)\right\rangle=\left\langle\varphi f,k_{z_j}^{\alpha,p}\right\rangle=\varphi(z_j)f(z_j)=\varphi(z_j)\left\langle f,k_{z_j}^{\alpha,p}\right\rangle=\left\langle f,\overline{\varphi(z_j)}k_{z_j}^{\alpha,p}\right\rangle.$$

Now if we choose $\varphi\in M_{B_p}$ so that $b_j=\overline{\varphi(z_j)}a_j$, then

$$\begin{aligned}\left\|\sum_{j=1}^{\infty}b_jk_{z_j}^{\alpha,p}\right\|_{B_{p'}}&=\left\|M_\varphi^*\left(\sum_{j=1}^{\infty}a_jk_{z_j}^{\alpha,p}\right)\right\|_{B_{p'}}\\&\leq\|M_\varphi\|\left\|\sum_{j=1}^{\infty}a_jk_{z_j}^{\alpha,p}\right\|_{B_{p'}}.\end{aligned}$$

We note in passing that the identity $M_\varphi^*\left(k_z^{\alpha,p}\right) = \overline{\varphi(z)} k_z^{\alpha,p}$ shows that multipliers are bounded:

$$\|M_\varphi\|_{op} = \left\|M_\varphi^*\right\|_{op} = \sup_{f \in B_{p'}} \frac{\left\|M_\varphi^* f\right\|}{\|f\|} \geq \sup_z \frac{\left\|M_\varphi^* k_z^{\alpha,p}\right\|}{\left\|k_z^{\alpha,p}\right\|} = \sup_z |\varphi(z)| = \|\varphi\|_{H^\infty}.$$

Next we prove the equivalence of (16) and (17), the arguments being short and essentially reversible. First, if the map $Tf = \left\{ \frac{f(z_j)}{\left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}} \right\}_{j=1}^\infty$ in (16) maps B_p *into* ℓ^p, then we have

$$\begin{aligned}
\left\| \sum_{j=1}^\infty a_j \frac{k_{z_j}^{\alpha,p}}{\left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}} \right\|_{B_{p'}} &= \sup_{\|f\|_{B_p}=1} \left| \left\langle f, \sum_{j=1}^\infty a_j \frac{k_{z_j}^{\alpha,p}}{\left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}} \right\rangle_{\alpha,p} \right| \\
&= \sup_{\|f\|_{B_p}=1} \left| \sum_{j=1}^\infty \frac{f(z_j)}{\left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}} \overline{a_j} \right| \\
&\leq \sup_{\|f\|_{B_p}=1} \left(\sum_{j=1}^\infty \left| \frac{f(z_j)}{\left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}} \right|^p \right)^{\frac{1}{p}} \left\| \{a_j\}_{j=1}^\infty \right\|_{\ell^{p'}} \\
&\leq C \left\| \{a_j\}_{j=1}^\infty \right\|_{\ell^{p'}}.
\end{aligned}$$

If the map T is also *onto*, then its adjoint T^*, given by

$$T^*\left(\{a_j\}_{j=1}^\infty\right) = \sum_{j=1}^\infty \overline{a_j} \frac{k_{z_j}^{\alpha,p}}{\left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}},$$

satisfies

$$\left\| T^*\left(\{a_j\}_{j=1}^\infty\right) \right\|_{B_{p'}} \geq c \left\| \{a_j\}_{j=1}^\infty \right\|_{\ell^{p'}},$$

which is the opposite inequality in (17), and completes the proof that (16) implies (17). Conversely, if the inequality $\lesssim$ in (17) holds, then

$$\begin{aligned}
&\left(\sum_{j=1}^\infty \left| \frac{f(z_j)}{\left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}} \right|^p \right)^{\frac{1}{p}} \qquad (24) \\
&= \sup_{\left\|\{a_j\}_{j=1}^\infty\right\|_{\ell^{p'}}=1} \left| \sum_{j=1}^\infty \frac{f(z_j)}{\left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}} \overline{a_j} \right| \\
&= \sup_{\left\|\{a_j\}_{j=1}^\infty\right\|_{\ell^{p'}}=1} \left| \sum_{j=1}^\infty \left\langle f, \frac{a_j k_{z_j}^{\alpha,p}}{\left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}} \right\rangle_{\alpha,p} \right| \\
&\leq \sup_{\left\|\{a_j\}_{j=1}^\infty\right\|_{\ell^{p'}}=1} \|f\|_{B_p} \left\| \sum_{j=1}^\infty a_j \frac{k_{z_j}^{\alpha,p}}{\left\|k_{z_j}^{\alpha,p}\right\|_{B_{p'}}} \right\|_{B_{p'}} \leq C \|f\|_{B_p},
\end{aligned}$$

and thus the map T in (16) is *into*. If the reverse inequality $\gtrsim$ in (17) also holds, then

$$\left\| T^{*}\left(\{a_j\}_{j=1}^{\infty}\right)\right\|_{B_{p'}} = \left\| \sum_{j=1}^{\infty} \overline{a_j} \frac{k_{z_j}^{\alpha,p}}{\|k_{z_j}^{\alpha,p}\|_{B_{p'}}} \right\|_{B_{p'}} \geq c \left\| \{a_j\}_{j=1}^{\infty} \right\|_{\ell^{p'}},$$

which shows that T is also *onto*.

Note We have shown in particular that the inequality $\lesssim$ in (17) implies that the map T in (16) is *into*. This will be used below.

The implication (17) implies (18) will now follow if we show that (16) implies (18). The Carleson embedding in (18) is a restatement that the map T in (16) is *into*. Indeed, the left side of (24) is $\|f\|_{L^p\left(\sum_{j=1}^{\infty}\|k_{z_j}^{\alpha,p}\|_{B_{p'}}^{-p}\delta_{z_j}\right)}$, and thus shows that the Carleson embedding in (18) holds. To obtain the separation condition, fix i and use that T is *onto* to obtain $f \in B_p$ satisfying $f(z_i) = 1$ and $f(z_j) = 0$ for $i \neq j$. It now follows from the open mapping theorem and the Hölder estimate

$$|f(z) - f(w)| \leq C \|f\|_{B_p} \beta(z,w)^{\frac{1}{p'}}, \qquad z, w \in \mathbb{B}_n, \tag{25}$$

that

$$\|f\|_{B_p} \leq C \|Tf\|_{\ell^p} = C \frac{|f(z_i)|}{\|k_{zi}^{\alpha,p}\|_{B_{p'}}} = C \frac{|f(z_i) - f(z_j)|}{\|k_{z_i}^{\alpha,p}\|_{B_{p'}}} \leq C \|f\|_{B_p} \frac{\beta(z_i, z_j)^{\frac{1}{p'}}}{\beta(z_i, 0)^{\frac{1}{p'}}},$$

for all $i \neq j$.

2.2.1 The necessity of separation and Carleson

Now we turn to proving the more difficult necessity implication (20) implies (18). First we dispose of the easy part - namely that the separation condition in (18) follows from (20). Indeed, by (22), (20) and (25) we have

$$\begin{aligned}
(1 + \beta(0, z_i))^{\frac{1}{p'}} \approx \left\| k_{z_i}^{\alpha,p} \right\|_{B_{p'}} &\leq C \left\| k_{z_i}^{\alpha,p} - k_{z_j}^{\alpha,p} \right\|_{B_{p'}} \\
&= C \sup_{\|f\|_{B_p}=1} \left| \left\langle f, k_{z_i}^{\alpha,p} - k_{z_j}^{\alpha,p} \right\rangle_{\alpha,p} \right| \\
&= C \sup_{\|f\|_{B_p}=1} |f(z_i) - f(z_j)| \\
&\leq C\beta(z_i, z_j)^{\frac{1}{p'}}.
\end{aligned}$$

It remains to prove that the Carleson embedding follows from (20). For this, we show that (20) implies (17) for both $1 < p < 1 + \frac{1}{n-1}$ and $p = 2$, and also that (20) implies the inequality $\lesssim$ of (17) for $p > 2$. The note above then yields that the map T in (16) is *into*, which we showed above is a restatement of the Carleson embedding.

2.2.2 The case $1 < p < 1 + \frac{1}{n-1}$

Here we prove the implication (20) implies (17) for the special case $1 < p < 1 + \frac{1}{n-1}$. Given $1 < p < 1 + \frac{1}{n-1}$, we make the choice $-1 < \alpha < \infty$ to satisfy

$$p = \frac{n+1+\alpha}{n+\alpha}, \tag{26}$$

which accounts for our restriction $1 < p < 1 + \frac{1}{n-1}$. Note that $p' = n + 1 + \alpha$, so that

$$\begin{aligned} \frac{n+1+\alpha}{p} &= n + \alpha, \\ \frac{n+1+\alpha}{p'} &= 1. \end{aligned}$$

Thus in this case we have $\mathcal{R}^{\alpha}_{\frac{n+1+\alpha}{p}} = \mathcal{R}^{\alpha}_{n+\alpha}$ and $\mathcal{R}^{\alpha}_{\frac{n+1+\alpha}{p'}} = \mathcal{R}^{\alpha}_{1}$ where α is as in (26), so that

$$\begin{aligned} \langle f, g \rangle_{\alpha,p} &= \left\langle \mathcal{R}^{\alpha}_{n+\alpha} f, \mathcal{R}^{\alpha}_{1} g \right\rangle_{A^2_\alpha} \\ &= \int_{\mathbb{B}_n} \left(1 - |z|^2\right)^{n+\alpha} \mathcal{R}^{\alpha}_{n+\alpha} f(z) \overline{\left(1 - |z|^2\right) \mathcal{R}^{\alpha}_{1} g(z)} d\lambda_n(z). \end{aligned}$$

The point of the choice of p in (26) is that

$$\mathcal{R}^{\alpha}_{1} k^{\alpha,p}_{w}(z) = \frac{1}{1 - \overline{w} \cdot z}$$

has positive real part in the ball. Let $\{z_j\}_{j=1}^{\infty}$ be a sequence in the ball $\mathbb{B}_n$. We will need the following two results.

Lemma 8 *(Lemma 3.1 in [12]) If $\{f_j\}_{j=1}^{\infty}$ is an unconditional basic sequence of* positive *functions in $L^q(d\mu)$, $1 < q < \infty$, then*

$$\left\| \sum_{j=1}^{\infty} |a_j f_j| \right\|_{L^q(d\mu)} \approx C_q \left\| \sup_{j \geq 1} |a_j f_j| \right\|_{L^q(d\mu)} \approx C_q \left(\sum_{j=1}^{\infty} |a_j|^q \|f_j\|^q_{L^q(d\mu)} \right)^{\frac{1}{q}}.$$

Proof. For convenience we sketch Böe's proof, which we will need to adapt to holomorphic trees later anyway. Since the f_n are positive and unconditional in $L^q(d\mu)$, we have, letting $\{r_j(t)\}_{j=1}^{\infty}$ denote the Rademacher functions,

$$\left\| \sum_{j=1}^{\infty} |a_j f_j| \right\|_{L^q(d\mu)} \leq \left\| \sum_{j=1}^{\infty} |a_j| f_j \right\|_{L^q(d\mu)} \leq C \left\| \sum_{j=1}^{\infty} r_j(t) a_j f_j \right\|_{L^q(d\mu)}$$

for all $t \in [0,1]$ (since $|a_j| = |r_j(t) a_j|$). Now average the q^{th} power of this inequality over $t \in [0,1]$, and use Khinchine's inequality to obtain

$$\begin{aligned}\left\| \sum_{j=1}^{\infty} |a_j f_j| \right\|_{L^q(d\mu)}^q &\leq C^q \int_0^1 \left\| \sum_{j=1}^{\infty} r_j(t) a_j f_j \right\|_{L^q(d\mu)}^q dt \\ &= C^q \int \int_0^1 \left| \sum_{j=1}^{\infty} r_j(t) a_j f_j \right|^q dt d\mu \\ &\leq C_q^q \int \left\| \{a_j f_j\}_{j=1}^{\infty} \right\|_{\ell^2}^q d\mu.\end{aligned}$$

Since $\left\| \{a_j f_j\}_{j=1}^{\infty} \right\|_{\ell^2} \leq \left\| \{a_j f_j\}_{j=1}^{\infty} \right\|_{\ell^1}^{\frac{1}{2}} \left\| \{a_j f_j\}_{j=1}^{\infty} \right\|_{\ell^\infty}^{\frac{1}{2}}$, we have by the Cauchy-Schwartz inequality,

$$\left\| \sum_{j=1}^{\infty} |a_j f_j| \right\|_{L^q(d\mu)}^q \leq C_q^q \left(\int \left\| \{a_j f_j\}_{j=1}^{\infty} \right\|_{\ell^1}^q d\mu \right)^{\frac{1}{2}} \left(\int \left\| \{a_j f_j\}_{j=1}^{\infty} \right\|_{\ell^\infty}^q d\mu \right)^{\frac{1}{2}},$$

which yields the inequality

$$\left\| \sum_{j=1}^{\infty} |a_j f_j| \right\|_{L^q(d\mu)} \leq C_q \left\| \sup_{j \geq 1} |a_j f_j| \right\|_{L^q(d\mu)}.$$

Thus the expressions

$$\left\| \left(\sum_{j=1}^{\infty} |a_j f_j|^r \right)^{\frac{1}{r}} \right\|_{L^q(d\mu)}$$

are all comparable for $1 < r < \infty$, and the choice $r = q$ yields the final equivalence in the lemma.

Lemma 9 *For* $-1 < \alpha < \infty$, $1 < q < \infty$ *and* $F \in H(\mathbb{B}_n)$ *with* $\operatorname{Im} F(0) = 0$,

$$\left(\int_{\mathbb{B}_n} |F(z)|^q d\nu_\alpha(z) \right)^{\frac{1}{q}} \approx \left(\int_{\mathbb{B}_n} |\operatorname{Re} F(z)|^q d\nu_\alpha(z) \right)^{\frac{1}{q}}. \tag{27}$$

Proof. The Korànyi-Vagi theorem (Theorem 6.3.1 in [29]) shows the equivalence of the left and right hand sides in (27) when the measure $d\nu_\alpha(z)$ on the ball $\mathbb{B}_n$ is replaced by surface measure $d\sigma(z)$ on the sphere $\partial\mathbb{B}_n$ (and F is say a polynomial). Note that $d\sigma$ corresponds to $\lim_{\alpha \to -1} d\nu_\alpha$. This immediately yields (27) by an integration in polar coordinates.

Now suppose that (20) holds. Since $p' = n+1+\alpha$, we have from (15) that

$$\mathcal{R}_1^\alpha k_w^{\alpha,p}(z) = \frac{1}{1-\overline{w}\cdot z}. \tag{28}$$

We now compute that

$$\begin{aligned}
\left\| \sum_{j=1}^\infty a_j \frac{k_{z_j}^{\alpha,p}}{\|k_{z_j}^{\alpha,p}\|_{B_{p'}}} \right\|_{B_{p'}} &= \left\| \left(1-|z|^2\right) \mathcal{R}_1^\alpha \left(\sum_{j=1}^\infty a_j \frac{k_{z_j}^{\alpha,p}}{\|k_{z_j}^{\alpha,p}\|_{B_{p'}}} \right) \right\|_{L^{p'}(d\lambda_n)} \\
&= \left\| \left(1-|z|^2\right) \left(\sum_{j=1}^\infty a_j \frac{\mathcal{R}_1^\alpha k_{z_j}^{\alpha,p}}{\|k_{z_j}^{\alpha,p}\|_{B_{p'}}} \right) \right\|_{L^{p'}(d\lambda_n)} \\
&= \left\| \sum_{j=1}^\infty a_j \left\| k_{z_j}^{\alpha,p} \right\|_{B_{p'}}^{-1} \frac{1}{1-\overline{z_j}\cdot z} \right\|_{L^{p'}(d\nu_\alpha)}
\end{aligned}$$

since $p' = n+1+\alpha$. Now by the lemmas above, and using $p' = n+1+\alpha$ and

$$f_j = \left\| k_{z_j}^{\alpha,p} \right\|_{B_{p'}}^{-1} \operatorname{Re} \frac{1}{1-\overline{z_j}\cdot z} > 0,$$

we continue with

$$\begin{aligned}
\left\| \sum_{j=1}^\infty a_j \left\| k_{z_j}^{\alpha,p} \right\|_{B_{p'}}^{-1} \frac{1}{1-\overline{z_j}\cdot z} \right\|_{L^{p'}(d\nu_\alpha)} &\approx \left\| \sum_{j=1}^\infty a_j \left\| k_{z_j}^{\alpha,p} \right\|_{B_{p'}}^{-1} \operatorname{Re} \frac{1}{1-\overline{z_j}\cdot z} \right\|_{L^{p'}(d\nu_\alpha)} \\
&\approx \left(\sum_{j=1}^\infty |a_j|^{p'} \|f_j\|_{L^{p'}(d\nu_\alpha)}^{p'} \right)^{\frac{1}{p'}} \\
&\approx \left(\sum_{j=1}^\infty |a_j|^{p'} \right)^{\frac{1}{p'}},
\end{aligned}$$

since

$$\begin{aligned}
\|f_j\|_{L^{p'}(d\nu_\alpha)} &= \left\| k_{z_j}^{\alpha,p} \right\|_{B_{p'}}^{-1} \left\| \operatorname{Re} \frac{1}{1-\overline{z_j}\cdot z} \right\|_{L^{p'}(d\nu_\alpha)} \\
&\approx \left\| k_{z_j}^{\alpha,p} \right\|_{B_{p'}}^{-1} \left\| \frac{1}{1-\overline{z_j}\cdot z} \right\|_{L^{p'}(d\nu_\alpha)} \\
&= \left\| k_{z_j}^{\alpha,p} \right\|_{B_{p'}}^{-1} \left\| \left(1-|z|^2\right) \mathcal{R}_1 k_{z_j}^{\alpha,p} \right\|_{L^{p'}(d\lambda_n)} \\
&= \left\| k_{z_j}^{\alpha,p} \right\|_{B_{p'}}^{-1} \left\| k_{z_j}^{\alpha,p} \right\|_{B_{p'}} = 1
\end{aligned}$$

upon using the second lemma above once more. This completes the proof of condition (17) in the case $1 < p < 1 + \frac{1}{n-1}$.

2.2.3 The case $p \geq 2$

Here we show that (20) implies the inequality $\lesssim$ in (17) for $p > 2$, and also that (20) implies (17) for $p = 2$. First we claim that the unconditional basis condition (20) and Khinchine's inequality yield the inequality

$$\left\| \sum_{j=1}^{\infty} a_j \frac{k_{z_j}^{\alpha,p}}{\|k_{z_j}^{\alpha,p}\|_{B_{p'}}} \right\|_{B_{p'}} \leq C \left(\sum_{j=1}^{\infty} |a_j|^{p'} \right)^{\frac{1}{p'}}$$

for $p \geq 2$, and with equality in the case $p = 2$. To see this, we compute using first (20) and then Khinchine, that for any $m > \frac{n}{p'}$, we have

$$\begin{aligned}
\left\| \sum_{j=1}^{\infty} a_j \frac{k_{z_j}^{\alpha,p}}{\|k_{z_j}^{\alpha,p}\|_{B_{p'}}} \right\|_{B_{p'}}^{p'} &\approx \int_0^1 \left\| \sum_{j=1}^{\infty} a_j \frac{k_{z_j}^{\alpha,p}}{\|k_{z_j}^{\alpha,p}\|_{B_{p'}}} r_j(t) \right\|_{B_{p'}}^{p'} dt \\
&= \int_0^1 \int_{\mathbb{B}_n} \left| \sum_{j=1}^{\infty} a_j \frac{\left(1-|z|^2\right)^m \mathcal{R}_m^{\alpha} k_{z_j}^{\alpha,p}(z)}{\|k_{z_j}^{\alpha,p}\|_{B_{p'}}} r_j(t) \right|^{p'} d\lambda_n(z)\, dt \\
&\approx \int_{\mathbb{B}_n} \left(\sum_{j=1}^{\infty} \left| a_j \frac{\left(1-|z|^2\right)^m \mathcal{R}_m^{\alpha} k_{z_j}^{\alpha,p}(z)}{\|k_{z_j}^{\alpha,p}\|_{B_{p'}}} \right|^2 \right)^{\frac{p'}{2}} d\lambda_n(z).
\end{aligned}$$

Since $\frac{p'}{2} \leq 1$, we continue with

$$\begin{aligned}
\left\| \sum_{j=1}^{\infty} a_j \frac{k_{z_j}^{\alpha,p}}{\|k_{z_j}^{\alpha,p}\|_{B_{p'}}} \right\|_{B_{p'}}^{p'} &\leq C \int_{\mathbb{B}_n} \sum_{j=1}^{\infty} \left| a_j \frac{\left(1-|z|^2\right)^m \mathcal{R}_m^{\alpha} k_{z_j}^{\alpha,p}(z)}{\|k_{z_j}^{\alpha,p}\|_{B_{p'}}} \right|^{p'} d\lambda_n(z) \\
&= \sum_{j=1}^{\infty} |a_j|^{p'} \left\| k_{z_j}^{\alpha,p} \right\|_{B_{p'}}^{-p'} \int_{\mathbb{B}_n} \left| \left(1-|z|^2\right)^m \mathcal{R}_m^{\alpha} k_{z_j}^{\alpha,p}(z) \right|^{p'} d\lambda_n(z) \\
&= \sum_{j=1}^{\infty} |a_j|^{p'},
\end{aligned}$$

which is the inequality $\lesssim$ in (17). In the case $p = 2$ we have equality, and so then (17).

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Taiwan lecture 3

Wednesday July 6 2005

1 Introduction

In this talk we consider Carleson measures for the analytic Besov-Sobolev spaces $B_p^\sigma(\mathbb{B}_n)$, $0 \leq \sigma < \infty$, $1 < p < \infty$, on the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$, consisting of those holomorphic functions f on the ball such that

$$\sum_{k=0}^{m-1} \left| f^{(k)}(0) \right| + \left\{ \int_{\mathbb{B}_n} \left| \left(1 - |z|^2\right)^{m+\sigma} f^{(m)}(z) \right|^p d\lambda_n(z) \right\}^{\frac{1}{p}} < \infty,$$

where $m+\sigma > \frac{n}{p}$, $d\lambda_n(z) = \left(1 - |z|^2\right)^{-n-1} dz$ is invariant measure on the ball with dz Lebesgue measure on $\mathbb{C}^n$, and $f^{(m)}$ is the m^{th} order complex derivative of f. Recall that this scale of spaces includes the Hardy space $H^2(\mathbb{B}_n) = B_2^{\frac{n}{2}}(\mathbb{D})$ with $\sigma = \frac{n}{2}$, the weighted Bergman spaces with $\sigma > \frac{n}{p}$, and the weighted Dirichlet-type spaces with $0 < \sigma < \frac{n}{p}$. The case $\sigma = \frac{1}{2}$ and $p = 2$ is the Drury-Arveson Hardy space that can be identified with the symmetric Fock space over $\mathbb{C}^n$ (see [10] and [19]). We will follow the development in Arcozzi, Rochberg and Sawyer [9].

For purposes of comparison, we also recall the Hardy-Sobolev scale of spaces $H_\alpha^p(\mathbb{B}_n)$, $0 < p < \infty$, $\alpha \in \mathbb{R}$, consisting of all holomorphic functions f in the unit ball whose radial derivative $R^\alpha f$ of order α belongs to the Hardy space $H^p(\mathbb{B}_n)$ ($R^\alpha f = \sum_{k=0}^\infty (k+1)^\alpha f_k$ if $f = \sum_{k=0}^\infty f_k$ is the homogeneous expansion of f). When $p = 2$ the Hardy-Sobolev scale coincides with the Besov-Sobolev scale and we have

$$B_2^\sigma(\mathbb{B}_n) = H_\alpha^2(\mathbb{B}_n), \qquad \sigma + \alpha = \frac{n}{2}, 0 \leq \sigma \leq \frac{n}{2}.$$

Thus σ measures the order of antiderivative required to belong to the Dirichlet space $B_2(\mathbb{B}_n)$, and $\alpha = \frac{n}{2} - \sigma$ measures the order of derivative that belongs to the Hardy space $H^2(\mathbb{B}_n)$. These equalities fail for $p \neq 2$, but are replaced by the strict inclusions

$$B_p^\sigma(\mathbb{B}_n) \subsetneqq H_{\frac{n}{p}-\sigma}^p(\mathbb{B}_n), \qquad 1 < p < 2,\ 0 \leq \sigma < \frac{n}{p}, \tag{1}$$

$$H^p_{\frac{n}{p}-\sigma}(\mathbb{B}_n) \subsetneqq B^\sigma_p(\mathbb{B}_n), \qquad 2 < p < \infty,\ 0 \le \sigma < \frac{n}{p}.$$

We will now construct the Bergman tree $\mathcal{T}_n$ that serves as a discrete model for function theory on the ball.

1.1 Construction of the Bergman tree $\mathcal{T}_n$

Let β be the Bergman metric on the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$. Note that the set

$$\mathcal{S}_r = \partial B_\beta(0,r) = \{z \in \mathbb{B}_n : \beta(0,z) = r\}$$

is a Euclidean sphere (with different radius) centered at the origin for each $r > 0$. In fact, by (1.40) in [39] we have $\beta(0,z) = \tanh^{-1}|z|$, and so

$$\begin{aligned} 1 - |z|^2 &= 1 - \tanh^2 \beta(0,z) \qquad (2)\\ &= \frac{4}{e^{2\beta(0,z)} + 2 + e^{-2\beta(0,z)}} \\ &\approx 4e^{-2\beta(0,z)} \end{aligned}$$

for $\beta(0,z)$ large. We recall the following elementary abstract construction from [8] (Lemma 7 on page 18)

Lemma 1 *Let (X,d) be a separable metric space and $\lambda > 0$. There is a denumerable set of points $E = \{x_j\}_{j=1}^{\infty \text{ or } J}$ and a corresponding set of Borel subsets Q_j of X satisfying*

$$\begin{aligned} X &= \cup_{j=1}^{\infty \text{ or } J} Q_j, \qquad (3)\\ Q_i \cap Q_j &= \phi, \qquad i \ne j, \\ B(x_j,\lambda) &\subset Q_j \subset B(x_j, 2\lambda), \qquad j \ge 1. \end{aligned}$$

We refer to the sets Q_j as unit *qubes* centered at x_j. In [8], we applied Lemma 1 to the spheres $\mathcal{S}_r$ for $r > 0$ as follows. Fix *structural constants* $\theta, \lambda > 0$. For $N \in \mathbb{N}$, apply the lemma to the metric space $(\mathcal{S}_{N\theta}, \beta)$ to obtain points $\{z_j^N\}_{j=1}^J$ and unit qubes $\{Q_j^N\}_{j=1}^J$ in $\mathcal{S}_{N\theta}$ satisfying (3).

However, we now wish to facilitate the definition of an equivalence relation that identifies qubes "lying in the same complex line intersected with the sphere". To achieve this, we recall the projective space $\mathbb{C}P(n-1)$ consisting of all complex circles $[\zeta] = \{e^{is}\zeta : e^{is} \in \mathbb{T}\}$, $\zeta \in \partial\mathbb{B}_n$, in the unit sphere that was introduced in [4]. They defined the induced Koranyi metric on $\mathbb{C}P(n-1)$ by

$$d([\eta],[\zeta]) = \inf\left\{d\left(e^{is}\eta, e^{it}\zeta\right) : e^{is}, e^{it} \in \mathbb{T}\right\}$$

where $d(\eta,\zeta) = |1 - \overline{\eta}\cdot\zeta|^{\frac{1}{2}}$. We scale this construction to the sphere $\mathcal{S}_r$ by defining $\mathbb{P}_r$ to be the projective space of complex circles $[\zeta] = \{e^{is}\zeta : e^{is} \in \mathbb{T}\}$, $\zeta \in \mathcal{S}_r$, in the sphere $\mathcal{S}_r$ with induced Bergman metric

$$\beta([\eta],[\zeta]) = \inf\left\{\beta\left(e^{is}\eta, e^{it}\zeta\right) : e^{is}, e^{it} \in \mathbb{T}\right\}.$$

For $N \in \mathbb{N}$, we now apply Lemma 1 to the projective metric space $(\mathbb{P}_{N\theta}, \beta)$ to obtain projective points (complex circles) $\left\{\mathsf{w}_j^N\right\}_{j=1}^J$ in $\mathbb{P}_{N\theta}$ and unit projective qubes $\left\{\mathsf{Q}_j^N\right\}_{j=1}^J$ contained in $\mathbb{P}_{N\theta}$ satisfying (3). For each N and j we define points $\left\{z_{j,i}^N\right\}_{i=1}^M$ on the complex circle w_j^N that are approximately distance one from their neighbours in the Bergman metric: $\beta\left(z_{j,i}^N, z_{j,i+1}^N\right) \approx 1$ for $1 \leq i \leq M$ $(z_{j,M+1}^N = z_{j,1}^N)$. We then define corresponding *qubes* $\left\{Q_{j,i}^N\right\}_i$ so that $\mathsf{Q}_j^N = \cup_i Q_{j,i}^N$, and so that (3) holds for the collection $\left\{Q_{j,i}^N\right\}_{j,i}$.

For $z \in \mathbb{B}_n$, let $P_r z$ denote the radial projection of z onto the sphere $\mathcal{S}_r$ (not to be confused with the orthogonal projection P_a defined above). We now define subsets $K_{j,i}^N$ of $\mathbb{B}_n$ by $K_1^0 = \{z \in \mathbb{B}_n : \beta(0,z) < \theta\}$ and

$$K_{j,i}^N = \left\{z \in \mathbb{B}_n : N\theta \leq d(0,z) < (N+1)\theta,\ P_{N\theta} z \in Q_{j,i}^N\right\}, \qquad N \geq 1 \text{ and } j,i \geq 1.$$

We define corresponding points $c_{j,i}^N \in K_{j,i}^N$ by

$$c_{j,i}^N = P_{\left(N+\frac{1}{2}\right)\theta}\left(z_{j,i}^N\right).$$

We will refer to the subset $K_{j,i}^N$ of $\mathbb{B}_n$ as a unit *kube* centered at $c_{j,i}^N$ (while K_1^0 is centered at 0). Similarly we define unit *projective kubes* $\mathsf{K}_j^N = \cup_i K_{j,i}^N$ with centre $\mathsf{c}_j^N = P_{\left(N+\frac{1}{2}\right)\theta}\left(\mathsf{w}_j^N\right)$.

Define a tree structure on the collection of all unit projective kubes

$$\mathcal{R}_n = \left\{\mathsf{K}_j^N\right\}_{N \geq 0, j \geq 1}$$

by declaring that K_i^{N+1} is a child of K_j^N, written $\mathsf{K}_i^{N+1} \in \mathcal{C}\left(\mathsf{K}_j^N\right)$, if the projection $P_{N\theta}\left(\mathsf{w}_i^{N+1}\right)$ of the circle w_i^{N+1} onto the sphere $\mathcal{S}_{N\theta}$ lies in the projective qube Q_j^N. In the case $N = 0$, we declare every kube K_j^1 to be a child of the root kube K_1^0.

We will now define a tree structure on the collection of unit kubes

$$\mathcal{T}_n = \left\{K_{j,i}^N\right\}_{N \geq 0 \text{ and } j,i \geq 1}$$

that is compatible with the above tree structure on the collection of projective kubes $\mathcal{R}_n$. To this end, we reindex the qubes $\left\{K_{j,i}^N\right\}_{N \geq 0 \text{ and } j,i \geq 1}$ as $\left\{K_j^N\right\}_{N \geq 0, j \geq 1}$ and define an equivalence relation $\sim$ on the reindexed collection $\left\{K_j^N\right\}_j$ by identifying kubes that lie in the same projective kube: $K_i^N \sim K_k^N$ if and only if there is a projective kube K_j^N such that $K_i^N, K_k^N \in \mathsf{K}_j^N$. Given $K_i^N \in \mathcal{T}_n$, we denote by $\left[K_i^N\right]$ the equivalence class of K_i^N, which can of course be identified with a unit projective kube in $\mathcal{R}_n$. Define the tree structure on $\mathcal{T}_n$ by declaring that K_i^{N+1} is a child of K_j^N, written $K_i^{N+1} \in \mathcal{C}\left(K_j^N\right)$, if the projection $P_{N\theta}\left(z_i^{N+1}\right)$ of z_i^{N+1} onto the sphere $\mathcal{S}_{N\theta}$ lies in the qube Q_j^N. Note that by construction, it follows that $\left[K_i^{N+1}\right]$ is then also a child of $\left[K_j^N\right]$ in $\mathcal{R}_n$. In the case $N=0$, we declare every kube K_j^1 to be a child of the root kube K_1^0. We will typically write α,β,γ etc. to denote elements K_j^N of the tree $\mathcal{T}_n$ when the correspondence with the unit ball $\mathbb{B}_n$ is immaterial. We will write K_α for the kube K_j^N and c_α for its center c_j^N when the correspondence matters. Sometimes we will further abuse notation by using α to denote the center $c_\alpha = c_j^N$ of the kube $K_\alpha = K_j^N$. Motivated by the above definition $\mathsf{K}_j^N = \cup_i K_{j,i}^N$, we will often refer to the elements K_j^N of the tree $\mathcal{R}_n$ as rings. Finally, for $\alpha \in \mathcal{T}_n$, we denote by $[\alpha]$ the ring in $\mathcal{R}_n$ that corresponds to the equivalence class of α. The following compatibility relation holds:

$$\beta \leq \alpha \Longrightarrow [\beta] \leq [\alpha]\,. \tag{4}$$

One can think of the ring tree $\mathcal{R}_n$ as a "quotient tree" of the Bergman tree $\mathcal{T}_n$ by the one-parameter family of slice rotations $z \to e^{is}z$, $e^{is} \in \mathbb{T}$.

Remark 1 *The set of points in $\mathcal{T}_n$ is neither a zero set for $B_p(\mathbb{B}_n)$, nor for $M_{B_p(\mathbb{B}_n)}$, and hence the restriction maps from $B_p(\mathbb{B}_n)$ and $M_{B_p(\mathbb{B}_n)}$ to sequences on $\mathcal{T}_n$ are one-to-one. Indeed, if a holomorphic function f in the ball vanishes on $\mathcal{T}_n$, then the admissible limits f^* of f on $\partial\mathbb{B}_n$ are zero whenever they exist, and thus f vanishes identically on the ball if it is in the Nevanlinna class ([30]: Theorem 5.6.4).*

1.1.1 Unitary rotations of $\mathcal{T}_n$ and its quotient tree $\mathcal{R}_n$

For each $w \in \mathbb{B}_n$ define $\alpha(w) \in \mathcal{T}_n$ to be the unique tree element such that $w \in K_{\alpha(w)}$. Let $\mathcal{U}_n$ be the unitary group with Haar measure dU. Recall that we may identify α with the center c_α of the Bergman kube K_α. If we define $K_{U^{-1}\alpha} = U^{-1}K_\alpha$, then $\{K_{U^{-1}\alpha}\}_{\alpha\in\mathcal{T}_n} \equiv \{U^{-1}K_\alpha\}_{\alpha\in\mathcal{T}_n}$ is the Bergman grid rotated by U^{-1}, and we denote by $U^{-1}\mathcal{T}_n$ the corresponding tree. The same construction applies to obtain the rotated ring tree $U^{-1}\mathcal{R}_n$, and the compatibility relation (4) persists between $U^{-1}\mathcal{T}_n$ and $U^{-1}\mathcal{R}_n$ since $[U^{-1}\alpha] = U^{-1}[\alpha]$.

1.2 Carleson measures

We show that Carleson measures for the Besov-Sobolev space $B_p^\sigma(\mathbb{B}_n)$ are characterized by the tree condition

$$\sum_{\beta\geq\alpha}\left[2^{\sigma d(\beta)}I^*\mu(\beta)\right]^{p'} \leq CI^*\mu(\alpha) < \infty, \qquad \alpha \in \mathcal{T}_n, \tag{5}$$

for the two ranges (9) and (10) below:

$$1 < p < 1 + \frac{1-\sigma}{n-1+\sigma}, \qquad 0<\sigma<1,$$
$$\frac{1}{1-\sigma} < p < 2 + \frac{1-2\sigma}{n-1+\sigma}, \qquad 0<\sigma<\frac{1}{2}.$$

In the case $p = 2$, $0 \leq \sigma < \frac{n}{2}$, where the Hardy and Besov scales coincide, is in fact the case that Carleson measures for the Hardy-Besov-Sobolev space $H^2_{\frac{n}{2}-\sigma}(\mathbb{B}_n) = B_2^\sigma(\mathbb{B}_n)$ are characterized by the tree condition (5) *if and only if* $0 \leq \sigma < \frac{1}{2}$. More generally, if $1 < p \leq 2$, $\sigma \geq \frac{1}{p}$, then by the results in [17], there is a positive measure μ on the ball that is Carleson for $H^p_{\frac{n}{p}-\sigma}(\mathbb{B}_n)$, and by (1) so also for $B_p^\sigma(\mathbb{B}_n)$, but that is *not* Carleson for a certain potential space $\mathcal{P}^p_{\frac{n}{p}-\sigma}(\mathbb{B}_n)$, and hence fails to satisfy the tree condition (5). Our positive result, that the Carleson measures for $B_p^\sigma(\mathbb{B}_n)$ are characterized by the tree condition, is only valid for $0 \leq \sigma < \frac{1}{p'}$, leaving the gap $\sigma \in \left[\frac{1}{p'}, \frac{1}{p}\right)$ where we do not know if the tree condition characterizes Carleson measures for $B_p^\sigma(\mathbb{B}_n)$.

Given a positive measure μ on the ball, we denote by $\widehat{\mu}$ the associated measure on the Bergman tree $\mathcal{T}_n$ given by $\widehat{\mu}(\alpha) = \int_{K_\alpha} d\mu$ for $\alpha \in \mathcal{T}_n$. We will often write $\mu(\alpha)$ for $\widehat{\mu}(\alpha)$ when no confusion should arise. Let $1 < p < \infty$ and $\sigma \geq 0$. In this section we show that μ is a $B_p^\sigma(\mathbb{B}_n)$-Carleson measure on $\mathbb{B}_n$ if $\widehat{\mu}$ is a $B_p^\sigma(\mathcal{T}_n)$-Carleson measure, i.e. if it satisfies

$$\left(\sum_{\alpha\in\mathcal{T}_n} If(\alpha)^p \mu(\alpha)\right)^{1/p} \leq C\left(\sum_{\alpha\in\mathcal{T}_n}\left[2^{-\sigma d(\alpha)} f(\alpha)\right]^p\right)^{1/p}, \qquad f \geq 0, \tag{6}$$

where $If(\alpha) = \sum_{\beta\leq\alpha} f(\beta)$. Here we define $B_p^\sigma(\mathcal{T}_n)$ to consist of all sequences F on the Bergman tree $\mathcal{T}_n$ satisfying

$$\|F\|^p_{B_p^\sigma(\mathcal{T}_n)} \equiv |F(0)|^p + \sum_{\alpha\in\mathcal{T}_n} |F(\alpha) - F(A\alpha)|^p < \infty,$$

where $A\alpha$ denotes the immediate predecessor of α. The Carleson embedding $B_p^\sigma(\mathcal{T}_n) \subset \ell^p(\mu)$ is easily seen to be equivalent to (6), and the dual of (6) is

$$\left(\sum_{\alpha\in\mathcal{T}_n}\left[2^{\sigma d(\alpha)} I^* g\mu(\alpha)\right]^{p'}\right)^{1/p'} \leq C\left(\sum_{\alpha\in\mathcal{T}_n} g(\alpha)^{q'} \mu(\alpha)\right)^{1/q'}, \qquad g \geq 0. \tag{7}$$

Theorem 3 shows that (6) is equivalent to the tree condition

$$\sum_{\beta\geq\alpha}\left[2^{\sigma d(\beta)} I^*\mu(\beta)\right]^{p'} \leq C I^*\mu(\alpha) < \infty, \qquad \alpha \in \mathcal{T}_n. \tag{8}$$

Conversely, in each of the ranges

$$1 < p < 1 + \frac{1-\sigma}{n-1+\sigma}, \qquad 0 < \sigma < 1, \tag{9}$$

and

$$\frac{1}{1-\sigma} < p < 2 + \frac{1-2\sigma}{n-1+\sigma}, \qquad 0 < \sigma < \frac{1}{2}, \tag{10}$$

we show that μ is $B_p^\sigma(\mathcal{T}_n)$-Carleson if μ is a $B_p^\sigma(\mathbb{B}_n)$-Carleson measure on $\mathbb{B}_n$. Necessity in the remaining ranges of p and σ is left open. Note that the two ranges above overlap if and only if $\sigma < \frac{1}{n+1}$.

Theorem 2 *Suppose $1 < p < \infty$, $\sigma \geq 0$ and that the structural constants λ, θ in the construction of $\mathcal{T}_n$ (subsection 2.2. of [8]) satisty $\lambda = 1$ and $\theta = \frac{\ln 2}{2}$. Let μ be a positive measure on the unit ball $\mathbb{B}_n$. Then with constants depending only on σ, p, n, conditions 2 and 3 below are equivalent, condition 3 is sufficient for condition 1, and provided that* either *p and σ satisfy (9),* or *p and σ satisfy (10), condition 3 is necessary for condition 1:*

1. *μ is a $B_p^\sigma(\mathbb{B}_n)$-Carleson measure on $\mathbb{B}_n$.*
2. *$\widehat{\mu} = \{\mu(\alpha)\}_{\alpha\in\mathcal{T}_n}$ is a $B_p^\sigma(\mathcal{T}_n)$-Carleson measure on the Bergman tree $\mathcal{T}_n$, i.e. (6) holds with $\mu(\alpha) = \int_{K_\alpha} d\mu$.*
3. *There is $C < \infty$ such that*

$$\sum_{\beta\geq\alpha} \left[2^{\sigma d(\beta)} I^*\mu(\beta)\right]^{p'} \leq C I^*\mu(\alpha) < \infty, \qquad \alpha \in \mathcal{T}_n.$$

We begin with the purely tree equivalence of conditions 2 and 3 in Theorem 2.

1.2.1 Unified proofs for trees

Here we will give a short proof that the two weight tree condition,

$$\sum_{\beta\in\mathcal{T}:\beta\geq\alpha} I^*\mu(\beta)^{p'}\omega(\beta) \leq C_0^{p'} I^*\mu(\alpha) < \infty, \qquad \alpha\in\mathcal{T}, \tag{11}$$

implies the dual Besov-Carleson embedding,

$$\sum_{\alpha\in\mathcal{T}} I^*(g\mu)(\alpha)^{p'}\omega(\alpha) \leq C^{p'} \sum_{\alpha\in\mathcal{T}} g(\alpha)^{p'}\mu(\alpha), \qquad g\geq 0, \tag{12}$$

for finite positive measures μ and ω on the tree $\mathcal{T}$. Note that the Besov-Carleson embedding is

$$\sum_{\alpha\in\mathcal{T}} I(f\omega)(\alpha)^p\mu(\alpha) \leq C^p \sum_{\alpha\in\mathcal{T}} f(\alpha)^p\omega(\alpha), \qquad f\geq 0.$$

Moreover, we will unify this result and the well-known equivalence of the Hardy-Carleson embedding on the tree,

$$\sum_{\alpha\in\mathcal{T}} \left(\frac{1}{|\mathcal{S}(\alpha)|_\nu}\int_{\mathcal{S}(\alpha)} f d\nu\right)^p \sigma(\alpha) \leq C^p \int_{\mathcal{G}_\mathcal{T}} f^p d\nu, \qquad f\geq 0 \text{ on } \mathcal{G}_\mathcal{T}, \tag{13}$$

with the simple condition on geodesics,

$$\sum_{\beta \geq \alpha} \sigma(\beta) \leq C_0^p \left|\mathcal{S}(\alpha)\right|_\nu, \qquad \alpha \in \mathcal{T}, \tag{14}$$

where σ is a finite positive measure on $\mathcal{T}$, ν is a finite positive measure on the set $\mathcal{G}_{\mathcal{T}}$ of maximal geodesics of $\mathcal{T}$ starting at the root, and $\mathcal{S}(\alpha)$ denotes the collection of all geodesics in $\mathcal{G}_{\mathcal{T}}$ passing through α (i.e. that are eventually in $S(\alpha)$).

The unification is achieved by viewing each of the measures μ, ω, σ and ν as living in the closure $\mathcal{T}^* = \mathcal{T} \cup \mathcal{G}_{\mathcal{T}}$ of the tree $\mathcal{T}$, with μ, ω and σ supported in $\mathcal{T}$, and ν supported in $\mathcal{G}_{\mathcal{T}}$. If we let $\mathcal{S}^*(\alpha) = S(\alpha) \cup \mathcal{S}(\alpha)$ be the union of the successor set $S(\alpha)$ with its boundary geodesics, then we can rewrite (12) as

$$\int_{\mathcal{T}} \left(\frac{1}{\left|\mathcal{S}^*(\alpha)\right|_\mu} \int_{\mathcal{S}^*(\alpha)} g d\mu \right)^{p'} \left|\mathcal{S}^*(\alpha)\right|_\mu^{p'} d\omega(\alpha) \leq C^{p'} \int_{\mathcal{T}^*} g^{p'} d\mu, \qquad g \geq 0 \text{ on } \mathcal{T}^*, \tag{15}$$

and we can rewrite (13) as

$$\int_{\mathcal{T}} \left(\frac{1}{\left|\mathcal{S}^*(\alpha)\right|_\nu} \int_{\mathcal{S}^*(\alpha)} f d\nu \right)^{p} d\sigma(\alpha) \leq C^{p} \int_{\mathcal{T}^*} f^{p} d\nu, \qquad f \geq 0 \text{ on } \mathcal{T}^*. \tag{16}$$

Thus we see that the first inequality (15) has exactly the same form as the second (16), but with $\left|\mathcal{S}^*(\cdot)\right|_\mu^{p'} d\omega(\cdot)$ in place of $d\sigma$, μ in place of ν, and p' in place of p. Note that the integrations on the left are over $\mathcal{T}$, where the averages on $\mathcal{S}^*(\alpha)$ are defined. Moreover, the tree condition (11) is just the simple condition (14) for the measures $\left|\mathcal{S}^*(\cdot)\right|_\mu^{p'} d\omega(\cdot)$ and μ:

$$\sum_{\beta \geq \alpha} \left|\mathcal{S}^*(\beta)\right|_\mu^{p'} \omega(\beta) \leq C_0^p \left|\mathcal{S}^*(\alpha)\right|_\mu, \qquad \alpha \in \mathcal{T}.$$

In fact, if one permits ν in the second inequality (16) above to live in all of the closure $\mathcal{T}^*$, then we can characterize (16) by a simple condition, and if one permits σ to live in all of $\mathcal{T}^*$ as well, then the corresponding maximal inequality is characterized by a simple condition. The following theorem will play a pivotal role below in characterizing Carleson measures for the Drury-Arveson Hardy space $B_2^{\frac{1}{2}}(\mathbb{B}_n)$.

Theorem 3 *Let $1 < p < \infty$. Inequality (16) holds if and only if*

$$\left|S(\alpha)\right|_\sigma \leq C_0^p \left|\mathcal{S}^*(\alpha)\right|_\nu, \qquad \alpha \in \mathcal{T}. \tag{17}$$

More generally, if both σ and ν live in $\mathcal{T}^*$, then the maximal inequality

$$\int_{\mathcal{T}^*} \mathcal{M}f(\zeta)^p d\sigma(\zeta) \leq C^p \int_{\mathcal{T}^*} |f|^p d\nu, \qquad \textit{for all } f \textit{ on } \mathcal{T}^*, \tag{18}$$

where

$$\mathcal{M}f(\zeta) = \mathcal{M}(fd\nu)(\zeta) = \sup_{\alpha\in\mathcal{T}:\alpha\geq\zeta} \frac{1}{|\mathcal{S}^*(\alpha)|_\nu} \int_{\mathcal{S}^*(\alpha)} |f|\, d\nu,$$

holds if and only if

$$|\mathcal{S}^*(\alpha)|_\sigma \leq C_0^p\, |\mathcal{S}^*(\alpha)|_\nu, \qquad \alpha \in \mathcal{T}. \tag{19}$$

Proof. The necessity of (17) for (16), and also (19) for (18), follows upon setting $f = \chi_{\mathcal{S}^*(\alpha)}$ in the respective inequality. To see that (19) is sufficient for (18), which includes the assertion that (17) is sufficient for (16), note that the sublinear map $\mathcal{M}$ is bounded with norm 1 from $L^\infty(\mathcal{T}^*;\nu)$ to $L^\infty(\mathcal{T}^*;\sigma)$, and is weak type $1-1$ with constant C_0 by (19). Indeed,

$$\{\zeta \in \mathcal{T}^* : \mathcal{M}f(\zeta) > \lambda\} \subset \cup\{\mathcal{S}^*(\alpha) : \alpha \in \mathcal{T} \text{ and } \mathcal{M}f(\alpha) > \lambda\},$$

and if we let $\lambda > 0$ and denote by Γ the minimal elements in $\{\alpha \in \mathcal{T} : \mathcal{M}f(\alpha) > \lambda\}$, then

$$\begin{aligned} |\{\zeta \in \mathcal{T}^* : \mathcal{M}f(\zeta) > \lambda\}|_\sigma &\leq \sum_{\alpha\in\Gamma} |\mathcal{S}^*(\alpha)|_\sigma \leq C_0^p \sum_{\alpha\in\Gamma} |\mathcal{S}^*(\alpha)|_\nu \\ &\leq C_0^p \sum_{\alpha\in\Gamma} \lambda^{-1} \int_{\mathcal{S}^*(\alpha)} |f|\, d\nu \leq C_0^p \lambda^{-1} \int_{\mathcal{T}^*} |f|\, d\nu. \end{aligned}$$

Marcinkiewicz interpolation now completes the proof.

The proof actually yields the following more general inequality.

Theorem 4 *Let $1 < p < \infty$. Then*

$$\int_{\mathcal{T}^*} \mathcal{M}f(\zeta)^p\, d\sigma(\zeta) \leq C^p \int_{\mathcal{T}^*} |f|^p\, \mathcal{M}(d\sigma)\, d\nu, \qquad \textit{for all } f \textit{ on } \mathcal{T}^*.$$

1.3 Proof of the Carleson measure theorem

Theorem 3 above yields the equivalence of conditions 2 and 3 in Theorem 2, and we will consider the necessity and sufficiency of condition 3 for condition 1 separately in the next two subsections. Note that when the structural constants λ, θ in the construction of the Bergman tree $\mathcal{T}_n$ (see subsection 2.2 of [8]) satisfy $\lambda = 1$ and $\theta = \frac{\ln 2}{2}$, we have that the dimension of $\mathcal{T}_n$ is n, and

$$1 - |z|^2 \approx e^{-2\beta(0,z)} \approx e^{-2\theta d(\alpha)} = 2^{-d(\alpha)} \tag{20}$$

for $z \in K_\alpha$.

First we dualize the Carleson embedding by computing its adjoint relative to the pairing

$$\begin{aligned}\langle f,g\rangle_{\alpha,p}^{\sigma} &= \left\langle \mathcal{R}^{\alpha}_{\frac{n+1+\alpha}{p}-\sigma} f, \mathcal{R}^{\alpha}_{\frac{n+1+\alpha}{p'}-\sigma} g \right\rangle_{\alpha} = \int_{\mathbb{B}_n} \mathcal{R}^{\alpha}_{\frac{n+1+\alpha}{p}-\sigma} f(z)\, \overline{\mathcal{R}^{\alpha}_{\frac{n+1+\alpha}{p'}-\sigma} g(z)} d\nu_{\alpha}(z) \\ &= \int_{\mathbb{B}_n} \left\{ \left(1-|z|^2\right)^{\frac{n+1+\alpha}{p}} \mathcal{R}^{\alpha}_{\frac{n+1+\alpha}{p}-\sigma} f(z) \right\} \overline{\left\{ \left(1-|z|^2\right)^{\frac{n+1+\alpha}{p'}} \mathcal{R}^{\alpha}_{\frac{n+1+\alpha}{p'}-\sigma} g(z) \right\}} d\lambda_n(z),\end{aligned}$$

and then restate the dual inequality as

$$\left\| S_{\mu}^{\sigma} g \right\|_{L^{p'}(\lambda_n)} \leq C \left\| g \right\|_{L^{q'}(\mu)}, \qquad g \in L^{q'}(\mu), \tag{21}$$

where the operator S_{μ}^{σ} is given by

$$S_{\mu}^{\sigma} g(w) = \int_{\mathbb{B}_n} \left(1-|w|^2\right)^{-\sigma} \left(\frac{1-|w|^2}{1-\overline{z}\cdot w} \right)^{\frac{n+1+\alpha}{p'}+\sigma} g(z)\, d\mu(z).$$

1.3.1 Necessity in the range $1 < p < 1 + \frac{1-\sigma}{n-1+\sigma}, 0 \leq \sigma < 1$

Let $0 \leq \sigma < 1$. Suppose that μ is a $B_p^{\sigma}(\mathbb{B}_n)$-Carleson measure on $\mathbb{B}_n$ where $1 < p < 1 + \frac{1-\sigma}{n-1+\sigma}$. Choose $\alpha > -1$ so that $\frac{n+1+\alpha}{p'} = 1 - \sigma$. We then obtain from (21) that

$$\int_{\mathbb{B}_n} \left| \int_{\mathbb{B}_n} \left(1-|w|^2\right)^{-\sigma} \operatorname{Re}\left(\frac{1-|w|^2}{1-\overline{z}\cdot w} \right) g(z)\, d\mu(z) \right|^{p'} d\lambda_n(w) \leq \left\| S_{\mu}^{\sigma} g \right\|_{L^{p'}(\lambda_n)}^{p'} \leq C \left(\int_{\mathbb{B}_n} |g|^{p'} d\mu \right)$$

for all $g \geq 0$. The tree inequality (6) now follows as in the one-dimensional case in [7]. Indeed, fix $\alpha \in \mathcal{T}_n$ and let $g = \sum_{\alpha \in n} g(\alpha) \chi_{K_{\alpha}}$. Here g is constant on K_{α} with value $g(\alpha)$. Then since

$$\operatorname{Re}\left(\frac{1-|w|^2}{1-\overline{z}\cdot w} \right) \geq c > 0, \qquad w \in K_{\beta}, z \in S_{\beta}$$

for $\beta \geq \alpha$, $\operatorname{Re}\left(\frac{1-|w|^2}{1-\overline{z}\cdot w} \right) \geq 0$ otherwise, and $1-|w|^2 \approx 2^{-d(\beta)}$ for $w \in K_{\beta}$ (since $\theta = \frac{\ln 2}{2}$), we obtain

$$\begin{aligned} c^{p'} \left\| 2^{\sigma d} I^* (g\mu) \right\|_{L^{p'}(\mathcal{T}_n)}^{p'} &= \sum_{\alpha \in \mathcal{T}_n} \left(2^{\sigma d(\alpha)} \sum_{\beta \geq \alpha} c(g\mu)(K_{\beta}) \right)^{p'} \\ &\leq C \int_{\mathbb{B}_n} \left(\int \left(1-|w|^2\right)^{-\sigma} \operatorname{Re}\left(\frac{1-|w|^2}{1-\overline{z}\cdot w} \right) g(z)\, d\mu(z) \right)^{p'} d\lambda_n(w) \end{aligned}$$

$$\leq C\left(\int_{\mathbb{B}_n}|g|^{q'}d\mu\right)^{p'/q'}$$

$$= C\left(\sum_{\alpha\in\mathcal{T}_n}|g(\alpha)|^{q'}\mu(\alpha)\right)^{\frac{p'}{q'}} = C\|g\|^{p'}_{L^{q'}(\mathcal{T}_n)},$$

which yields (7), the dual of inequality (6), and hence (8).

Unfortunately, this elegant proof breaks down for $p \geq 1+\frac{1-\sigma}{n-1+\sigma}$, since we can no longer choose $\alpha > -1$ so that $\theta = \frac{n+1+\alpha}{p'} \in (0,1]$ forces $\operatorname{Re}\left(\frac{1-|w|^2}{1-\overline{z}\cdot w}\right)^{\theta} > 0$.

1.3.2 Sufficiency in the range $1 < p < \infty$, $\sigma \geq 0$

Suppose that $\widehat{\mu}$ satisfies the tree inequality (6). Since $\widehat{\mu} \leq \widehat{\widetilde{\mu}}$, where $\widetilde{\mu}$ is the discretization of μ, we now replace μ by $\widetilde{\mu}$ in the dual inequality (21) and consider

$$S^{\sigma}_{\widetilde{\mu}}g\left(w\right) = \int_{\mathbb{B}_n}\left(1-|w|^2\right)^{-\sigma}\left(\frac{1-|w|^2}{1-\overline{z}\cdot w}\right)^{\frac{n+1+\alpha}{p'}+\sigma} g\left(z\right)d\widetilde{\mu}\left(z\right), \qquad w\in\mathbb{B}_n,$$

We will show that the positive operator $T^{\sigma}_{\widetilde{\mu}}$ given by

$$T^{\sigma}_{\widetilde{\mu}}g\left(w\right) = \int_{\mathbb{B}_n}\left(1-|w|^2\right)^{-\sigma}\left(\frac{1-|w|^2}{|1-w\cdot\overline{z}|}\right)^{\frac{n+1+\alpha}{p'}+\sigma} g\left(z\right)d\widetilde{\mu}\left(z\right), \qquad w\in\mathbb{B}_n,$$

is bounded from $L^{q'}\left(\widetilde{\mu}\right)$ to $L^{p'}\left(\lambda_n\right)$, i.e.

$$\left(\int_{\mathbb{B}_n}\left|T^{\sigma}_{\widetilde{\mu}}g\left(w\right)\right|^{p'}d\lambda_n\left(w\right)\right)^{\frac{1}{p'}} \leq C\left(\int_{\mathbb{B}_n}g^{q'}d\widetilde{\mu}\right)^{\frac{1}{q'}}, \qquad g\geq 0. \tag{22}$$

With this done, it follows that $\widetilde{\mu}$ is a (B^{σ}_p, q)-Carleson measure, and hence also μ.

From Lemma 5 below with $\tau=\sigma$, $s=n+1+\alpha$, $r=p'$ and

$$f\left(z\right) = g\left(z\right)\frac{d\widetilde{\mu}}{dz}\left(z\right) = g\left(z\right)\sum_{\alpha\in\mathcal{T}_n}\mu\left(K_\alpha\right)\left(1-|z|^2\right)^{-n-1}\chi_{K_\alpha}\left(z\right),$$

we obtain with $\widehat{T}$ as in Lemma 5,

$$\left\|T^{\sigma}_{\widetilde{\mu}}g\right\|_{L^{p'}(d\lambda_n)} = \left\|\widehat{T}\left(g\widetilde{\mu}\right)\right\|_{L^{p'}(d\lambda_n)} \leq C_{p'}\left(\sum_{\alpha\in\mathcal{T}_n}\left[2^{\sigma d(\alpha)}I^*\left(g\widetilde{\mu}\right)\left(\alpha\right)\right]^{p'}\right)^{\frac{1}{p'}},$$

where

$$\begin{aligned} I^*\left(g\widetilde{\mu}\right)\left(\alpha\right) &= \sum_{\beta\in\mathcal{T}_n:\beta\geq\alpha}\int_{K_\beta}gd\widetilde{\mu} = \sum_{\beta\in\mathcal{T}_n:\beta\geq\alpha}\mu\left(K_\beta\right)\int_{K_\beta}gd\lambda_n \\ &\approx \sum_{\beta\in\mathcal{T}_n:\beta\geq\alpha}\widehat{\widetilde{\mu}}\left(\beta\right)\widehat{g}\left(\beta\right) = I^*\left(\widehat{g}\widehat{\widetilde{\mu}}\right)\left(\alpha\right), \end{aligned}$$

and $\widehat{g}(\alpha) = \int_{K_\alpha} g d\lambda_n$. The dual tree inequality (7) holds for $\widehat{\mu}$, and it is an easy exercise using

$$I^*\left(g\widehat{\widehat{\mu}}\right)(\alpha) = \sum_{\beta \geq \alpha} g(\beta) \sum_{d(\beta,\gamma) \leq N} \mu(\gamma) \leq \sum_{\beta \geq A^N \alpha} g(\beta) C_N \mu(\beta) \leq C_N I^*(g\widehat{\mu})\left(A^N \alpha\right)$$

to show that the dual tree inequality then also holds for $\widehat{\widehat{\mu}}$, but with a larger constant, i.e.

$$\left(\sum_{\alpha \in \mathcal{T}_n} \left[2^{\sigma d(\alpha)} I^*\left(\widehat{g}\widehat{\widehat{\mu}}\right)(\alpha)\right]^{p'}\right)^{\frac{1}{p'}} \leq C \left(\sum_{\alpha \in \mathcal{T}_n} \widehat{g}(\alpha)^{q'} \widehat{\widehat{\mu}}(\alpha)\right)^{\frac{1}{q'}}, \qquad \widehat{g} \geq 0.$$

Finally, since $\widetilde{\mu}$ is constant on K_α, we have

$$\widehat{g}(\alpha)^{q'} \widehat{\widehat{\mu}}(\alpha) = \widetilde{\mu}(K_\alpha) \left(\int_{K_\alpha} g d\lambda_n\right)^{q'} \leq \widetilde{\mu}(K_\alpha) \int_{K_\alpha} g^{q'} d\lambda_n = \int_{K_\alpha} g^{q'} d\widetilde{\mu},$$

for all $\alpha \in \mathcal{T}_n$, and hence

$$\left(\sum_{\alpha \in \mathcal{T}_n} \widehat{g}(\alpha)^{q'} \widehat{\widehat{\mu}}(\alpha)\right)^{\frac{1}{q'}} \leq C \left(\int_{\mathbb{B}_n} g^{q'} d\widetilde{\mu}\right)^{\frac{1}{q'}}.$$

Combining these inequalities establishes (22) as required.

Lemma 5 *For $\tau \geq 0$, $1 < r < \infty$, $s + \tau r > n$ and ν a positive Borel measure on $\mathbb{B}_n$ define*

$$\widehat{T}\nu(w) = \int_{\mathbb{B}_n} \frac{\left(1 - |w|^2\right)^{\frac{s}{r}}}{|1 - w \cdot \overline{z}|^{\frac{s+\tau r}{r}}} d\nu(z).$$

Then we have

$$\left\|\widehat{T}f\right\|_{L^r(d\lambda_n)} \leq C_r \left(\sum_{\alpha \in \mathcal{T}_n} \left[e^{2\theta\tau d(\alpha)} I^* f(\alpha)\right]^r\right)^{\frac{1}{r}},$$

where $I^*\nu(\alpha) = \sum_{\beta \in \mathcal{T}_n : \beta \geq \alpha} \nu(\beta)$ and $\nu(\beta) = \int_{K_\beta} d\nu$.

Note that the discretization of g used in the previous argument corresponds to the discretization of the measure $g d\lambda_n$ defined in the above lemma. We omit the technical proof of this lemma, and this completes our proof that condition 3 is sufficient for condition 1 in Theorem 2.

1.3.3 Necessity in the range $\frac{1}{1-\sigma} < p < 2 + \frac{1-2\sigma}{n-1+\sigma}$, $0 \leq \sigma < \frac{1}{2}$

We omit the lengthy T^*T argument needed to prove this. We note however the main estimate used in the proof:

$$\operatorname{Re}\left(\frac{1}{1-\overline{z}\cdot z'}\right)^{\eta} \geq c_\eta 2^{\eta d(\alpha\wedge\beta)}, \qquad z \in K_\alpha, z' \in K_\beta, \tag{23}$$

for a positive constant c_η, provided $0 < \eta < 1$. Note that c_η tends to 0 as $\eta \to 1$, so that we cannot use $\eta = 1$ even though $\operatorname{Re}\frac{1}{1-\overline{z}\cdot z'} > 0$ on the ball. It is this limitation on η that results in the restriction (10).

1.4 The Drury-Arveson Hardy space $B_2^{\frac{1}{2}}(\mathbb{B}_n)$

The above theorem just misses capturing the Drury-Arveson Hardy space $B_2^{\frac{1}{2}}(\mathbb{B}_n)$. If we take $p = 2$, $\sigma = \frac{1}{2}$ and $\eta = 1$ in the above proof, then the case $\eta = 1$ of (23) is weakened to the inequality

$$\operatorname{Re}\frac{1}{1-\overline{z}\cdot z'} = \frac{\operatorname{Re}(1-\overline{z}\cdot z')}{|1-\overline{z}\cdot z'|^2} \geq c + c2^{2d(\alpha\wedge\alpha')-d([\alpha]\wedge[\alpha'])}, \qquad z \in K_\alpha, z' \in K_{\alpha'}, \tag{24}$$

which does not lead to the tree condition. We will however modify the proof so as to give a characterization in Theorem 7 below of the Carleson measures for $B_2^{\frac{1}{2}}(\mathbb{B}_n)$ in terms of the simple condition (31) and "split" tree condition (52) given below. We will proceed initially via the following three **steps**:

Step 1 If a positive measure μ is $B_2^{\frac{1}{2}}(\mathbb{B}_n)$-Carleson, then the bilinear inequality

$$\left|\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\left(\frac{1}{1-\overline{z}\cdot z'}\right) f(z')\,d\mu(z')\,g(z)\,d\mu(z)\right| \leq C\|f\|_{L^2(\mu)}\|g\|_{L^2(\mu)}, \tag{25}$$

holds for all $f, g \in L^2(\mu)$.

Step 2 If a positive measure μ satisfies the Dirichlet tree condition, i.e. (8) for $p = 2$ and $\sigma = 0$;

$$\sum_{\beta\geq\alpha} I^*\mu(\beta)^2 \leq CI^*\mu(\alpha) < \infty, \qquad \alpha \in \mathcal{T}_n, \tag{26}$$

then μ is $B_2^{\frac{1}{2}}(\mathbb{B}_n)$-Carleson if and only if the bilinear inequality

$$\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\left(\operatorname{Re}\frac{1}{1-\overline{z}\cdot z'}\right) f(z')\,d\mu(z')\,g(z)\,d\mu(z) \leq C\|f\|_{L^2(\mu)}\|g\|_{L^2(\mu)}, \tag{27}$$

holds for all $f, g \geq 0$.

Step 3 Suppose a positive measure μ satisfies (26) as well as the following growth condition on Bergman balls: for each $R \geq 1$, there is a positive constant C_R such that

$$\mu\left(B_\beta\left(w,R\right)\right) \leq C_R \mu\left(B_\beta\left(w,1\right)\right), \qquad w \in \mathbb{B}_n, \tag{28}$$

where $B_\beta\left(w,t\right) = \left\{z \in \mathbb{B}_n : \beta\left(z,w\right) < t\right\}$ is the Bergman ball of radius t about w. Then the bilinear inequality (27) is equivalent to the discrete inequality,

$$\sum_{\alpha \in \mathcal{T}_n} \left|T_\mu g\left(\alpha\right)\right|^2 \mu\left(\alpha\right) \leq C \sum_{\alpha \in \mathcal{T}_n} \left|g\left(\alpha\right)\right|^2 \mu\left(\alpha\right), \qquad g \geq 0, \tag{29}$$

where T_μ is the positive linear operator on the tree given by,

$$T_\mu g\left(\alpha\right) = \sum_{\beta \in \mathcal{T}_n} 2^{2d(\alpha\wedge\beta) - d([\alpha]\wedge[\beta])} g\left(\beta\right) \mu\left(\beta\right), \qquad \alpha \in \mathcal{T}_n, \tag{30}$$

and where $\mathcal{T}_n$ ranges over all unitary rotations of a fixed Bergman tree (and $[\alpha] \in \mathcal{R}_n$ which ranges over the corresponding unitary rotations of the associated ring tree).

Using the above three **steps** we can characterize Carleson measures for the Drury-Arveson Hardy space $B_2^{\frac{1}{2}}\left(\mathbb{B}_n\right)$ by the Dirichlet tree conditon (26) together with either (27) or (29). As an alternative to the Dirichlet tree conditon (26), we can also use the the simple condition

$$2^{d(\alpha)} I^* \mu\left(\alpha\right) \leq C, \qquad \alpha \in \mathcal{T}_n, \tag{31}$$

(recall that $p\sigma = 1$ and that $\theta = \frac{\ln 2}{2}$ so that $1 - |w|^2 \approx 2^{-d(\alpha)} = 2^{-p\sigma d(\alpha)}$ for $w \in K_\alpha$ by (20)), which is far stronger than the Dirichlet tree condition (26) in general:

$$\begin{aligned}
\sum_{\beta \geq \alpha} I^* \mu\left(\beta\right)^2 &= \sum_{\beta \geq \alpha} \sum_{\gamma,\gamma' \geq \beta} \mu\left(\gamma\right) \mu\left(\gamma'\right) = \sum_{\gamma,\gamma' \geq \alpha} \left[1 + d\left(\alpha, \gamma \wedge \gamma'\right)\right] \mu\left(\gamma\right) \mu\left(\gamma'\right) \qquad (32) \\
&= \sum_{\gamma \geq \alpha} \mu\left(\gamma\right) \sum_{\gamma' \geq \alpha} \left[1 + d\left(\alpha, \gamma \wedge \gamma'\right)\right] \mu\left(\gamma'\right) \\
&\leq \sum_{\gamma \geq \alpha} \mu\left(\gamma\right) \sum_{j=0}^{d(\alpha,\gamma)} \left(1 + d\left(\alpha,\gamma\right) - j\right) I^* \mu\left(A^j \gamma\right) \\
&\leq \sum_{\gamma \geq \alpha} \mu\left(\gamma\right) \sum_{j=0}^{d(\alpha,\gamma)} \left(1 + d\left(\alpha,\gamma\right) - j\right) C 2^{j - d(\gamma)} \\
&\leq C 2^{-d(\alpha)} I^* \mu\left(\alpha\right) \leq C I^* \mu\left(\alpha\right).
\end{aligned}$$

Recall that $\widehat{\mu}\left(\alpha\right) = \mu\left(\alpha\right) = \int_{K_\alpha} d\mu$ for $\alpha \in \mathcal{T}_n$.

Theorem 6 *Let μ be a positive measure on the ball $\mathbb{B}_n$. Then the following conditions are equivalent:*

1. *μ is a Carleson measure on the Drury-Arveson space $B_2^{\frac{1}{2}}(\mathbb{B}_n)$,*
2. *$\widehat{\mu}$ satisfies both (31) and (27),*
3. *$\widehat{\mu}$ satisfies both (31) and (29),*
4. *$\widehat{\mu}$ satisfies both (26) and (27),*
5. *$\widehat{\mu}$ satisfies both (26) and (29).*

In Theorem 7 of the next subsubsection, we will complete the characterization of Carleson measures for the Drury-Arveson space by giving a necessary and sufficient condition, namely (52) below, for (29) in the presence of (31).

Proof. If μ is $B_2^{\frac{1}{2}}(\mathbb{B}_n)$-Carleson, then it is also $B_2^{\sigma}(\mathbb{B}_n)$-Carleson for all $0 \leq \sigma \leq \frac{1}{2}$, in particular for $\sigma = 0$, so that (26) holds. Moreover, (21) with $g = \chi_{S(\alpha)}$ yields the simple condition (31). Indeed (we drop the $\widehat{}$ from $\widehat{\mu}$ when no confusion should arise):

$$
\begin{aligned}
I^*\mu(\alpha) &\geq c\left\|\chi_{S(\alpha)}\right\|^2_{L^2(\mu)} \geq c\left\|S_\mu^{\frac{1}{2}}\chi_{S(\alpha)}\right\|^2_{L^2(\lambda_n)} \\
&= c\int_{\mathbb{B}_n}\left|\int_{S(\alpha)}\frac{\left(1-|w|^2\right)^{\frac{n+1+\alpha}{2}}}{(1-\overline{z}\cdot w)^{\frac{n+1+\alpha}{2}+\frac{1}{2}}}d\mu(z)\right|^2\frac{dw}{\left(1-|w|^2\right)^{n+1}} \\
&\geq c\int_{w\in\cup_{\beta\leq\alpha}K_\beta}\left|\int_{S(\alpha)}\frac{\left(1-|w|^2\right)^{\frac{n+1+\alpha}{2}}}{(1-\overline{z}\cdot w)^{\frac{n+1+\alpha}{2}+\frac{1}{2}}}d\mu(z)\right|^2\frac{dw}{\left(1-|w|^2\right)^{n+1}} \\
&\approx \int_{w\in\cup_{\beta\leq\alpha}K_\beta}\left|\int_{S(\alpha)}d\mu\right|^2\frac{dw}{\left(1-|w|^2\right)^{n+2}} \\
&\approx I^*\mu(\alpha)^2\left(1-|w|^2\right)^{-1} \approx I^*\mu(\alpha)^2\, 2^{d(\alpha)}.
\end{aligned}
$$

Note that we may choose structural constants for the Bergman tree so that the successor sets $S(\alpha)$ are sufficiently narrow that $Re\,(1-\overline{z}\cdot w)^{\frac{n+1+\alpha}{2}+\frac{1}{2}} \approx |1-\overline{z}\cdot w|^{\frac{n+1+\alpha}{2}+\frac{1}{2}}$ for $z \in S(\alpha)$ and $w \in \cup_{\beta\leq\alpha}K_\beta$.

Conversely, suppose that (26) holds. We now define an associated measure $\mu^\natural$ that will satisfy (28) as well as (26):

$$
d\mu^\natural(z) = \sum_{N=1}^{\infty} 2^{-NA}\sum_{\alpha\in\mathcal{T}_n}\left(\sum_{d(\alpha,\beta)\leq N}\mu(K_\beta)\right)\chi_{K_\alpha}(z)\,d\lambda_n(z),
$$

where $A > 0$ will be chosen large enough that the Dirichlet tree condition (26) holds for the measure $\mu^{\natural}$. One way to see that such an A exists is to note that for each $N \geq 1$, the measure

$$d\mu_N(z) = \sum_{\alpha \in \mathcal{T}_n} \left(\sum_{d(\alpha,\beta) \leq N} \mu(K_\beta) \right) \chi_{K_\alpha}(z)\, d\lambda_n(z)$$

is $B_2(\mathbb{B}_n)$-Carleson with norm at most 2^{CnN} times that of μ. Clearly $\mu^{\natural}$ satisfies (28) as well. Then if either (27) or (29) holds, we obtain that $\mu^{\natural}$ is $B_2^{\frac{1}{2}}(\mathbb{B}_n)$-Carleson by applying **step 2** or **3** respectively to the measure $\mu^{\natural}$ in place of μ. Next, we note that (28) implies $\widetilde{\mu} \leq \mu^{\natural}$ and so $\widetilde{\mu}$ is $B_2^{\frac{1}{2}}(\mathbb{B}_n)$-Carleson, and hence also μ is $B_2^{\frac{1}{2}}(\mathbb{B}_n)$-Carleson.

Given the three **steps**, this completes the proof of Theorem 6.

We now turn our attention to proving some parts of the three **steps**.

Proof. (**Step 1**) We use the presentation of H_n^2 given by the kernel function $k(w,z) = \frac{1}{1-\overline{w}\cdot z}$ on $\mathbb{B}_n$ as given earlier. Now μ is H_n^2-Carleson if and only if the inclusion map T is bounded from H_n^2 to $L^2(\mu)$, hence if and only if the adjoint T^* is bounded from $L^2(\mu)$ to H_n^2, i.e.

$$\langle T^* f, T^* f \rangle_{H_n^2} \leq C \|f\|^2_{L^2(\mu)}, \qquad f \in L^2(\mu). \tag{33}$$

We claim that

$$T^* f(z) = \int k_w(z) f(w)\, d\mu(w), \qquad z \in \mathbb{B}_n. \tag{34}$$

Indeed, if μ is the delta mass δ_ζ, then

$$T^* f(z) = \langle T^* f, k_z \rangle_{H_n^2} = \langle f, T k_z \rangle_{L^2(\mu)} = f(\zeta)\, \overline{k_z(\zeta)} = f(\zeta)\, k_\zeta(z) = \int k_w(z) f(w)\, d\mu(w).$$

This also holds when μ is a finite linear combination of points masses, and a limiting argument shows it holds in general. From (34) we obtain

$$\begin{aligned}
\langle T^* f, T^* f \rangle_{H_n^2} &= \left\langle \int k_w f(w)\, d\mu(w), \int k_{w'} f(w')\, d\mu(w') \right\rangle_{H_n^2} \\
&= \int \int \langle k_w, k_{w'} \rangle_{H_n^2} f(w)\, d\mu(w)\, f(w')\, d\mu(w') \\
&= \int \int k_w(w') f(w)\, d\mu(w)\, f(w')\, d\mu(w') \\
&= \int \int \frac{1}{1 - \overline{w} \cdot w'} f(w)\, d\mu(w)\, f(w')\, d\mu(w'),
\end{aligned}$$

which in particular proves (25) if we substitute this in (33).

However, using that $\langle T^* f, T^* f\rangle_{H_n^2}$ is real, we obtain that μ is H_n^2-Carleson if and only if

$$\int\int\left(\operatorname{Re}\frac{1}{1-\overline{w}\cdot w'}\right) f(w)\,d\mu(w)\,f(w')\,d\mu(w') \leq C\|f\|^2_{L^2(\mu)}, \qquad f\geq 0,$$

which is equivalent to (27). Thus we have also proved **step 2** without recourse to (26), thus simplifying some of the arguments below.

Proof. (**Step 2**) If μ is $B_2^{\frac{1}{2}}(\mathbb{B}_n)$-Carleson, then (25) and hence also (27) holds, and we showed above that (26) holds.

Conversely, suppose that (26) and (27) hold. Given $g \in L^2(\mu)$, we compute that $\left\|S_\mu^{\frac{1}{2}} g\right\|^2_{L^2(\lambda_n)}$ is

$$\begin{aligned}
&\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\left\{\int_{\mathbb{B}_n}\frac{\left(1-|w|^2\right)^{\frac{n+1+\alpha}{2}}}{\left(1-\overline{z}\cdot w\right)^{\frac{n+2+\alpha}{2}}}\frac{\left(1-|w|^2\right)^{\frac{n+1+\alpha}{2}}}{\left(1-z'\cdot\overline{w}\right)^{\frac{n+2+\alpha}{2}}}d\lambda_n(w)\right\} g(z)\,\overline{g(z')}d\mu(z)\,d\mu(z') \\
&=\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\left\{\int_{\mathbb{B}_n}\frac{\left(1-|w|^2\right)^{\alpha}}{\left[\left(1-\overline{z}\cdot w\right)\left(1-z'\cdot\overline{w}\right)\right]^{\frac{n+2+\alpha}{2}}}dw\right\} g(z)\,\overline{g(z')}d\mu(z)\,d\mu(z') \\
&=\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}K_\alpha(z,z')\,g(z)\,\overline{g(z')}d\mu(z)\,d\mu(z'),
\end{aligned}$$

where we have denoted the integral in braces above by $K_\alpha(z,z')$.

A calculation shows that

$$K_\alpha(z,z') = c_\alpha\frac{1}{1-\overline{z}\cdot z'} + c'_\alpha \log\frac{1}{1-\overline{z}\cdot z'} + b_\alpha(z,z'), \tag{35}$$

where c_α, c'_α are positive and $b_\alpha(z,z')$ is a bounded function for $z, z' \in \mathbb{B}_n$.

Suppose now that $f, g \geq 0$. Then using the asymptotic estimate (35) with $\alpha = 0$ or 1, together with the fact that $\left\|S_\mu^{\frac{1}{2}} g\right\|^2_{L^2(\lambda_n)}$ is real, we obtain the bound

$$\begin{aligned}
\left\|S_\mu^{\frac{1}{2}} g\right\|^2_{L^2(\lambda_n)} &= c_\alpha\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\left\{\operatorname{Re}\frac{1}{1-\overline{z}\cdot z'}\right\} g(z)\,\overline{g(z')}d\mu(z)\,d\mu(z') \\
&+\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\left\{c'_\alpha\ln\left|\frac{1}{1-\overline{z}\cdot z'}\right| + \operatorname{Re} b_\alpha(z,z')\right\} g(z)\,\overline{g(z')}d\mu(z)\,d\mu(z') \\
&\leq C\|g\|^2_{L^2(\mu)},
\end{aligned}\tag{36}$$

by using (27) on the first term after the equal sign, and using the Dirichlet tree condition (26) on the second term. Indeed, using inequality (83) in Subsubsection

5.2.1 of [8] together with the argument used for (41) below, we see that the second term after the equal sign in (36) is dominated by

$$C\int_{\mathcal{U}_n}\left\{\sum_{\alpha,\alpha'\in\mathcal{T}_n} d\left(\alpha\wedge\alpha'\right)g\left(U^{-1}\alpha\right)\mu\left(U^{-1}\alpha\right)g\left(U^{-1}\alpha'\right)\mu\left(U^{-1}\alpha'\right)\right\}dU$$
$$=C\int_{\mathcal{U}_n}\left\{\sum_{\alpha\in\mathcal{T}_n} I^*g\mu\left(U^{-1}\alpha\right)^2\right\}dU.$$

We have thus proved (21), and hence that μ is a $B_2^{\frac{1}{2}}\left(\mathbb{B}_n\right)$-Carleson measure.

Proof. (**Step 3**) We first use (26) and (28) to discretize the bilinear inequality (27) to the following discrete bilinear inequality taken over all unitary rotations $U^{-1}\mathcal{T}_n$ of the Bergman tree $\mathcal{T}_n$:

$$\sum_{\alpha,\alpha'\in U^{-1}\mathcal{T}_n} 2^{2d(\alpha\wedge\alpha')-d([\alpha]\wedge[\alpha'])}f\left(\alpha\right)\mu\left(\alpha\right)g\left(\alpha'\right)\mu\left(\alpha'\right) \tag{37}$$
$$\leq C\left\{\sum_{\alpha\in U^{-1}\mathcal{T}_n} f\left(\alpha\right)^2\mu\left(\alpha\right)\right\}^{\frac{1}{2}}\left\{\sum_{\alpha'\in U^{-1}\mathcal{T}_n} g\left(\alpha'\right)^2\mu\left(\alpha'\right)\right\}^{\frac{1}{2}},$$

for all $f,g\geq 0$ on $U^{-1}\mathcal{T}_n$, and for all $U\in\mathcal{U}_n$. At a crucial point in the argument below, we need to identify the distance $1-\left|\overline{z}\cdot z'\right|^2$ in terms of the tree structure, and this is what leads to the associated ring tree $\mathcal{R}_n$ and the quantity $d\left([\alpha]\wedge[\alpha']\right)$. Recall that a *slice* of the ball $\mathbb{B}_n$ is the intersection of the ball with a complex line through the origin. In particular, every point $z\in\mathbb{B}_n\setminus\{0\}$ lies in a unique slice

$$S_z=\left\{\left(e^{i\theta}z_1,...,e^{i\theta}z_n\right):\theta\in[0,2\pi)\right\}.$$

We define two elements α and α' of the Bergman tree $\mathcal{T}_n$ to be *slice-related* if $\alpha\sim\alpha'$ where $\sim$ is the equivalence relation introduced earlier. Now given $\alpha,\alpha'\in\mathcal{T}_n$, let

$$[o,\alpha]=\{o,\alpha_1,...,\alpha_m=\alpha\}\text{ and }[o,\alpha']=\{o,\alpha_1',...,\alpha_{m'}'=\alpha'\}$$

be the geodesics from the root to α,α' respectively. We then have from (4) that α_k and α_k' are slice-related if and only if $k\leq d\left([\alpha]\wedge[\alpha']\right)$.

It may help the reader to visualize $d\left([\alpha]\wedge[\alpha']\right)$ in the following way. Imagine that each slice S is thickened to a *slab* $\mathcal{S}$ of width one in the Bergman metric. Thus in the Euclidean metric, a slab $\mathcal{S}$ is a lens whose "thickness" at any point is roughly the square root of the distance the boundary of the ball $\partial\mathbb{B}_n$. Moreover, given $z\in\mathbb{B}_n$, we denote by $\mathcal{S}_z$ the slab corresponding to the slice S_z, but truncated by intersecting it with $B\left(0,|z|\right)$. The slabs $\mathcal{S}_{c_\alpha}$ and $\mathcal{S}_{c_{\alpha'}}$ associated with the unique slices S_{c_α} and $S_{c_{\alpha'}}$ through c_α and $c_{\alpha'}$ will intersect in a "disc" of radius roughly $d([\alpha]\wedge[\alpha'])$ in the Bergman metric. Note that from this picture that $\alpha_{d([\alpha]\wedge[\alpha'])}$ is the *exit point* $E_{\alpha'}\alpha$ of the geodesic $[o,\alpha]$ from the slab $\mathcal{S}_{\alpha'}$ associated to the slice $S_{\alpha'}$ through $c_{\alpha'}$, and similarly, $\alpha'_{d([\alpha]\wedge[\alpha'])}$ is the exit point $E_\alpha\alpha'$ of the geodesic $[o,\alpha']$ from the slab $\mathcal{S}_\alpha$. Both points have the same distance from the root. Note that we can also define $E_{\alpha'}\alpha$ as the intersection of the geodesic $[o,\alpha]$ with the ring $[\alpha]\wedge[\alpha']$, which we will denote by $E_{[\alpha]\wedge[\alpha']}\alpha$. Finally, note that since $d([\alpha]\wedge[\alpha'])=d(E_{\alpha'}\alpha)=d(E_\alpha\alpha')$ and $\alpha\wedge\alpha'=\alpha_\ell$ where $\ell=\max\{k:\alpha_k=\alpha'_k\}$, we have that $d([\alpha]\wedge[\alpha'])$ satisfies

$$d(\alpha\wedge\alpha')\leq d([\alpha]\wedge[\alpha'])\leq\min\{d(\alpha),d(\alpha')\}. \tag{38}$$

The key feature of the quantity $d([\alpha]\wedge[\alpha'])$ is that $2^{-d([\alpha]\wedge[\alpha'])}$ is essentially $1-|\overline{z}\cdot z'|^2$ for $z\in K_\alpha$, $z'\in K_{\alpha'}$. More precisely, for each $z,z'\in\mathbb{B}_n$, there is a subset Σ of the unitary group $\mathcal{U}_n$ with Haar measure $|\Sigma|\geq c>0$ and satisfying

$$c2^{-d\left(\left[U^{-1}\alpha(z)\right]\wedge\left[U^{-1}\alpha(z')\right]\right)}\leq 1-|\overline{z}\cdot z'|^2\leq C2^{-d([\alpha(z)]\wedge[\alpha(z')])},\qquad U\in\Sigma. \tag{39}$$

The main inequalities used in establishing the equivalence of (27) and (37) are (24), i.e.

$$\operatorname{Re}\frac{1}{1-\overline{z}\cdot z'}\geq c+c2^{2d(\alpha\wedge\alpha')-d([\alpha]\wedge[\alpha'])},\qquad z\in K_\alpha, z'\in K_{\alpha'}, \tag{40}$$

for all $\alpha,\alpha'\in U^{-1}\mathcal{T}_n$, $U\in\mathcal{U}_n$, together with a converse obtained by averaging over all unitary rotations $U^{-1}\mathcal{T}_n$ of the Bergman tree $\mathcal{T}_n$,

$$\operatorname{Re}\frac{1}{1-\overline{z}\cdot z'}\leq C+C\int_{\mathcal{U}_n}2^{2d(\alpha(Uz)\wedge\alpha(Uz'))-d([\alpha(Uz)]\wedge[\alpha(Uz')])}dU. \tag{41}$$

This latter inequality is analogous to similar inequalities in Euclidean space used to control an operator by translations of its dyadic version. These inequalities follow from

$$\operatorname{Re}\frac{1}{1-\overline{z}\cdot z'}=\frac{\operatorname{Re}(1-\overline{z}\cdot z')}{|1-\overline{z}\cdot z'|^2}\approx\frac{1-|\overline{z}\cdot z'|^2}{|1-\overline{z}\cdot z'|^2}+1. \tag{42}$$

With all this, we can complete the proof of the equivalence of (27) and (37) in the presence of (26) and (28).

Now (37) can be rewritten as

$$\sum_{\alpha\in\mathcal{T}_n}f(\alpha)\{T_\mu g(\alpha)\}\mu(\alpha)\leq C\|f\|_{\ell^2(\mu)}\|g\|_{\ell^2(\mu)}, \tag{43}$$

for all $f,g\geq 0$ on $\mathcal{T}_n$, and where T_μ is given in (30):

$$T_\mu g(\alpha)=\sum_{\alpha'\in\mathcal{T}_n}2^{2d(\alpha\wedge\alpha')-d([\alpha]\wedge[\alpha'])}g(\alpha')\mu(\alpha').$$

Upon using the Cauchy-Schwartz inequality and taking the supremum over all f with $\|f\|_{\ell^2(\mu)}=1$ in (43), we obtain the equivalence of (43) and the discrete inequality (29), where $\mathcal{T}_n$ ranges over all unitary rotations of a fixed Bergman tree.

1.4.1 A characterization of Carleson measures for $B_2^{\frac{1}{2}}$

The bilinear inequality associated with (29) is

$$\sum_{\alpha\in\mathcal{T}_n} f(\alpha)\, T_\mu g(\alpha)\,\mu(\alpha) = \sum_{\alpha,\beta\in\mathcal{T}_n} 2^{2d(\alpha\wedge\beta)-d([\alpha]\wedge[\beta])} f\mu(\alpha)\, g\mu(\beta) \tag{44}$$

$$\leq C\left(\sum_{\alpha\in\mathcal{T}_n} f(\alpha)^2\mu(\alpha)\right)^{\frac{1}{2}}\left(\sum_{\alpha\in\mathcal{T}_n} g(\alpha)^2\mu(\alpha)\right)^{\frac{1}{2}}.$$

We rewrite the left hand side as

$$\sum_{A,B\in\mathcal{R}_n}\sum_{\substack{\alpha\in A\\ \beta\in B}} 2^{2d(\alpha\wedge\beta)-d(A\wedge B)} f\mu(\alpha)\, g\mu(\beta) \tag{45}$$

$$= \sum_{C\in\mathcal{R}_n}\sum_{\substack{A,B\in\mathcal{R}_n\\ A\wedge B=C}}\sum_{\substack{\alpha\in A\\ \beta\in B}} \frac{2^{2d(\alpha\wedge\beta)}}{2^{d(C)}} f\mu(\alpha)\, g\mu(\beta).$$

Define the projection P_C from functions $h=\{h(\alpha)\}_{\alpha\in C}$ on the ring A to functions $P_C h$ on the ring C (provided $C\leq A$) by

$$P_C h = \left\{\sum_{\substack{\alpha\in A\\ \alpha\geq\gamma}} h(\alpha)\right\}_{\gamma\in C}.$$

We also define the "Poisson kernel" $\mathbb{P}_C$ at scale C to be the mapping taking functions $h=\{h(\gamma')\}_{\gamma'\in C}$ on C to functions $\mathbb{P}_C h=\{\mathbb{P}_C h(\gamma)\}_{\gamma\in C}$ on C given by

$$\mathbb{P}_C h = \left\{\sum_{\gamma'\in A} \frac{2^{2d(\gamma\wedge\gamma')}}{2^{d(C)}} h(\gamma')\right\}_{\gamma\in C}.$$

Now if f_A denotes the restriction $\chi_A f$ of f to the ring A, we can write (45) as aproximately

$$\sum_{C\in\mathcal{R}_n}\sum_{\substack{A,B\in\mathcal{R}_n\\ A\wedge B=C}}\sum_{\gamma\in C} \mathbb{P}_C\left(P_C(f_A\mu)\right)(\gamma)\, P_C(g_B\mu)(\gamma)$$

$$\equiv \sum_{C\in\mathcal{R}_n}\sum_{\substack{A,B\in\mathcal{R}_n\\ A\wedge B=C}} \left\langle \mathbb{P}_C\left(P_C(f_A\mu)\right), P_C(g_B\mu)\right\rangle_C,$$

where the inner product $\langle F,G\rangle_C$ is given by $\sum_{\gamma\in C} F(\gamma)\,G(\gamma)$. At this point we notice that the Poisson kernel

$$\mathbb{P}_C(\gamma,\gamma') = \frac{2^{2d(\gamma\wedge\gamma')}}{2^{d(C)}}$$

is a geometric sum of averaging operators $\mathbb{A}_C^k$ with kernel

$$\mathbb{A}_C^k(\gamma,\gamma') = 2^{d(C)-k}\chi_{\{d(\gamma\wedge\gamma')=d(C)-k\}},$$

namely

$$\mathbb{P}_C(\gamma,\gamma') = \sum_{k=0}^{d(C)} 2^{-k}\mathbb{A}_C^k(\gamma,\gamma'). \tag{46}$$

We now consider the bilinear inequality with $\mathbb{P}_C$ replaced by $\mathbb{A}_C^0$:

$$\sum_{C\in\mathcal{R}_n}\sum_{\substack{A,B\in\mathcal{R}_n\\ A\wedge B=C}}\left\langle \mathbb{A}_C^0\left(P_C(f_A\mu)\right), P_C(g_B\mu)\right\rangle_C \leq C\|f\|_{\ell^2(\mu)}\|g\|_{\ell^2(\mu)}. \tag{47}$$

The left side of (47) is

$$\begin{aligned}\sum_{C\in\mathcal{R}_n} 2^{d(C)} &\sum_{\substack{A,B\in\mathcal{R}_n\\ A\wedge B=C}}\langle P_C(f_A\mu), P_C(g_B\mu)\rangle_C \\ &= \sum_{C\in\mathcal{R}_n} 2^{d(C)}\sum_{\gamma\in C}\sum_{\substack{A,B\in\mathcal{R}_n\\ A\wedge B=C}} I^*(f_A\mu)(\gamma)\, I^*(g_B\mu)(\gamma).\end{aligned}$$

For fixed $\gamma\in C$, we dominate the sum $\sum_{\substack{A,B\in\mathcal{R}_n\\ A\wedge B=C}}$ above by

$$\begin{aligned}\sum_{\substack{A,B\in\mathcal{R}_n\\ A\wedge B=C}} I^*(f_A\mu)(\gamma)\, I^*(g_B\mu)(\gamma) \leq\; & I^*(f\mu)(\gamma)(g\mu)(\gamma) \\ &+(f\mu)(\gamma)\, I^*(g\mu)(\gamma) \\ &+\sum_{\substack{\delta,\delta'\in\mathcal{C}(\gamma)\\ [\delta]\neq[\delta']}} I^*(f\mu)(\delta)\, I^*(g\mu)(\delta').\end{aligned}$$

The first two terms easily satisfy the bilinear inequality using only the simple condition (31). Indeed,

$$\begin{aligned}\sum_{C\in\mathcal{R}_n} 2^{d(C)}\sum_{\gamma\in C} I^*(f\mu)(\gamma)(g\mu)(\gamma) &= \sum_{\gamma\in\mathcal{T}_n} 2^{d(\gamma)} I^*(f\mu)(\gamma)(g\mu)(\gamma) \\ &= \sum_{\gamma\in\mathcal{T}_n} I\left(2^d f\mu\right)(\gamma)\, g(\gamma)\,\mu(\gamma) \\ &\leq \left\|I\left(2^d f\mu\right)\right\|_{\ell^2(\mu)}\|g\|_{\ell^2(\mu)}.\end{aligned}$$

Now the inequality $\left\|I\left(2^d f\mu\right)\right\|_{\ell^2(\mu)} \leq C\|f\|_{\ell^2(\mu)}$ can be rewritten

$$\sum_{\gamma\in\mathcal{T}_n} Ih(\gamma)^2\,\mu(\gamma) \leq C\sum_{\gamma\in\mathcal{T}_n} h(\gamma)^2\, 2^{-2d(\gamma)}\mu(\gamma),$$

which is equivalent to the condition

$$\sum_{\gamma\geq\alpha} I^*\mu(\gamma)^2\, 2^{2d(\gamma)}\mu(\gamma) \leq CI^*\mu(\alpha), \qquad \alpha\in\mathcal{T}_n,$$

which in turn is trivially implied by the simple condition $I^*\mu(\gamma)^2\, 2^{2d(\gamma)} \leq C^2$.

It remains then to consider the "split" bilinear inequality

$$\sum_{\gamma\in\mathcal{T}_n} 2^{d(\gamma)} \sum_{\substack{\delta,\delta'\in\mathcal{C}(\gamma)\\ [\delta]\neq[\delta']}} I^*(f\mu)(\delta)\, I^*(g\mu)(\delta') \leq C\,\|f\|_{\ell^2(\mu)}\,\|g\|_{\ell^2(\mu)}, \tag{48}$$

or equivalently the corresponding quadratic inequality obtained by setting $f=g$:

$$\sum_{\gamma\in\mathcal{T}_n} 2^{d(\gamma)} \sum_{\substack{\delta,\delta'\in\mathcal{C}(\gamma)\\ [\delta]\neq[\delta']}} I^*(f\mu)(\delta)\, I^*(f\mu)(\delta') \leq C\sum_{\alpha\in\mathcal{T}_n} f(\alpha)^2\,\mu(\alpha). \tag{49}$$

Note that we have the following necessary condition for (49) that is obtained by taking $f=\chi_{S(\alpha)}$ in (49):

$$\sum_{\gamma\geq\alpha} 2^{d(\gamma)} \sum_{\substack{\delta,\delta'\in\mathcal{C}(\gamma)\\ [\delta]\neq[\delta']}} I^*\mu(\delta)\, I^*\mu(\delta') \leq CI^*\mu(\alpha), \qquad \alpha\in\mathcal{T}_n. \tag{50}$$

We now show that (50) and (31) together imply (49). To see this write the left side of (49) as

$$\sum_{\gamma\in\mathcal{T}_n} 2^{d(\gamma)} \sum_{\substack{\delta,\delta'\in\mathcal{C}(\gamma)\\ [\delta]\neq[\delta']}} I^*\mu(\delta)\, I^*\mu(\delta')\,\frac{I^*(f\mu)(\delta)\, I^*(f\mu)(\delta')}{I^*\mu(\delta)\, I^*\mu(\delta')},$$

and using the symmetry in δ,δ' we bound it by

$$\sum_{\gamma\in\mathcal{T}_n} 2^{d(\gamma)} \sum_{\substack{\delta,\delta'\in\mathcal{C}(\gamma)\\ [\delta]\neq[\delta']}} I^*\mu(\delta)\, I^*\mu(\delta')\left(\frac{I^*(f\mu)(\delta)}{I^*\mu(\delta)}\right)^2.$$

By Theorem 3, this last term is dominated by the right side of (49) provided

$$\begin{aligned}\sum_{\gamma\geq\alpha} 2^{d(\gamma)} \sum_{\substack{\delta,\delta'\in\mathcal{C}(\gamma)\\ [\delta]\neq[\delta']}} I^*\mu(\delta)\, I^*\mu(\delta') + 2^{d(A\alpha)} \sum_{\substack{\delta'\in\mathcal{C}(A\alpha)\\ [\alpha]\neq[\delta']}} I^*\mu(\alpha)\, I^*\mu(\delta') \\ \leq CI^*\mu(\alpha), \qquad \alpha\in\mathcal{T}_n.\end{aligned} \tag{51}$$

Now the necessary condition (50) shows that the first sum in (51) is at most $CI^*\mu(\alpha)$, while the simple condition (31) yields $2^{d(A\alpha)}I^*\mu(\delta')\leq C$, which shows that the second sum in (51) is at most $CI^*\mu(\alpha)$. This completes the proof that (47) holds when both (50) and (31) hold.

The averaging operators $\mathbb{A}_C^k$ for $k>0$ are handled similarly, and we obtain the following theorem where

$$\mathcal{G}^{(k)}(\gamma)=\left\{(\eta,\eta')\in\mathcal{C}^{(k)}(\gamma)\times\mathcal{C}^{(k)}(\gamma):\eta\wedge\eta'=\gamma,[A\eta]=[A\eta'],[\eta]\neq[\eta']\right\}.$$

Theorem 7 *A positive measure μ on the ball $\mathbb{B}_n$ is $B_2^{\frac{1}{2}}(\mathbb{B}_n)$-Carleson if and only if μ satisfies the simple condition (31) and the following split tree condition,*

$$\sum_{k\geq 0}\sum_{\gamma\geq\alpha}2^{d(\gamma)-k}\sum_{(\delta,\delta')\in\mathcal{G}^{(k)}(\gamma)}I^*\mu(\delta)\,I^*\mu(\delta')\leq CI^*\mu(\alpha),\qquad\alpha\in\mathcal{T}_n,\tag{52}$$

taken over all unitary rotations of the Bergman tree $\mathcal{T}_n$.

1.4.2 Related inequalities

Since

$$2d(\alpha\wedge\alpha')-d([\alpha]\wedge[\alpha'])=d(\alpha)-d(\alpha,E_\alpha\alpha'),$$

where $E_\alpha\alpha'$ denotes the exit point of the geodesic $[0,\alpha']$ from the slab $\mathcal{S}_{c_\alpha}$, we may rewrite the operator T_μ as the following variant of a fractional integral:

$$T_\mu g(\alpha)=\sum_{\alpha'\in\mathcal{T}_n}2^{d(\alpha)-d(\alpha,E_\alpha\alpha')}g(\alpha')\,\mu(\alpha').$$

Inequality (38),

$$d(\alpha\wedge\alpha')\leq d([\alpha]\wedge[\alpha'])\leq\min\{d(\alpha),d(\alpha')\},$$

has the following interpretation relative to the kernel $K(\alpha,\alpha')=2^{2d(\alpha\wedge\alpha')-d([\alpha]\wedge[\alpha'])}$. If we replace $d([\alpha]\wedge[\alpha'])$ by the lower bound $d(\alpha\wedge\alpha')$ in the kernel $K(\alpha,\alpha')$, then T_μ becomes

$$T_\mu g(\alpha)=\sum_{\alpha'\in\mathcal{T}_n}2^{d(\alpha\wedge\alpha')}g(\alpha')\,\mu(\alpha'),\tag{53}$$

whose boundedness on $\ell^2(\mu)$ is equivalent to μ being a Carleson measure for $B_2^{\frac{1}{2}}(\mathcal{T}_n)$, which is in turn equivalent to the tree condition (8) with $\sigma=\frac{1}{2}$ and $p=2$ (Alternatively, the above kernel is the discretization of the continuous kernel $\left|\frac{1}{1-\overline{z}\cdot z'}\right|$, whose Carleson measures are characterized by the tree condition). This observation provides another proof in the case of the Drury-Arveson space $B_2^{\frac{1}{2}}(\mathbb{B}_n)$, of a more general result that shows the tree condition characterizes Carleson measures supported on a tangential manifold (when $d([\alpha]\wedge[\alpha'])=d(\alpha\wedge\alpha')$ for α,α' in the support of the measure). In addition, we can see from this observation that the simple condition (31) is not sufficient for μ to be a

$B_2^{\frac{1}{2}}(\mathcal{T}_n)$-Carleson measure. Indeed, let $\mathcal{Y}$ be any dyadic subtree of $\mathcal{T}_n$ with the properties that the two children α_+ and α_- of each $\alpha \in \mathcal{Y}$ are also children of α in $\mathcal{T}_n$, and such that no two tree elements in $\mathcal{Y}$ are equivalent. Now let μ be any measure supported on $\mathcal{Y}$ that satisfies the simple condition

$$2^{d(\alpha)} I^* \mu(\alpha) \leq C, \qquad \alpha \in \mathcal{Y},$$

but not the tree condition

$$\sum_{\beta \in \mathcal{Y}: \beta \geq \alpha} \left[2^{\sigma d(\beta)} I^* \mu(\beta)\right]^{p'} \leq C I^* \mu(\alpha) < \infty, \qquad \alpha \in \mathcal{Y}.$$

For $\alpha, \alpha' \in \mathcal{Y}$, we have $d([\alpha] \wedge [\alpha']) = d(\alpha \wedge \alpha')$, and so μ is a $B_2^{\frac{1}{2}}(\mathcal{T}_n)$-Carleson measure if and only if the operator T in (53) is bounded on $\ell^2(\mu)$, which is equivalent to the above tree condition, which we have chosen to fail. Finally, to transplant this example to the ball $\mathbb{B}_n$, we take $d\mu(z) = \sum_{\alpha \in \mathcal{Y}} \mu(\alpha) \delta_{c_\alpha}(z)$ and show that the above tree condition fails on a positive proportion of the rotated trees $U^{-1}\mathcal{T}_n$, $U \in \mathcal{U}_n$.

If on the other hand, we replace $d([\alpha] \wedge [\alpha'])$ by the upper bound $\min\{d(\alpha), d(\alpha')\}$ in the kernel $K(\alpha, \alpha')$, then T_μ becomes

$$T_\mu g(\alpha) = \sum_{\alpha' \in \mathcal{T}_n} 2^{2d(\alpha \wedge \alpha') - \min\{d(\alpha), d(\alpha')\}} g(\alpha') \mu(\alpha'), \tag{54}$$

whose boundedness on $\ell^2(\mu)$ is shown in Theorem 8 below to be implied by the simple condition (31). Thus we see that the simple condition (31) characterizes Carleson measures supported on a slice (when $d([\alpha] \wedge [\alpha']) = \min\{d(\alpha), d(\alpha')\}$ for α, α' in the support of the measure). In particular, this provides a new proof that the simple condition (31) characterizes Carleson measures for the Hardy space $H^2(\mathbb{D}) = B_2^{\frac{1}{2}}(\mathbb{D})$ in the unit disc.

Finally, when μ is invariant, the operator T_μ in (30) is bounded on $\ell^2(\mu)$ if and only if μ is finite. To see this we need the "Poisson kernel" estimate

$$\sum_{\beta \in B} 2^{2d(\alpha \wedge \beta)} \approx 2^{d(B) + d([\alpha] \wedge B)}, \qquad \alpha \in \mathcal{T}_n, B \in \mathcal{R}_n. \tag{55}$$

Using (55) we compute that $T_\mu 1$ is bounded (and hence a Schur function). Thus T_μ is bounded on $\ell^\infty(\mu)$ with norm at most $\|\mu\|$, and by duality also on $\ell^1(\mu)$. Interpolation now yields that T_μ is bounded on $\ell^2(\mu)$ with norm at most $\|\mu\|$.

Theorem 8 *A positive measure μ satisfies the bilinear inequality*

$$\sum_{\alpha, \alpha' \in \mathcal{T}_n} 2^{2d(\alpha \wedge \alpha') - \min\{d(\alpha), d(\alpha')\}} f(\alpha) \mu(\alpha) g(\alpha') \mu(\alpha') \leq C \|f\|_{\ell^2(\mathcal{T}_n; \mu)} \|g\|_{\ell^2(\mathcal{T}_n; \mu)}, \tag{56}$$

if μ satisfies the simple condition (31).

Remark 2 *Using the argument on pages 538-542 of [31], it can be shown that the bilinear inequality (56) holds if and only if the following pair of dual conditions hold:*

$$\sum_{\beta\geq\alpha}\left|\mathcal{I}2^{d}\left(\chi_{S(\alpha)}\mu\right)(\beta)\right|^{2}\mu(\beta)\leq C\sum_{\beta\geq\alpha}\mu(\beta)<\infty,\qquad\alpha\in\mathcal{T}_{n},\tag{57}$$
$$\sum_{\beta\geq\alpha}\left|2^{d}\mathcal{I}\left(\chi_{S(\alpha)}2^{-d}\mu\right)(\beta)\right|^{2}\mu(\beta)\leq C\sum_{\beta\geq\alpha}2^{-2d(\beta)}\mu(\beta),\qquad\alpha\in\mathcal{T}_{n},$$

where $\mathcal{I}$ is the fractional integral of order one on the Bergman tree given by,

$$\mathcal{I}\nu(\alpha)=\sum_{\beta\in\mathcal{T}_{n}}2^{-d(\alpha,\beta)}\nu(\beta),\qquad\alpha\in\mathcal{T}_{n}.\tag{58}$$

The following simple sufficient condition of Schur type is used for the proof. Recall that a measure space (Z,μ) is σ-finite if $Z=\cup_{N=1}^{\infty}Z_{N}$ where $\mu(Z_{N})<\infty$, and that a function k on $Z\times Z$ is σ-bounded if $Z=\cup_{N=1}^{\infty}Z_{N}$ where k is bounded on $Z_{n}\times Z_{n}$.

Lemma 9 *(Vinogradov-Seničkin Test, pg 151of [27]) Let (Z,μ) be a σ-finite measure space and k a nonnegative σ-bounded function on $Z\times Z$ satisfying*

$$\int\int_{Z\times Z}k(s,t)\,k(s,x)\,d\mu(s)\leq M\left(\frac{k(t,x)+k(x,t)}{2}\right)\qquad\text{for }\mu\text{-a.e. }(t,x)\in Z\times Z.\tag{59}$$

Then the linear map T defined by

$$Tg(s)=\int_{Z}k(s,t)\,g(t)\,d\mu(t)$$

is bounded on $L^{2}(\mu)$ with norm at most M.

Proof. Let $Z=\cup_{N=1}^{\infty}Z_{N}$ where $\mu(Z_{N})<\infty$ and k is bounded on $Z_{N}\times Z_{N}$. The kernels

$$k_{N}(s,t)=k(s,t)\,\chi_{Z_{N}\times Z_{N}}(s,t)$$

satisfy (59) uniformly in N, and the corresponding operators $T_{N}g(s)=\int_{Z}k_{N}(s,t)\,g(t)\,d\mu(t)$ are bounded on $L^{2}(\mu)$ (with norms depending on $\mu(Z_{N})$ and the bound for k on $Z_{N}\times Z_{N}$). However, (59) for k_{N} implies that the integral kernel of the operator $T_{N}^{*}T_{N}$ is dominated pointwise by $\frac{M}{2}$ times that of $T_{N}^{*}+T_{N}$, and this gives $\|T_{N}\|^{2}=\|T_{N}^{*}T_{N}\|\leq\frac{M}{2}\|T_{N}^{*}+T_{N}\|\leq M\|T_{N}\|$, and hence $\|T_{N}\|\leq M$. Now let $N\to\infty$ and use the monotone convergence theorem to obtain $\|T\|\leq M$.

Remark 3 *If $k(x,y)=k(y,x)$ is symmetric, then (59) ensures that for any choice of a, $k(a,\cdot)$ can be used as a test function for Schur's Lemma.*

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Taiwan lecture 4

Friday July 8 2005

1 Introduction

We recall our interpolation theorems for Besov spaces.

Theorem 1 *Let* $1 < p < \infty$, $\alpha > -1$ *and* $k_w^{\alpha,p}(z)$ *be the reproducing kernel for* B_p *relative to the pairing* $\langle \cdot, \cdot \rangle_{\alpha,p}$. *Let* $\{z_j\}_{j=1}^{\infty}$ *be a sequence in the unit ball* $\mathbb{B}_n$. *Then the following conditions are equivalent.*

1. $\{z_j\}_{j=1}^{\infty}$ *interpolates* B_p:
$$\text{The map } f \to \left\{ \frac{f(z_j)}{\|k_{z_j}^{\alpha,p}\|_{B_{p'}}} \right\}_{j=1}^{\infty} \text{ takes } B_p \text{ boundedly into and onto } \ell^p. \tag{1}$$

2. *The following norm equivalence holds:*
$$\left\| \sum_{j=1}^{\infty} a_j \frac{k_{z_j}^{\alpha,p}}{\|k_{z_j}^{\alpha,p}\|_{B_{p'}}} \right\|_{B_{p'}} \approx \left(\sum_{j=1}^{\infty} |a_j|^{p'} \right)^{\frac{1}{p'}}. \tag{2}$$

3. *The following separation condition and Carleson embedding hold:*
$$\beta(z_i, 0) \leq C\beta(z_i, z_j), i \neq j \text{ and} \tag{3}$$
$$\sum_{j=1}^{\infty} \left\| k_{z_j}^{\alpha,p} \right\|_{B_{p'}}^{-p} \delta_{z_j} \text{ is a } B_p\text{-Carleson measure.}$$

Theorem 2 *Let* $1 < p < \infty$, $\alpha > -1$ *and* $k_w^{\alpha,p}(z)$ *be the reproducing kernel for* B_p *relative to the pairing* $\langle \cdot, \cdot \rangle_{\alpha,p}$. *Let* $\{z_j\}_{j=1}^{\infty}$ *be a sequence in the unit ball* $\mathbb{B}_n$. *If* $p \in \left(1, 2 + \frac{1}{n-1}\right)$, *then each of conditions (4) and (6) below is equivalent to the three conditions in Theorem 1. In general, for* $1 < p < \infty$, *(6) implies (4) implies (5). For* $p > 2n$ *(3) implies (4). If* $p \in \left(1, 1 + \frac{1}{n-1}\right) \cup [2, \infty)$, *we also have that (5) implies (3):*

1. $\{z_j\}_{j=1}^{\infty}$ *interpolates* M_{B_p}:

The map $f \to \{f(z_j)\}_{j=1}^{\infty}$ *takes* M_{B_p} *boundedly into and onto* ℓ^{∞}. (4)

2. $\left\{k_{z_j}^{\alpha,p}\right\}_{j=1}^{n}$ *is an unconditional basic sequence in* $B_{p'}$:

$$\left\|\sum_{j=1}^{\infty} b_j k_{z_j}^{\alpha,p}\right\|_{B_{p'}} \leq C \left\|\sum_{j=1}^{\infty} a_j k_{z_j}^{\alpha,p}\right\|_{B_{p'}}, \quad \text{whenever } |b_j| \leq |a_j|. \tag{5}$$

3. $\{z_j\}_{j=1}^{\infty} = \left\{c_{\alpha_j}\right\}_{j=1}^{\infty}$ *where* $\{\alpha_j\}_{j=1}^{\infty}$ *is a sequence in a Bergman tree* $\mathcal{T}_n$ *satisfying*

$$\begin{aligned} &\beta(z_i, 0) \leq C\beta(z_i, z_j), i \neq j \text{ and} \\ &\sum_{j=1}^{\infty} (1 + d(\alpha_j, o))^{1-p} \delta_{\alpha_j} \text{ satisfies the tree condition (??).} \end{aligned} \tag{6}$$

1.1 Multiplier space sufficiency

Here we prove that (6) implies (4) for $1 < p < \infty$, and also that (3) implies (4) for $p > 2n$, beginning with the proof that the multiplier interpolation property (4) follows from (6). We generalize the main ideas in Böe's one-dimensional proof to the unit ball $\mathbb{B}_n$. First we give the following characterization of multipliers in terms of Carleson measures.

Theorem 3 *Let* $\varphi \in H^{\infty}(\mathbb{B}_n) \cap B_p^{\sigma}(\mathbb{B}_n)$ *and* $m + \sigma > \frac{n}{p}$. *Then* φ *is a pointwise multiplier on* $B_p^{\sigma}(\mathbb{B}_n)$, *i.e.* $\|\varphi f\|_{B_p^{\sigma}} \leq C\|f\|_{B_p^{\sigma}}$ *for all* $f \in B_p^{\sigma}(\mathbb{B}_n)$, *if and only if*

$$\left|\left(1 - |z|^2\right)^{m+\sigma} \varphi^{(m)}(z)\right|^p d\lambda_n(z)$$

is a $B_p^{\sigma}(\mathbb{B}_n)$*-Carleson measure on* $\mathbb{B}_n$.

Proof. Using

$$\|\varphi f\|_{B_p^{\sigma}}^p = \int_{\mathbb{B}_n} \left|\left(1 - |z|^2\right)^{m+\sigma} (\varphi f)^{(m)}(z)\right|^p d\lambda_n(z),$$

together with

$$(\varphi f)^{(m)}(z) = \varphi^{(m)}(z) f(z) + \ldots + \varphi(z) f^{(m)}(z),$$

we see that the last term $\varphi(z) f^{(m)}(z)$ can be handled using the boundedness of φ, while the first term is handled by the Carleson embedding. The intermediate terms are handled by an interpolation argument - see [8].

For $z \in \mathbb{B}_n$ and $\beta < 1$, define the region V_z^β by

$$V_z^\beta = \left\{ w \in \mathbb{B}_n : |1 - \overline{w} \cdot Pz| \leq (1 - |z|)^\beta \right\},$$

where Pz denotes the radial projection of z onto the sphere $\partial\mathbb{B}_n$. The intersection of V_z^β with the complex line $\mathbb{C}z$ through z and the origin is

$$\left\{ w \in \mathbb{B}_n \cap \mathbb{C}z : |w - Pz| \leq (1 - |z|)^\beta \right\},$$

and the intersection of V_z^β with the sphere $\partial\mathbb{B}_n$ is an "ellipse" with radius $(1 - |z|)^\beta$ in the radial tangential direction, and radius $(1 - |z|)^{\frac{\beta}{2}}$ in the complex tangential directions. Using arguments in Marshall and Sundberg [24], the separation condition in (3) implies the following geometric separation conditions.

Lemma 4 *Suppose the separation condition in (3) holds. Then there are constants* $0 < \beta < 1 < \beta\eta < \eta$ *such that if* $V_{z_i}^\beta \cap V_{z_j}^\beta \neq \phi$ *and* $|z_j| \geq |z_i|$, *then* $z_i \notin V_{z_j}^\beta$ *and*

$$(1 - |z_j|) \leq (1 - |z_i|)^\eta . \tag{7}$$

We now fix constants β and η as in Lemma 4, and write $V_z = V_z^\beta$.

Lemma 7 below is the key construction in the sufficiency proof and is motivated by the formula ((1.35) in [39])

$$R^{s-n,n}\left(\frac{1}{(1 - \overline{w} \cdot z)^{1+s}}\right) = \frac{1}{(1 - \overline{w} \cdot z)^{n+1+s}},$$

valid for s not a negative integer. The point is that if we define

$$\Gamma_s g(z) \equiv \int_{\mathbb{B}_n} \frac{g(w)\left(1 - |w|^2\right)^s}{(1 - \overline{w} \cdot z)^{1+s}} dw \tag{8}$$

for a given (not necessarily holomorphic) function g, then with $\varphi(z) = \Gamma_s g(z)$, so that φ is essentially an n-fold antiderivative of g, we have

$$R^{s-n,n}\varphi(z) = R^{s-n,n}\Gamma_s g(z) = \int_{\mathbb{B}_n} \frac{g(w)\left(1 - |w|^2\right)^s}{(1 - \overline{w} \cdot z)^{n+1+s}} dw, \tag{9}$$

and by the reproducing formula valid for $\operatorname{Re} s > -1$, we also have that

$$R^{s-n,n}\varphi(z) = c_{n,s} \int_{\mathbb{B}_n} \frac{R^{s-n,n}\varphi(w)\left(1 - |w|^2\right)^s}{(1 - \overline{w} \cdot z)^{n+1+s}} dw.$$

Thus $R^{s-n,n}\varphi(w)$ behaves morally like $g(w)$, and this provides flexibility in choosing g so that φ has desirable algebraic multiplier properties on the one hand, while controlling the multiplier norm of φ on the other hand. Indeed, by Theorem 3, the multiplier norm is equivalent to the Carleson norm of $\left|\left(1-|z|^2\right)^n \nabla^n \varphi(z)\right|^p d\lambda_n(z)$, which can in turn be dominated by the "tree condition" norm of $\left|\left(1-|z|^2\right)^n g(z)\right|^p d\lambda_n(z)$ by the lemma in the next subsubsection.

Finally, we note that the construction of an approximate zero-one Dirichlet multiplier in Lemma 7 below (which results in a holomorphic function that is close to one on a Carleson tent and close to zero away from it) is a substitute for Jones' clever construction of an exact zero-one Hardy multiplier in the disk using Blaschke products. In the analogous result on the Bergman tree, the Dirichlet multiplier construction is given simply by defining a sequence $h^j = \{h^j(\alpha)\}_{\alpha \in \mathcal{T}_n}$ to satisfy $h^j(\alpha_j) = 1$ and to decrease linearly to 0 on a sufficiently small stretch of the geodesic preceding α, and then "linearly" to 0 off the geodesic as well. See [7] for such constructions.

Terminology We say that a measure μ on $\mathbb{B}_n$ satisfies the *tree condition* if its discretization $\widetilde{\mu}$ does.

It is here that we first use the tree condition in a significant way.

1.1.1 Transformation of Carleson measures

Lemma 5 *(analogue of Lemma 2.4 in [12]) Suppose that g satisfies the following reverse Hölder condition on Bergman kubes,*

$$\left(\int_{K_\alpha} |g(z)|^p d\lambda_n(z)\right)^{\frac{1}{p}} \leq C_0 \int_{K_\alpha} |g(z)| d\lambda_n(z), \qquad \alpha \in \mathcal{T}_n, \tag{10}$$

and that the measure

$$d\mu(z) = \left|\left(1-|z|^2\right)^n g(z)\right|^p d\lambda_n(z)$$

satisfies the tree condition with norm C_1. Then for s sufficiently large, both

$$\left|\left(1-|z|^2\right)^n R^{s-n,n}\Gamma_s g(z)\right|^p d\lambda_n(z)$$

and

$$\left|\left(1-|z|^2\right)^n \nabla^n \Gamma_s g(z)\right|^p d\lambda_n(z)$$

satisfy the tree condition with norms at most $C(C_0+C_1)$.

Note that it then follows that both measures in the conclusion of the lemma are B_p-Carleson measures. The proof of the lemma uses only Schur's test and lengthy calculations involving discretization of functions and measures, and unitary rotations of the Bergman tree. We emphasize again that this is where the tree condition is used in a significant way. The multiplier norm of $\varphi(z) = \Gamma_s g(z)$ is the same as the Carleson measure norm of

$$\left|\left(1-|z|^2\right)^{m+\sigma} \nabla^m \Gamma_s g(z)\right|^p d\lambda_n(z),$$

and the tree condition norm of this latter expression is estimated by the tree condition norm of

$$\left|\left(1-|z|^2\right)^n g(z)\right|^p d\lambda_n(z).$$

Following Böe [12] one can prove the following alternate version of Lemma 5, where the tree condition is replaced by the B_p-Carleson measure condition, for the range $p > 2n$ with $s > n - \frac{1}{p'}$. This version will be instrumental in proving the implication (3) implies (4) for $p > 2n$ below.

Lemma 6 *(another analogue of Lemma 2.4 in [B]) Suppose that*

$$\sup_{\zeta \in \mathbb{B}_n} \left|\left(1-|\zeta|^2\right)^n g(\zeta)\right| \leq C_0, \tag{11}$$

and that the measure

$$d\mu(z) = \left|\left(1-|z|^2\right)^n g(z)\right|^p d\lambda_n(z)$$

is a Carleson measure for B_p with norm C_1. Then for $p > 2n$ and $s > n - \frac{1}{p'}$, both

$$\left|\left(1-|z|^2\right)^n R^{s-n,n} \Gamma_s g(z)\right|^p d\lambda_n(z)$$

and

$$\left|\left(1-|z|^2\right)^n \nabla^n \Gamma_s g(z)\right|^p d\lambda_n(z)$$

are also Carleson measures for B_p, and with norms at most $C(C_0 + C_1)$.

The simpler Lemma 6 uses the characterization of B_p given by Theorem 6.28 of [39], which states that

$$\|f\|_{B_p}^p \approx \int_{\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|f(z) - f(w)|^p}{|1 - \overline{w} \cdot z|^{2(n+1+t)}} d\nu_t(z)\, d\nu_t(w), \tag{12}$$

provided $t > -1$ and $p > \begin{cases} 1, & n = 1 \\ 2n & n > 1 \end{cases}$. In order to obtain the full range $1 < p < \infty$ when $n > 1$, we instead need to use the the tree condition to obtain Lemma 5.

1.1.2 Multiplier approximations

The next lemma constructs a holomorphic function that is close to 1 on the Carleson region associated to a point $w \in \mathbb{B}_n$, and decays appropriately away from the Carleson region. We follow Böe's proof in [12], which adapts a real-variable argument of Marshall and Sundberg in [24] to produce a holomorphic multiplier approximation. Given $\beta < \rho < \alpha < 1$, we will use the cutoff function $c_{\rho,\alpha}$ defined by

$$c_{\rho,\alpha}(\gamma) = \begin{cases} 0 & \text{for } \gamma < \rho \\ \frac{\gamma-\rho}{\alpha-\rho} & \text{for } \rho \leq \gamma \leq \alpha \\ 1 & \text{for } \alpha < \gamma \end{cases} . \tag{13}$$

Lemma 7 *(analogue of Lemma 4.1 in [12]) Suppose $s > -1$. There are ρ and α satisfying $\beta < \rho < \alpha < 1$ such that for every $w \in \mathbb{B}_n$, we can find a function g_w so that*

$$\varphi_w(z) = \Gamma_s g_w(z) = \int_{\mathbb{B}_n} \frac{g_w(\zeta)\left(1-|\zeta|^2\right)^s}{\left(1-\overline{\zeta}\cdot z\right)^{1+s}} d\zeta$$

satisfies

$$\begin{cases} \varphi_w(w) &= 1 \\ \varphi_w(z) &= c_{\rho,\alpha}(\gamma_w(z)) + O\left(\left(\log \frac{1}{1-|w|^2}\right)^{-1}\right), z \in V_w \\ |\varphi_w(z)| &\leq C\left(\log \frac{1}{1-|w|^2}\right)^{1-p}, \qquad\qquad z \notin V_w \end{cases} , \tag{14}$$

where $\gamma_w(z)$ is defined by

$$|1 - \overline{z}\cdot Pw| = \left(1-|w|^2\right)^{\gamma_w(z)}$$

and $c_{\rho,\alpha}$ is as in (13). Furthermore we have the estimate

$$\int_{\mathbb{B}_n} \left|\left(1-|\zeta|^2\right)^n g_w(\zeta)\right|^p d\lambda_n(\zeta)\, d\zeta \leq C\left(\log \frac{1}{1-|w|^2}\right)^{1-p}. \tag{15}$$

The final estimate (15) will lead to the Carleson measure estimate for

$$\left|\left(1-|z|^2\right)^n g(z)\right|^p d\lambda_n(z).$$

Remark 1 *The proof of Lemma 7 shows that the third estimate in (14) can be vastly improved, and also holds for a larger range of z; namely there is $\beta < \beta_1 < \rho$ such that*

$$|\varphi_w(z)| \leq C\left(\log \frac{1}{1-|w|^2}\right)^{-1} \left(1-|w|^2\right)^{(\rho-\beta_1)(1+s)}, \qquad z \notin V_w^{\beta_1}.$$

This fact will be used in the proof of Lemma 8 below.

Proof. Define $g_w(\zeta)$ by

$$\frac{g_w(\zeta)\left(1-|\zeta|^2\right)^s}{\left(1-\overline{\zeta}\cdot w\right)^{1+s}} = K\left(\log\frac{1}{1-|w|^2}\right)^{-1}\left|1-\overline{\zeta}\cdot Pw\right|^{-n-1}, \tag{16}$$

when ζ lives in the annular sector $\mathcal{S}$ centred at Pw given as the intersection of the annulus

$$\mathcal{A} = \mathcal{A}_w = \left\{\zeta\in\mathbb{B}_n : \left(1-|w|^2\right)^\alpha \le \left|1-\overline{\zeta}\cdot Pw\right| \le \left(1-|w|^2\right)^\rho\right\} \tag{17}$$

and the cone

$$\mathcal{C} = \mathcal{C}_w = \left\{\zeta\in\mathbb{B}_n : \left|\operatorname{Im}\left(\zeta\cdot\overline{Pw}\right)\right| + \left|\zeta-\left(\zeta\cdot\overline{Pw}\right)Pw\right|^2 \le c\left(1-\left|\overline{\zeta}\cdot Pw\right|\right)\right\},$$

where c is a suitably small constant. Define $g_w(\zeta)=0$ otherwise. The following observation will be used repeatedly.

Remark 2 *The cone $\mathcal{C}_w$ corresponds to the geodesic in the Bergman tree $\mathcal{T}_n$ joining the root to the "boundary point" Pw. To see this, consider the case $w=(t,0,...,0)$ and $\zeta=\left(re^{i\theta},\zeta'\right)$ with $re^{i\theta}=x+iy$, so that* $\operatorname{Im}\left(\zeta\cdot\overline{Pw}\right)=y$, $\zeta-\left(\zeta\cdot\overline{Pw}\right)Pw=(0,\zeta')$ *and* $1-\left|\overline{\zeta}\cdot Pw\right|=1-r$.

Now choose K so that $\varphi_w(w)=1$, i.e.

$$K = \left(\log\frac{1}{1-|w|^2}\right)\left(\int_{\mathcal{S}}\left|1-\overline{\zeta}\cdot Pw\right|^{-n-1}d\zeta\right)^{-1},$$

which satisfies

$$K \approx K_{\alpha,\rho,n} = \frac{c_n}{\alpha-\rho} \tag{18}$$

since the annular sector

$$\mathcal{E}_a = \left\{\zeta\in\mathbb{B}_n : a\le\left|1-\overline{\zeta}\cdot Pw\right|\le 2a\right\}\cap\mathcal{C}$$

is comparable to a Bergman ball of radius one, $\left|1-\overline{\zeta}\cdot Pw\right|^{-n-1}d\zeta$ is comparable to invariant measure $d\lambda_n(\zeta)$ on $\mathcal{E}_a$, and $\mathcal{S}\approx\cup_{j=0}^J\mathcal{E}_{2^j\left(1-|w|^2\right)^\alpha}$ where

$$J = \log\frac{\left(1-|w|^2\right)^\rho}{\left(1-|w|^2\right)^\alpha} = (\rho-\alpha)\log\left(1-|w|^2\right).$$

Note also that

$$\left|1-\overline{\zeta}\cdot Pw\right| \approx \left|1-\overline{\zeta}\cdot w\right| \approx 1-|\zeta|^2, \qquad \zeta\in\mathcal{S}, \tag{19}$$

and so g_w satisfies the estimate

$$\left|g_w(\zeta)\right| \le C\left(\log\frac{1}{1-|w|^2}\right)^{-1}\left|1-\overline{\zeta}\cdot Pw\right|^{-n}, \qquad \zeta\in\mathbb{B}_n. \tag{20}$$

Now fix $z \in V_w$ and set

$$\begin{aligned} E_1 &= \left\{\zeta\in\mathbb{B}_n : \left|1-\overline{\zeta}\cdot Pw\right| \le \left(1-|w|^2\right)^{\gamma_w(z)}\right\}, \\ E_2 &= \mathbb{B}_n\setminus E_1 = \left\{\zeta\in\mathbb{B}_n : \left|1-\overline{\zeta}\cdot Pw\right| > \left(1-|w|^2\right)^{\gamma_w(z)}\right\}. \end{aligned}$$

Thus the common boundary of E_1 and E_2 passes through z. The main contribution to $\varphi_w(z)$ will come from integration over E_2. Thus we write

$$\varphi_w(z) = \int_{E_1}\frac{g_w(\zeta)\left(1-|\zeta|^2\right)^s}{\left(1-\overline{\zeta}\cdot z\right)^{1+s}}d\zeta + \int_{E_2}\frac{g_w(\zeta)\left(1-|\zeta|^2\right)^s}{\left(1-\overline{\zeta}\cdot z\right)^{1+s}}d\zeta = I + II.$$

By (19), (20) and the definition of $\gamma_w(z)$, term I is dominated by $C\left(\log\frac{1}{1-|w|^2}\right)^{-1}$ times

$$\int_{\left\{\left(1-|w|^2\right)^\alpha \le \left|1-\overline{\zeta}\cdot Pw\right| \le |1-\overline{z}\cdot Pw|\right\}\cap\mathcal{C}}\left(\frac{1-|\zeta|^2}{\left|1-\overline{\zeta}\cdot z\right|}\right)^{1+s} d\lambda_n(\zeta),$$

which is at most a constant C since

$$\left|1-\overline{\zeta}\cdot z\right| \approx |1-\overline{z}\cdot Pw|, \qquad \zeta\in\mathcal{C}\cap E_1.$$

Thus we have

$$|I| \le C\left(\log\frac{1}{1-|w|^2}\right)^{-1}.$$

We now write

$$\begin{aligned} II &= \int_{E_2\cap\mathcal{S}}\frac{g_w(\zeta)\left(1-|\zeta|^2\right)^s}{\left(1-\overline{\zeta}\cdot z\right)^{1+s}}d\zeta \\ &= \int_{E_2\cap\mathcal{S}}\left\{\frac{g_w(\zeta)\left(1-|\zeta|^2\right)^s}{\left(1-\overline{\zeta}\cdot z\right)^{1+s}} - \frac{g_w(\zeta)\left(1-|\zeta|^2\right)^s}{\left(1-\overline{\zeta}\cdot w\right)^{1+s}}\right\}d\zeta \\ &\quad + \int_{E_2\cap\mathcal{S}}\frac{g_w(\zeta)\left(1-|\zeta|^2\right)^s}{\left(1-\overline{\zeta}\cdot w\right)^{1+s}}d\zeta \\ &= III + IV. \end{aligned}$$

The point of isolating term IV is that the variable z occurs there only in the exponent $\gamma_w(z)$, and this leads to the following exact calculation. Using (18), and that g_w is supported in $\mathcal{S}$, we calculate that term IV is $\left(\log \frac{1}{1-|w|^2}\right)^{-1}$ times

$$K\int_{\left\{\left(1-|w|^2\right)^{\gamma_w(z)}\leq\left|1-\overline{\zeta}\cdot Pw\right|\leq\left(1-|w|^2\right)^{\rho}\right\}\cap\mathcal{C}}\left|1-\overline{\zeta}\cdot Pw\right|^{-n-1}d\zeta=\frac{\gamma_w(z)-\rho}{\alpha-\rho}\log\frac{1}{1-|w|^2}$$

in the case $\rho<\gamma_w(z)<\alpha$. We also have $IV=0$ in the case $\gamma_w(z)<\rho$, and $IV=\log\frac{1}{1-|w|^2}$ in the case $\alpha<\gamma_w(z)$. This gives the estimate

$$IV=c_{\rho,\alpha}\left(\gamma_w(z)\right)+O\left(\left(\log\frac{1}{1-|w|^2}\right)^{-1}\right),\qquad z\in V_w.$$

Using

$$\left|\frac{1}{\left(1-\overline{\zeta}\cdot z\right)^{1+s}}-\frac{1}{\left(1-\overline{\zeta}\cdot w\right)^{1+s}}\right|\leq C\frac{|z-w|}{\left(1-|\zeta|^2\right)^{2+s}}$$

together with (19) and (20), we obtain that

$$\begin{aligned}|III|&\leq C\int_{E_2\cap\mathcal{S}}\left|g_w(\zeta)\right|\left(1-|\zeta|^2\right)^{-2}|z-w|\,d\zeta\\&\leq C\,|z-w|\left(\log\frac{1}{1-|w|^2}\right)^{-1}\int_{E_2\cap\mathcal{S}}\left(1-|\zeta|^2\right)^{-1}d\lambda_n(\zeta)\\&\leq C\left(\log\frac{1}{1-|w|^2}\right)^{-1},\end{aligned}$$

as required. This completes the proof of the second estimate in (14).

We now turn to the third estimate in (14). For $z\notin V_w$, we have $\left|1-\overline{\zeta}\cdot z\right|\geq c\left(1-|w|^2\right)^{\beta}$ for $\zeta\in\mathcal{S}$, and thus

$$\begin{aligned}\left|\varphi_w(z)\right|&\leq\left(\log\frac{1}{1-|w|^2}\right)^{-1}\int_{\mathcal{S}}\left(\frac{1-|\zeta|^2}{\left|1-\overline{\zeta}\cdot z\right|}\right)^{1+s}d\lambda_n(\zeta)\\&\leq C\left(\log\frac{1}{1-|w|^2}\right)^{-1}\left(1-|w|^2\right)^{(\rho-\beta)(1+s)}\\&\leq C_p\left(\log\frac{1}{1-|w|^2}\right)^{1-p}.\end{aligned}$$

Finally, the estimate (15) is a calculation using (19), (20) the definition of the support of g_w. Indeed, the left side of (15) is at most

$$C\int_{\mathcal{S}}\left(\log\frac{1}{1-|w|^2}\right)^{-p}d\lambda_n(\zeta)\leq C\left(\log\frac{1}{1-|w|^2}\right)^{1-p}.$$

The next lemma uses Lemma 5 to construct inductively a holomorphic function whose restriction to the sequence $\{z_j\}_{j=1}^{\infty}$ approximates an arbitrarily prescribed bounded sequence $\{\xi_j\}_{j=1}^{\infty}$.

Lemma 8 *(analogue of Lemma 4.2 in [12]) Suppose $s > -1$, that $\{\xi_j\}_{j=1}^{\infty} \in \ell^{\infty}$ and let $0 < \delta < 1$. Let φ_j, g_j and γ_j correspond to z_j as in Lemma 7 and with the same s. Then there is $\{a_i\}_{i=1}^{\infty} \in \ell^{\infty}$ such that $\varphi = \sum_{i=1}^{\infty} a_i \varphi_i$ satisfies*

$$\left\| \{\xi_j - \varphi(z_j)\}_{j=1}^{\infty} \right\|_{\ell^{\infty}} < \delta \left\| \{\xi_j\}_{j=1}^{\infty} \right\|_{\ell^{\infty}} \tag{21}$$

and

$$\|\{a_i\}_{i=1}^{\infty}\|_{\ell^{\infty}}, \|\varphi\|_{H^{\infty}(\mathbb{B}_n)} \le C \left\| \{\xi_j\}_{j=1}^{\infty} \right\|_{\ell^{\infty}}. \tag{22}$$

Remark 3 *The series $\sum_{i=1}^{\infty} a_i \varphi_i$ in Lemma 8 converges absolutely for each $z \in \mathbb{B}_n$. In fact, the proof below will show that (using $\#\mathcal{G}_\ell \le C\beta(0, z_\ell)$)*

$$\sum_{i=1}^{\infty} |\varphi_i(z)| \le C \left(1 + \log \frac{1}{1 - |z|^2}\right), \qquad z \in \mathbb{B}_n.$$

Remark 4 *The construction in the proof below shows that both the sequence $\{a_i\}_{i=1}^{\infty}$ and the function φ depend linearly on the data $\{\xi_j\}_{j=1}^{\infty}$.*

Proof. We follow the proof of Lemma 4.2 in [12]. Let $\left\| \{\xi_j\}_{j=1}^{\infty} \right\|_{\ell^{\infty}} = 1$. We first choose J so large that

$$\sup_{j \ge J} \left(\log \frac{1}{1 - |z_j|}\right)^{-1} + \sum_{j=J}^{\infty} \left(\log \frac{1}{1 - |z_j|}\right)^{1-p} < \varepsilon, \tag{23}$$

where $\varepsilon > 0$ will be determined later. Note that the series above converges by the Carleson embedding. By standard arguments in [24], we may discard the finitely many points $\{z_j\}_{j=1}^{J-1}$. Thus we may assume that $J = 1$ in (23).

Order the points $\{z_j\}_{j=1}^{\infty}$ so that $1 - |z_{j+1}| \le 1 - |z_j|$ for $j \ge 1$. We now define a "forest structure" on the index set $\mathbb{N}$ by declaring that j is a child of i (or that i is a parent of j) provided that

$$\begin{aligned} & i < j, \\ & V_{z_j} \subset V_{z_i}, \\ & V_{z_j} \subsetneq V_{z_k} \text{ for } i < k < j. \end{aligned} \tag{24}$$

Note that a child j chooses the "nearest" parent i if we have competing indices i and i' with $V_{z_j} \subset V_{z_i} \cap V_{z_{i'}}$. We define a partial order associated with this parent-child relationship by declaring that j is a successor of i (or that i is a predecessor of j) if there is a "chain" of indices $\{i = k_1, k_2, ..., k_m = j\} \subset \mathbb{N}$ such that $k_{\ell+1}$ is a child of k_ℓ for $1 \leq \ell < m$. Under this partial ordering, $\mathbb{N}$ decomposes into a disjoint union of trees. Thus associated to each index $\ell \in \mathbb{N}$, there is a unique tree containing ℓ and, unless ℓ is the root of the tree, a unique parent $P(\ell)$ of ℓ in that tree. Denote by $\mathcal{G}_\ell$ the unique geodesic joining the root of the tree to ℓ. We will now define the coefficients $\{a_i\}_{i=1}^\infty$ of $\varphi = \sum_{i=1}^\infty a_i \varphi_i$, where φ_i is the function φ_{z_i} in Lemma 7 with w there replaced by z_i, by considering separately the indices in each tree of the forest $\mathbb{N}$.

Let $\mathcal{Y}$ be a tree in the forest $\mathbb{N}$ with root k_0. For each $k \in \mathcal{Y} \setminus \{k_0\}$, define $\beta_k \in [0,1]$ by

$$\beta_k = c\left(\gamma_{P(k)}(z_k)\right),$$

where the functions $c = c_{\rho,\alpha}$ and $\gamma_j = \gamma_{z_j}$ are defined as in the statement of Lemma 7 with w there replaced by z_j. Note that by Lemma 7 with $w = z_{P(k)}$, we have the estimate

$$\begin{aligned}\varphi_{P(k)}(z_k) &= c\left(\gamma_{P(k)}(z_k)\right) + O\left(\left(\log \frac{1}{1-\left|z_{P(k)}\right|^2}\right)^{-1}\right) \\ &= \beta_k + O\left(\left(\log \frac{1}{1-\left|z_{P(k)}\right|^2}\right)^{-1}\right),\end{aligned}$$

which can serve as motivation for the definition of the coefficients given below in (26). Indeed, with gross oversimplification, what we want is

$$\begin{aligned}\xi_k &= \varphi(z_k) \approx a_k \varphi_k(z_k) + a_{P(k)} \varphi_{P(k)}(z_k) + ... \\ &\approx a_k + a_{P(k)} \beta_k + ...,\end{aligned}$$

which leads to (26).

We will now define numbers $\{a_k\}_{k \in \mathcal{Y}}$ by induction on the linear ordering in $\mathcal{Y}$ induced from the natural ordering of $\mathbb{N}$, so that

$$\begin{cases} |a_k| & \leq 2 \\ \left|\sum_{i \in \mathcal{G}_k \setminus \{k_0\}} \beta_i a_{P(i)}\right| & \leq 1 \end{cases} \tag{25}$$

holds for all $k \in \mathcal{Y}$. First define $a_{k_0} = \xi_{k_0}$. Now fix $\ell \in \mathcal{Y} \setminus \{k_0\}$ and assume that a_k has been defined for all $k \in \mathcal{Y}$ for which $k < \ell$ so that (25) holds for all $k \in \mathcal{Y}$ for which $k < \ell$. We now define a_ℓ by

$$a_\ell = \xi_\ell - \sum_{i \in \mathcal{G}_\ell \setminus \{k_0\}} \beta_i a_{P(i)}. \tag{26}$$

Of crucial importance is the observation that the geodesics $\mathcal{G}_\ell$ and $\mathcal{G}_{P(\ell)}$ are related by

$$\mathcal{G}_{P(\ell)} = \mathcal{G}_\ell \setminus \{\ell\},$$

i.e. if $\mathcal{G}_\ell = [k_0, k_1, ..., k_{m-1}, k_m]$ with $k_m = \ell$, then $k_{m-1} = P(\ell)$ and $\mathcal{G}_{P(\ell)} = [k_0, k_1, ..., k_{m-1}]$. At this point the reader should draw a picture. The region $V_{z_{k_j}} = V_{z_{k_j}}^{\beta}$ is essentially a Carleson tent with vertex not at z_{k_j}, but rather at the point $z_{k_j}^{\beta}$ lying on the ray through z_{k_j} and having distance $\left(1 - \left|z_{k_j}\right|\right)^{\beta}$ to the boundary, much larger than the distance $1 - \left|z_{k_j}\right|$ from z_{k_j} to the boundary. Note that z_{k_j} can have infinitely many children in the tree $\mathcal{Y}$, one of which is the point $z_{k_{j+1}}$ having $V_{z_{k_{j+1}}} \subset V_{z_{k_j}}$ with $1 - \left|z_{k_{j+1}}\right| \ll 1 - \left|z_{k_j}\right|$. Finally, note that β_{j+1} equals 1 if $z_{k_{j+1}}$ lies within the smaller Carleson tent $V_{z_{k_j}}^{\alpha}$, β_{j+1} equals 0 if $z_{k_{j+1}}$ lies outside the Carleson tent $V_{z_{k_j}}^{\rho}$, and β_{j+1} is defined linearly in $(0,1)$ for $z_{k_{j+1}}$ within the annulus of Carleson tents $V_{z_{k_j}}^{\rho} \setminus V_{z_{k_j}}^{\alpha}$. By the induction assumption and the fact that $P(\ell) \in \mathcal{Y}$ and $P(\ell) < \ell$, we have

$$\left| \sum_{i \in \mathcal{G}_{P(\ell)} \setminus \{k_0\}} \beta_i a_{P(i)} \right| \leq 1.$$

We have from (26) and the above that

$$\begin{aligned}
\left| \sum_{i \in \mathcal{G}_\ell \setminus \{k_0\}} \beta_i a_{P(i)} \right| &= \left| \left(\sum_{i \in \mathcal{G}_{P(\ell)} \setminus \{k_0\}} \beta_i a_{P(i)} \right) + \beta_\ell \left(\xi_{P(\ell)} - \sum_{i \in \mathcal{G}_{P(\ell)} \setminus \{k_0\}} \beta_i a_{P(i)} \right) \right| \\
&= \left| \beta_\ell \xi_{P(\ell)} + (1 - \beta_\ell) \sum_{i \in \mathcal{G}_{P(\ell)} \setminus \{k_0\}} \beta_i a_{P(i)} \right| \\
&\leq \beta_\ell \left| \xi_{P(\ell)} \right| + (1 - \beta_\ell) \left| \sum_{i \in \mathcal{G}_{P(\ell)} \setminus \{k_0\}} \beta_i a_{P(i)} \right| \leq 1.
\end{aligned}$$

From this and (26) once more it immediately follows that $|a_\ell| \leq 2$, which shows that (25) holds for $k = \ell$ as well. This completes the inductive definition of the sequence $\{a_k\}_{k \in \mathcal{Y}}$ satisfying (25) on the tree $\mathcal{Y}$, and hence defines the entire sequence $\{a_i\}_{i=1}^{\infty}$.

We omit the proof that both (21) and (22) hold for the function $\varphi = \sum_{i=1}^{\infty} a_i \varphi_i$.

1.1.3 The proof of multiplier interpolation

Using Lemma 8, we first complete the proof that (6) implies (4) for $1 < p < \infty$. Fix $s > -1$, $0 < \delta < 1$ and $\left\{\xi_j\right\}_{j=1}^{\infty}$ with $\left\| \left\{\xi_j\right\}_{j=1}^{\infty} \right\|_{\ell^\infty} = 1$. Then by Lemma 8 there is $f_1 = \sum_{i=1}^{\infty} a_i^1 \varphi_i \in H^\infty(\mathbb{B}_n)$ such that $\left\| \left\{\xi_j - f_1(z_j)\right\}_{j=1}^{\infty} \right\|_{\ell^\infty} < \delta$ and $\left\| \{a_i^1\}_{i=1}^{\infty} \right\|_{\ell^\infty}, \|f_1\|_{H^\infty(\mathbb{B}_n)} \leq C$ where C is as in (22). Now apply Lemma 8 to the sequence $\left\{\xi_j - f_1(z_j)\right\}_{j=1}^{\infty}$ to obtain the existence of $f_2 = \sum_{i=1}^{\infty} a_i^2 \varphi_i \in H^\infty(\mathbb{B}_n)$

such that $\left\| \left\{ \xi_j - f_1(z_j) - f_2(z_j) \right\}_{j=1}^{\infty} \right\|_{\ell^\infty} < \delta^2$ and $\left\| \{a_i^2\}_{i=1}^{\infty} \right\|_{\ell^\infty}, \|f_2\|_{H^\infty(\mathbb{B}_n)} \leq C\delta$ where C is as in (22). Continuing inductively, we obtain $f_m = \sum_{i=1}^{\infty} a_i^m \varphi_i \in H^\infty(\mathbb{B}_n)$ such that

$$\left\| \left\{ \xi_j - \sum_{i=1}^{m} f_i(z_j) \right\}_{j=1}^{\infty} \right\|_{\ell^\infty} < \delta^m,$$
$$\|\{a_i^m\}_{i=1}^{\infty}\|_{\ell^\infty}, \|f_m\|_{H^\infty(\mathbb{B}_n)} \leq C\delta^{m-1}.$$

If we now take $\varphi = \sum_{m=1}^{\infty} f_m$, we have

$$\begin{aligned} \xi_j &= \varphi(z_j), \quad 1 \leq j < \infty, \\ \|\varphi\|_{H^\infty(\mathbb{B}_n)} &\leq C_\delta, \end{aligned} \tag{27}$$

as well as $\varphi = \sum_{i=1}^{\infty} a_i \varphi_i$ with $\|\{a_i\}_{i=1}^{\infty}\|_{\ell^\infty} \leq C_\delta$. Recall that the series $\varphi = \sum_{i=1}^{\infty} a_i \varphi_i$ converges absolutely by Remark 3, and depends linearly on the data $\{\xi_j\}_{j=1}^{\infty}$ by Remark 4 and the linear construction in this paragraph. Thus $\varphi \in H^\infty(\mathbb{B}_n)$ linearly interpolates the values $\{\xi_j\}_{j=1}^{\infty}$ on the sequence $\{z_j\}_{j=1}^{\infty}$, and it remains to prove that $\varphi \in M_{B_p}$. Recall that our function φ depends on our choice of $s > -1$.

By Theorem 3, $\varphi \in M_{B_p}$ will follow if we show that

$$\left\| \left| \left(1 - |z|^2\right)^n \nabla^n \varphi(z) \right|^p d\lambda_n(z) \right\|_{Carleson} \leq C. \tag{28}$$

Since

$$\varphi = \sum_{i=1}^{\infty} a_i \varphi_i = \sum_{i=1}^{\infty} a_i \Gamma_s g_i = \Gamma_s g$$

where $g = \sum_{i=1}^{\infty} a_i g_i$ with $\sup_{i \geq 1} |a_i| \leq C_\delta$, (28) will follow from Theorem **??** and Lemma 5 for s sufficiently large provided we show that (10) holds and that

$$\left| \left(1 - |z|^2\right)^n g(z) \right|^p d\lambda_n(z)$$

satisfies the tree condition. From the definition of g_i in (16), and the fact that the supports of the g_i are pairwise disjoint by the separation condition, we may assume that the reverse Hölder condition on Bergman balls in (10) holds. The tree condition estimate will follow from the next lemma.

Lemma 9 *With $s > n - \frac{1}{p'}$ and $g = \sum_{i=1}^{\infty} a_i g_i$ as above, we have*

$$\left\| \left| \left(1 - |z|^2\right)^n g(z) \right|^p d\lambda_n(z) \right\|_{tree\ condition} \leq C. \tag{29}$$

Proof. Inequality (29) follows from the estimate (15) as follows. If we discretize (15), we obtain with $w = z_i$ and $S(\alpha) \approx V_{z_i}$,

$$\sum_{\beta \in \mathcal{T}_n : \beta \geq \alpha} \left(1 - |c_\beta|^2\right)^{np} g_{z_i}(\beta)^p \leq C \left(\log \frac{1}{1 - |z_i|^2}\right)^{1-p}. \tag{30}$$

Denote the Carleson tent at c_β by $S(\beta) = T_\beta = \cup_{\gamma \geq \beta} K_\gamma$. We are assuming that the tree condition holds for the measure $\nu = \sum_{j=1}^\infty \left(\log \frac{1}{1-|z_j|^2}\right)^{1-p} \delta_{z_j}$, i.e.

$$\begin{aligned} &\sum_{\beta \in \mathcal{T}_n : \beta \geq \alpha} \left(\sum_{z_j \in T_\beta} \left(\log \frac{1}{1 - |z_j|^2}\right)^{1-p} \right)^{p'} \\ &= \sum_{\beta \in \mathcal{T}_n : \beta \geq \alpha} I^* \nu(\beta)^{p'} \\ &\leq C^{p'} I^* \nu(\alpha) = C^{p'} \sum_{z_j \in T_\alpha} \left(\log \frac{1}{1 - |z_j|^2}\right)^{1-p} < \infty, \qquad \alpha \in \mathcal{T}_n. \end{aligned} \tag{31}$$

If we now also discretize (29), we see that we must prove

$$\sum_{\beta \in \mathcal{T}_n : \beta \geq \alpha} I^* \mu(\beta)^{p'} \leq C^{p'} I^* \mu(\alpha) < \infty, \qquad \alpha \in \mathcal{T}_n,$$

where

$$\begin{aligned} \mu(\beta) &= \left| \left(1 - |c_\beta|^2\right)^n g(\beta) \right|^p, \\ g(\beta) &= \int_{K_\alpha} |g| \, d\lambda_n, \\ I^* \mu(\alpha) &= \sum_{\beta \in \mathcal{T}_n : \beta \geq \alpha} \mu(\beta), \end{aligned}$$

for all $g = \sum_{i=1}^\infty a_i g_i$ as above. Since $\nu = \sum_{j=1}^\infty \left(\log \frac{1}{1-|z_j|^2}\right)^{1-p} \delta_{z_j}$ satisfies the tree condition, it suffices to prove

$$\sum_{\beta \in \mathcal{T}_n : \beta \geq \alpha} I^* \mu(\beta)^{p'} \leq C^{p'} I^* \nu(\alpha), \qquad \alpha \in \mathcal{T}_n. \tag{32}$$

Indeed, (32) shows that $\mu + \nu$ satisfies the tree condition , and now the equivalence with the Carleson embedding on the tree shows that μ satisfies the tree condition as well (since the Carleson embedding is preserved for smaller measures).

Now $g=\sum_{i=1}^{\infty}a_ig_i$ where the supports of the $g_i=g_{z_i}$ are pairwise disjoint by the separation condition on $\{z_i\}_{i=1}^{\infty}$. Fix $\alpha\in\mathcal{T}_n$. Then we have

$$\begin{aligned}\sum_{\beta\in\mathcal{T}_n:\beta\geq\alpha} I^*\mu(\beta)^{p'} &= \sum_{\beta\in\mathcal{T}_n:\beta\geq\alpha}\left(\sum_{\gamma\in\mathcal{T}_n:\gamma\geq\beta}\mu(\gamma)\right)^{p'} \\ &= \sum_{\beta\in\mathcal{T}_n:\beta\geq\alpha}\left(\sum_{\gamma\in\mathcal{T}_n:\gamma\geq\beta}\left|\left(1-|c_\gamma|^2\right)^n\left(\sum_{i=1}^{\infty}a_ig_i\right)(\gamma)\right|^p\right)^{p'} \\ &= \sum_{\beta\in\mathcal{T}_n:\beta\geq\alpha}\left(\sum_{i:z_i\in T_\beta}|a_i|^p\sum_{\gamma\in\mathcal{T}_n:\gamma\geq\beta}\left|\left(1-|c_\gamma|^2\right)^n g_i(\gamma)\right|^p\right)^{p'}.\end{aligned}$$

Now we use (30) to dominate the last sum above by

$$\begin{aligned}& C\sum_{\beta\in\mathcal{T}_n:\beta\geq\alpha}\left(\sum_{i:z_i\in T_\beta}|a_i|^p\left(\log\frac{1}{1-|z_i|^2}\right)^{1-p}\right)^{p'} \\ &\leq C\left\|\{a_i\}_{i=1}^{\infty}\right\|_{\ell^\infty}^{pp'}\sum_{\beta\in\mathcal{T}_n:\beta\geq\alpha}\left(\sum_{i:z_i\in T_\beta}\left(\log\frac{1}{1-|z_i|^2}\right)^{1-p}\right)^{p'} \\ &\leq C\sum_{j:z_j\in T_\alpha}\left(\log\frac{1}{1-|z_j|^2}\right)^{1-p} = CI^*\nu(\alpha),\end{aligned}$$

where the final inequality follows from (31). This establishes (32), and completes the proof of Lemma 9.

With this done, we have completed the proof that (6) implies (4) for $1<p<\infty$.

We now prove that (3) implies (4) for $p>2n$. For this we will argue as above but with Lemma 5 replaced by Lemma 6, and with Lemma 9 replaced by the following analogue.

Lemma 10 *Suppose (3) holds. With $p>2n$, $s>n-\frac{1}{p'}$ and $g=\sum_{i=1}^{\infty}a_ig_i$ as above, we have*

$$\left\|\left|\left(1-|z|^2\right)^n g(z)\right|^p d\lambda_n(z)\right\|_{Carleson}\leq C. \tag{33}$$

Proof. Inequality (33) follows from estimate (15),

$$\int_{\mathbb{B}_n}\left|\left(1-|\zeta|^2\right)^n g_{z_i}(\zeta)\right|^p d\lambda_n(\zeta)\,d\zeta\leq C\left(\log\frac{1}{1-|z_i|^2}\right)^{1-p}, \tag{34}$$

as follows. Fix an index i. From Remark 2 we see that the support of g_{z_i} is essentially the union of a geodesic segment of Bergman kubes $K_1^i, K_2^i, ..., K_{M_i}^i$ where

$$M_i \approx (\alpha - \rho) \log \frac{1}{1 - |z_i|^2}.$$

Indeed, recall that the support g_{z_i} is contained in the intersection of the cone $\mathcal{C}_{z_i}$ and the annulus $\mathcal{A}_{z_i}$. Now for ζ in the cone $\mathcal{C}_{z_i}$, we have $\left|1 - \overline{\zeta} \cdot P z_i\right| \approx 1 - |\zeta|^2$, and thus for ζ in the annulus $\mathcal{A}_{z_i}$ as well, we have approximately

$$\log \frac{1}{1 - |\zeta|^2} \in \left(\rho \log \frac{1}{1 - |z_i|^2}, \alpha \log \frac{1}{1 - |z_i|^2} \right).$$

Thus $\zeta \in supp\ g_{z_i}$ lies in the union of those kubes in $\mathcal{T}_n$ along the geodesic joining the root to the "boundary point" $P z_i$, and having tree distance from the root lying roughly between $\rho\beta(0, z_i)$ and $\alpha\beta(0, z_i)$. Moreover, this segment can be continued to a longer sequence of adjacent Bergman kubes $K_1^i, K_2^i, ..., K_{M_i}^i, ...K_{J_i}^i = K_{z_i}$ connecting the support of g_{z_i} to the kube K_{z_i} containing z_i, and where

$$J_i \approx \log \frac{1}{1 - |z_i|^2}. \tag{35}$$

Choose $w_j \in K_j^i$ for $1 \le j < J_i$. Then we have for $z \in K_m^i$, $1 \le m \le M_i$, and $f \in B_p(\mathbb{B}_n)$,

$$\begin{aligned} |f(z)|^p &= \left| [f(z) - f(w_m)] + \sum_{j=m}^{J_i - 1} [f(w_j) - f(w_{j+1})] + [f(w_{J_i}) - f(z_i)] + f(z_i) \right|^p \\ &\lesssim |f(z) - f(w_m)|^p + (J_i)^{p-1} \sum_{j=1}^{J_i - 1} |f(w_j) - f(w_{j+1})|^p \\ &\quad + |f(w_{J_i}) - f(z_i)|^p + |f(z_i)|^p \\ &\le C (J_i)^{p-1} \sum_{j=1}^{J_i} \left(\max_{z_1, z_2 \in (K_j^i)^*} |f(z_1) - f(z_2)| \right)^p + C |f(z_i)|^p. \end{aligned}$$

Using this together with (34), (35) and the fact that the supports of the g_{z_i} are pairwise disjoint, we obtain

$$\begin{aligned} &\int_{\mathbb{B}_n} |f(z)|^p \left| \left(1 - |z|^2\right)^n g(z) \right|^p d\lambda_n(z) \\ &\le C \sum_i \sum_{j=1}^{J_i} \left(\max_{z_1, z_2 \in (K_j^i)^*} |f(z_1) - f(z_2)| \right)^p \\ &\quad + C \sum_i |f(z_i)|^p \left(\log \frac{1}{1 - |z_i|^2} \right)^{1-p}. \end{aligned}$$

Since the kubes $\left\{K_j^i\right\}$ are pairwise disjoint by Lemma 4, the first term on the right is dominated by

$$\sum_{\alpha\in\mathcal{T}_n}\left(\max_{z_1,z_2\in K_\alpha}|f(z_1)-f(z_2)|\right)^p\leq C\|f\|_{B_p(\mathbb{B}_n)}^p$$

by Theorem 6.30 of [Zhu]. The second term is dominated by $C\|f\|_{B_p(\mathbb{B}_n)}^p$ since we are assuming in (3) that $\sum_i\left(\log\frac{1}{1-|z_i|^2}\right)^{1-p}\delta_{z_i}$ is a $B_p(\mathbb{B}_n)$-Carleson measure. This completes the proof of Lemma 10.

Arguing as above, the proof that (3) implies (4) will follow from the characterization of Carleson measures on trees, together with Lemma 6 provided we show that (11) holds and that

$$\left|\left(1-|z|^2\right)^n g(z)\right|^p d\lambda_n(z)$$

is a $B_p(\mathbb{B}_n)$-Carleson measure. From the definition of g_i in (16), and the fact that the supports of the g_i are pairwise disjoint by the separation condition, we have that (11) holds. Lemma 10 above shows that $\left|\left(1-|z|^2\right)^n g(z)\right|^p d\lambda_n(z)$ is a $B_p(\mathbb{B}_n)$-Carleson measure for $p>2n$, and this completes the proof that (3) implies (4) for $p>2n$.

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Taiwan lecture 5

Monday July 10 2005

1 Introduction

Definition 1 *For $1 < p < \infty$ and $m \geq 0$, define the Besov space $B_{p,m}(\mathcal{T})$ on a tree $\mathcal{T}$ to consist of all sequences $f = \{f(\alpha)\}_{\alpha \in \mathcal{T}}$ such that*

$$\|f\|_{B_{p,m}(\mathcal{T})} = \left(\sum_{\alpha \in \mathcal{T}: d(\alpha) \geq m} \left| \left(2^{-d}\right)^m \left(2^d \bigtriangleup\right)^m f(\alpha) \right|^p \right)^{\frac{1}{p}} + \sum_{d(\alpha) \leq m-1} |f(\alpha)| < \infty.$$

Note that in comparing this definition to the standard definition of Besov spaces in the unit ball of $\mathbb{C}^n$, the term 2^{-d} plays the role of $\left(1 - |z|^2\right)$, and $2^d \bigtriangleup$ plays the role of gradient.

Remark 1 *The restrictions of linear functions on the ball fail to belong to $B_p(\mathcal{T}_n)$ for $p \leq 2n$, $n \geq 2$ (on the other hand, the analogous linear functions $f(\alpha) = 2^{-d(\alpha)}$ on the tree belong to $B_{p,2}(\mathcal{T}_n)$ for all $1 < p < \infty$). Indeed, if $f(z) = z_1$, then for most $\alpha \in \mathcal{T}_n$, in particular for those α at a distance at least $c > 0$ from the complex line $\mathbb{C}(1, 0, ..., 0)$, we have*

$$\sum_{\beta \in \mathcal{C}(\alpha)} |f(\beta) - f(\alpha)| = \sum_{\beta \in \mathcal{C}(\alpha)} |\beta_1 - \alpha_1| \approx e^{-d(\alpha)\theta},$$

where $\mathcal{C}(\alpha)$ denotes the set of children of α. By property 4 of Lemma **??**, *we have*

$$\#\{\alpha \in \mathcal{T}_n : d(\alpha) = N\} \approx e^{2nN\theta},$$

and thus

$$\|f\|^p_{B_p(\mathcal{T}_n)} = \|f\|^p_{B_{p,1}(\mathcal{T}_n)} \geq c \sum_{\alpha \in \mathcal{T}_n} \left(\sum_{\beta \in \mathcal{C}(\alpha)} |f(\beta) - f(\alpha)| \right)^p$$

$$\geq c \sum_{\alpha \in \mathcal{T}_n} \left(e^{-d(\alpha)\theta}\right)^p \approx \sum_{N=1}^{\infty} \sum_{\alpha \in \mathcal{T}_n : d(\alpha) = N} e^{-pN\theta}$$
$$\approx \sum_{N=1}^{\infty} e^{2nN\theta} e^{-pN\theta} = \infty$$

if $2n - p \geq 0$. *In fact, the restricion of this* f *fails to belong to* $B_{p,m}(\mathcal{T}_n)$*for any* $m \geq 1$ *and* $p \leq 2n$, $n \geq 2$.

What is needed now is a definition of Besov space on a tree that involves complex structure sufficient for higher order differences, or "derivatives", to be properly defined. This is introduced in the following subsection.

1.1 Structured trees

In this subsection we introduce an alternative definition of $B_{p,1}(\mathcal{T}_n)$ that is better adapted for generalization to those higher order differences that reflect the underlying complex structure of the Bergman tree. We must first interpret the fact that for a holomorphic function F in $B_p(\mathbb{B}_n)$, the differences $F(\beta) - F(\alpha)$ are *related* as β ranges over the children of a fixed α; namely they are close to $F'(\alpha)(\beta - \alpha)$. Thus we wish to define in a natural way the notion of a complex derivative f' of a complex-valued function f on the Bergman tree $\mathcal{T}_n$. It is convenient at this point to consider trees more general than $\mathcal{T}_n$, namely those with a complex structure.

An n-dimensional *complex structure* $\mathcal{V}$ on a tree $\mathcal{T}$ is a collection of n-vectors $\mathcal{V} = \{\mathbf{v}_\alpha\}_{\alpha \in \mathcal{T}}$, $\mathbf{v}_\alpha \in \mathbb{C}^n$. We can "immerse" the structured tree $(\mathcal{T}, \mathcal{V})$ in $\mathbb{C}^n$ by identifying $\alpha \in \mathcal{T}$ with the point $c(\alpha) = \mathbf{v}_o + \sum_{o < \beta \leq \alpha} \mathbf{v}_\beta \in \mathbb{C}^n$. For example, the standard embedding of the Bergman tree $\mathcal{T}_n$ in the ball arises in this way from the complex structure $\mathcal{V} = \{c_\alpha - c_{A\alpha}\}_{\alpha \in \mathcal{T}_n}$ on $\mathcal{T}_n$. In general however, the map $\alpha \to c(\alpha)$ need not be one-to-one, hence the term "immerse". Additional properties of $\mathcal{V}$ will be required below.

Define the (backward) difference operator $\triangle$ on functions f mapping the tree $\mathcal{T}$ to $\mathbb{C}$ by

$$\triangle f(\alpha) = f(\alpha) - f(A\alpha), \qquad \alpha \in \mathcal{T},$$

where $A\alpha$ denotes the predecessor, or immediate *A*ncestor, of α (we reserve P for projection). We denote the set of children of $\alpha \in \mathcal{T}$ by $\mathcal{C}(\alpha)$. We now assume that $\dim(\mathcal{T})$ is finite, so that in particular, there is an upper bound N for the branching number of the tree, i.e. $\#\mathcal{C}(\alpha) \leq N$ for all $\alpha \in \mathcal{T}_n$. Let $\{\alpha^j\}_{j=1}^{\#\mathcal{C}(\alpha)} = \mathcal{C}(\alpha)$ be an enumeration of the children of α. Then for $\alpha \in \mathcal{T}$ we define the linear map $L_\alpha : \mathbb{C}^n \to \mathbb{C}^N$ by

$$L_\alpha(\mathbf{w}) = \left(\mathbf{w} \cdot \mathbf{v}_{\alpha^j}\right)_{j=1}^N,$$

with the convention that $\alpha^j = \alpha$ and $\mathbf{v}_{\alpha^j} = 0$ if $\#\mathcal{C}(\alpha) < j \leq N$. We now make the assumption that L_α is *one-to-one* for all $\alpha \in \mathcal{T}$. Then given a complex-valued function f on $\mathcal{T}$ we can define its *complex derivative* $f'(\alpha) \in \mathbb{C}^n$ as follows. Let $\mathcal{P}_\alpha$ denote orthogonal projection of $\mathbb{C}^N$ onto the range of L_α, and let $\mathcal{Q}_\alpha = I - \mathcal{P}_\alpha$. Denote by $\mathcal{D}_\alpha f$ the N-vector of (forward) differences of f:

$$\mathcal{D}_\alpha f = \left(f\left(\alpha^j\right) - f(\alpha)\right)_{j=1}^N = \left(\triangle\, f\left(\alpha^j\right)\right)_{j=1}^N \in \mathbb{C}^N,$$

where $\triangle\, f(\alpha^j) = f(\alpha) - f(\alpha) = 0$ if $\#\mathcal{C}(\alpha) < j \leq N$ by our convention. Then we define

$$f'(\alpha) = L_\alpha^{-1}\mathcal{P}_\alpha(\mathcal{D}_\alpha f),$$

so that

$$\left(f\left(\alpha^j\right) - f(\alpha)\right)_{j=1}^N = \left(f'(\alpha)\cdot \mathbf{v}_{\alpha^j}\right)_{j=1}^N + \mathcal{Q}_\alpha(\mathcal{D}_\alpha f). \tag{1}$$

In the case of the natural complex structure introduced above on the Bergman tree $\mathcal{T}_n$, we have $\mathbf{v}_{\alpha^j} = \alpha^j - \alpha$ and (1) is thus an analogue of Taylor's formula of degree one on $\mathcal{T}_n$. We now make this more precise in the special case of the Bergman tree $\mathcal{T}_n$.

We also define the difference

$$\triangle\, \alpha = \alpha - A\alpha, \qquad \alpha \in \mathcal{T}_n,$$

where we identify α with the center c_α of the Bergman kube K_α. We view the set of differences

$$\mathcal{D}_\alpha f = \left(\triangle\, f\left(\alpha^j\right)\right)_{j=1}^N$$

as a vector of complex numbers of length N, i.e. $\mathcal{D}_\alpha f \in \mathbb{C}^N$, choosing N larger than the branching number at any element of the tree. This is simply a matter of convenience and we could just as well have worked with $\mathbb{C}^N$ replaced replaced by $\mathbb{C}^{\#\mathcal{C}(\alpha)}$ at each element α, but at the expense of more complicated notation. We also consider the corresponding family of differences

$$\left(\triangle\, \alpha^j\right)_{j=1}^N,$$

as a vector of points in $\mathbb{C}^n$ of length N, i.e. in $(\mathbb{C}^n)^N$.

The linear map L_α defined above sends $\mathbf{v} \in \mathbb{C}^n$ to the point

$$L_\alpha \mathbf{v} = \left(\mathbf{v}\cdot\left(\alpha^j - \alpha\right)\right)_{j=1}^N \in \mathbb{C}^N.$$

Note that for the Bergman tree, the map L_α is one-to-one since the collection of n-vectors $(\alpha^j - \alpha)_{j=1}^N$ has rank n if $\theta > 1$ is chosen large enough. Recall that $\mathcal{P}_\alpha$ is the orthogonal projection of $\mathbb{C}^N$ onto the range of L_α (which has dimension n

since L_α is one-to-one) and $\mathcal{Q}_\alpha = I - \mathcal{P}_\alpha$. The complex derivative $f'(\alpha)$ of f at the point α is then the unique vector $\mathbf{v}$ such that

$$L_\alpha \mathbf{v} = \mathcal{P}_\alpha (\mathcal{D}_\alpha f).$$

Thus we have

$$L_\alpha f'(\alpha) = \mathcal{P}_\alpha (\mathcal{D}_\alpha f) = \left(f'(\alpha) \cdot (\alpha^j - \alpha)\right)_{j=1}^N .$$

Now denote the radial and tangential components of $f'(\alpha)$ by $f'(\alpha) P_\alpha$ and $f'(\alpha) Q_\alpha$ respectively, where $P_\alpha z = \frac{z \cdot \overline{\alpha}}{|\alpha|^2}\alpha$, and $Q_\alpha = I - P_\alpha$. Here we are viewing $f'(\alpha)$ as belonging to the space $\mathcal{L}(\mathbb{C}^n, \mathbb{C})$ of linear maps from $\mathbb{C}^n$ to $\mathbb{C}$, and $\mathcal{P}_\alpha, \mathcal{Q}_\alpha$ as belonging to the corresponding space of linear maps $\mathcal{L}(\mathbb{C}^n, \mathcal{L}(\mathbb{C}^n, \mathbb{C}))$. Thus we have decomposed the difference set $\mathcal{D}_\alpha f$ as

$$\begin{aligned} \mathcal{D}_\alpha f &= \left(\triangle\, f(\alpha^j)\right)_{j=1}^N \\ &= \left(f'(\alpha) P_\alpha \cdot (\alpha^j - \alpha)\right)_{j=1}^N \\ &\quad + \left(f'(\alpha) Q_\alpha \cdot (\alpha^j - \alpha)\right)_{j=1}^N \\ &\quad + \mathcal{Q}_\alpha (\mathcal{D}_\alpha f). \end{aligned} \tag{2}$$

In our alternative definition of the Besov space $B_{p,1}(\mathcal{T}_n)$, we weight the various components of this decomposition in accordance with the complex structure the Bergman tree $\mathcal{T}_n$ inherits from its embedding in the unit ball $\mathbb{B}_n$.

Definition 2 *For $1 < p < \infty$, define the* holomorphic *Besov space $HB_{p,1}(\mathcal{T}_n)$ on $\mathcal{T}_n$ to consist of all complex-valued sequences $f = \{f(\alpha)\}_{\alpha \in \mathcal{T}_n}$ such that*

$$\|f\|^p_{HB_{p,1}(\mathcal{T}_n)} = |f(o)|^p + \sum_{\alpha \in \mathcal{T}_n} \left| e^{-2d(\alpha)\theta} f'(\alpha) P_\alpha + e^{-d(\alpha)\theta} f'(\alpha) Q_\alpha \right|^p$$
$$+ \sum_{\alpha \in \mathcal{T}_n} |\mathcal{Q}_\alpha \mathcal{D}_\alpha f|^p < \infty.$$

Remark 2 *The expression*

$$\left| e^{-2d(\alpha)\theta} \mathbf{v} P_\alpha + e^{-d(\alpha)\theta} \mathbf{v} Q_\alpha \right|$$

is the tree analogue of

$$\sqrt{(B(\alpha)\mathbf{v}) \cdot \overline{\mathbf{v}}} = \sqrt{\left(\frac{P_\alpha \mathbf{v}}{\left(1 - |\alpha|^2\right)^2} + \frac{Q_\alpha \mathbf{v}}{1 - |\alpha|^2} \right) \cdot \overline{\mathbf{v}}},$$

where $B(z) = \frac{1}{n+1} \frac{\partial^2}{\partial \overline{z_i} \partial z_j} \log \frac{1}{\left(1 - |z|^2\right)^{n+1}}$ is the Bergman Riemannian metric on tangent vectors $\mathbf{v}$ at the point z in $\mathbb{B}_n$ that leads to the Bergman distance β - see Chapter 1.5 of [39].

The equivalence of the norms $\|\cdot\|_{B_{p,1}(\mathcal{T}_n)}$ and $\|\cdot\|_{HB_{p,1}(\mathcal{T}_n)}$ for $1 < p < \infty$ and θ sufficiently large follows from the next result.

Definition 3 *For a vector* $\mathbf{v} \in \mathbb{C}^n$ *and* $\alpha \in \mathcal{T}_n$*, let*

$$|\mathbf{v}|_\alpha = \left| e^{-2d(\alpha)\theta} \mathbf{v} P_\alpha + e^{-d(\alpha)\theta} \mathbf{v} Q_\alpha \right| = \sqrt{(B(\alpha)\mathbf{v}) \cdot \overline{\mathbf{v}}}.$$

Lemma 4 *For* θ *sufficiently large in the construction of the Bergman tree* $\mathcal{T}_n$*, and for all* $1 < p < \infty$*, we have*

$$\left(\sum_{\alpha \in \mathcal{T}_n} |\mathcal{D}_\alpha f|^p \right)^{\frac{1}{p}} \approx \left(\sum_{\alpha \in \mathcal{T}_n} |f'(\alpha)|_\alpha^p \right)^{\frac{1}{p}} + \left(\sum_{\alpha \in \mathcal{T}_n} |\mathcal{Q}_\alpha \mathcal{D}_\alpha f|^p \right)^{\frac{1}{p}}.$$

Proof. Since $\mathcal{P}_\alpha$ and $\mathcal{Q}_\alpha$ (respectively P_α and Q_α) are orthogonal projections on $\mathbb{C}^N$ (respectively $\mathbb{C}^n$), we have

$$\begin{aligned} |\mathcal{D}_\alpha f|^2 &= |\mathcal{P}_\alpha \mathcal{D}_\alpha f|^2 + |\mathcal{Q}_\alpha \mathcal{D}_\alpha f|^2 \\ &= |L_\alpha f'(\alpha)|^2 + |\mathcal{Q}_\alpha \mathcal{D}_\alpha f|^2 \\ &\approx \left| e^{-2d(\alpha)\theta} f'(\alpha) P_\alpha \right|^2 + \left| e^{-d(\alpha)\theta} f'(\alpha) Q_\alpha \right|^2 + |\mathcal{Q}_\alpha \mathcal{D}_\alpha f|^2 \\ &= |f'(\alpha)|_\alpha^2 + |\mathcal{Q}_\alpha \mathcal{D}_\alpha f|^2, \end{aligned}$$

where the third line follows from

$$\begin{aligned} |\mathcal{P}_\alpha \mathcal{D}_\alpha f|^2 &= |L_\alpha f'(\alpha)|^2 \\ &= \left| \left\{ f'(\alpha) P_\alpha \cdot P_\alpha \left(\alpha^j - \alpha\right) + f'(\alpha) Q_\alpha \cdot Q_\alpha \left(\alpha^j - \alpha\right) \right\}_{j=1}^N \right|^2 \\ &\approx \left| e^{-2d(\alpha)\theta} f'(\alpha) P_\alpha \right|^2 + \left| e^{-d(\alpha)\theta} f'(\alpha) Q_\alpha \right|^2. \end{aligned}$$

To see this last equivalence, we note that both

$$\sum_{j=1}^N \left| P_\alpha \left(\alpha^j - \alpha\right) \right| \geq c e^{-2d(\alpha)\theta}, \tag{3}$$

$$\sum_{j=1}^N \left| Q_\alpha \left(\alpha^j - \alpha\right) \right| \geq c e^{-d(\alpha)\theta}.$$

The first inequality is obvious. The second inequality follows if $\theta > 1$ is chosen sufficiently large, since the set of projections onto the sphere $\mathcal{S}_{d(\alpha)\theta}$ of the children $\mathcal{C}(\alpha)$ is $e^{-2\theta}C_2$-dense in the qube $Q_j^{d(\alpha)} = K_\alpha \cap \mathcal{S}_{d(\alpha)\theta}$ corresponding to K_α. Indeed, there are roughly $e^{2n\theta}$ children of α whose projections onto the Bergman sphere $S_{d(\alpha)\theta}$ all lie in $Q_j^{d(\alpha)}$. Since the Bergman distance β is preserved by automorphisms, we see upon mapping matters to the origin that these projections are roughly $e^{-2\theta}$-dense in the Bergman distance. With this established for $e^{-2\theta}$ sufficiently small, we now see that the vectors $\{Q_\alpha(\alpha^j - \alpha)\}_{j=1}^N$ are sufficiently well distributed that (3) holds uniformly in α.

1.1.1 An abstract approach

The purpose of this short subsubsection is to illustrate the flexibility of defining spaces via "derivatives" on trees, by including a variety of classical spaces within a generalization of this framework.

Given a tree $\mathcal{T}$ with branching number bounded by N, we can more generally than above, suppose that we are given an m-dimensional complex vector space W, and for each $\alpha \in \mathcal{T}$, Hilbert space norms $[\cdot]_\alpha$ and $\{\cdot\}_\alpha$ on W and $\mathbb{C}^N$ respectively, and a one-to-one linear map L_α from W to $\mathbb{C}^N$. Then we can define a derivative $f'(\alpha) \in W$ by

$$f'(\alpha) = L_\alpha^{-1}\mathcal{P}_\alpha(\mathcal{D}_\alpha f),$$

where $\mathcal{D}_\alpha f$ is the (forward) difference set defined as above, and $\mathcal{P}_\alpha$ is orthogonal projection onto the range of L_α. With $\mathcal{Q}_\alpha = I - \mathcal{P}_\alpha$, define a norm on $\mathbb{C}^N$ by

$$|w|_\alpha^2 = \left[L_\alpha^{-1}\mathcal{P}_\alpha(w)\right]_\alpha^2 + \{\mathcal{Q}_\alpha(w)\}_\alpha^2.$$

Then we define a Besov space norm $\|f\|_{B_p(\mathcal{T})}$ by

$$\|f\|_{B_p(\mathcal{T})}^p = |f(o)|^p + \sum_{\alpha\in\mathcal{T}} |f'(\alpha)|_\alpha^p.$$

In the case $\mathcal{T} = \mathcal{T}_n$ and $W = \mathbb{C}^n$, with L_α defined as above and

$$\begin{aligned}
[\mathbf{v}]_\alpha &= |\mathbf{v}|_\alpha = \left|e^{-2d(\alpha)\theta}\mathbf{v}P_\alpha + e^{-d(\alpha)\theta}\mathbf{v}Q_\alpha\right|, \qquad \mathbf{v} \in \mathbb{C}^n,\\
\{w\}_\alpha &= |w|, \qquad w \in \mathbb{C}^N,
\end{aligned}$$

we obtain that $B_p(\mathcal{T})$ is the holomorphic Besov space $HB_{p,1}(\mathcal{T}_n)$ defined above. If however we take $W = \mathbb{C}^N$, $L_\alpha = I$ and the norms $[\cdot]_\alpha$ and $\{\cdot\}_\alpha$ to be the Euclidean norm on $\mathbb{C}^N$, then $B_p(\mathcal{T})$ is the abstract space $B_{p,1}(\mathcal{T})$ defined earlier.

For another example, suppose that $\mathcal{T}$ is a homogeneous tree with branching number N. Take W to be the orthogonal complement in $\mathbb{C}^N$ of the one-dimensional subspace $\mathbb{C}(1,1,...,1)$ generated by $(1,1,...,1)$, and let $[\cdot]_\alpha$ be the restriction of the Euclidean norm to W. Let L_α be the natural inclusion of W into $\mathbb{C}^N$. Finally, let $\{\cdot\}_\alpha$ be the trivial norm on $\mathbb{C}^N$ that is infinite on all nonzero vectors and vanishes on the zero vector. Then $B_p(\mathcal{T})$ is the martingale of all ℓ^p functions on the tree $\mathcal{T}$ satisfying

$$f(\alpha) = \frac{1}{N}\sum_{\beta\in\mathcal{C}(\alpha)} f(\beta), \qquad \alpha \in \mathcal{T}.$$

We will now construct holomorphic Besov spaces on Bergman trees $\mathcal{T}_n$ that model the Besov spaces $B_p(\mathbb{B}_n)$ on the ball.

2 Holomorphic Besov spaces on Bergman trees

Our goal here is to obtain a definition of a holomorphic Besov space $HB_{p,m}(\mathcal{T}_n)$ on the Bergman tree $\mathcal{T}_n$ so that the restriction map is bounded from $B_p(\mathbb{B}_n)$ to $HB_{p,m}(\mathcal{T}_n)$ in the range $p > \frac{2n}{m}$, while retaining as many of the properties of the abstract Besov space $B_{p,m}(\mathcal{T}_n)$ as possible. One essential property we wish to retain is that Carleson measures for $HB_{p,m}(\mathcal{T}_n)$ be characterized by the tree condition , as that is the condition needed to prove the sufficiency implication for multiplier interpolation on the ball. Another essential property is an appropriate positivity of "derivatives" of reproducing kernels for $HB_{p,m}(\mathcal{T}_n)$, as that is needed to prove the necessity of the tree condition for multiplier interpolation on the ball in the difficult range $1 + \frac{1}{n-1} \leq p < 2$.

Recall that the restriction map is bounded from $B_p(\mathbb{B}_n)$ to $B_{p,1}(\mathcal{T}_n)$ in the range $p > \widehat{n} = \frac{2n}{1}$. Of course the Carleson measures for $B_{p,1}(\mathcal{T}_n)$ are characterized by the tree condition, and the first order difference of the reproducing kernel for $B_{p,1}(\mathcal{T}_n)$ is nonnegative (the analogues of these latter two properties actually hold for *all* $B_{p,m}(\mathcal{T}_n)$). This demonstrates that the abstract Besov space $B_{p,1}(\mathcal{T}_n)$ has the properties desired of our holomorphic Besov space for $p > 2n$, and in view of Lemma 4, we have

$$HB_{p,1}(\mathcal{T}_n) = B_{p,1}(\mathcal{T}_n). \tag{4}$$

However, the abstract Besov spaces $B_{p,m}(\mathcal{T}_n)$ on trees do not capture the higher order derivatives of holomorphic functions in the ball. Indeed, higher order tree differences vanish on appropriate polynomial functions of $r^{-d(\alpha)}$ on the tree, but *not* on the restrictions to the tree of polynomials on the ball, even though the corresponding derivatives are identically zero on the ball. In particular, recall from Remark 1 that linear functions on the ball do not restrict to $B_{p,m}(\mathcal{T}_n)$ for $p \leq 2n$ for any $m \geq 1$.

We begin with the holomorphic Besov space $HB_{p,1}(\mathcal{T}_n)$ already defined, and derive its reproducing kernels relative to the duality pairing induced by the norm $\|\cdot\|_{HB_{p,1}(\mathcal{T}_n)}$, along with their positivity properties. In preparation for an inductive definition of the higher order Besov spaces $HB_{p,m}(\mathcal{T}_n)$, we must also derive the analogous theory of the Besov spaces $HB_{p,1}^{(k)}(\mathcal{T}_n)$ of k-tensors defined on $\mathcal{T}_n$, as we will view higher order complex tree derivatives $f^{(k)}(\alpha)$ as tensor-valued functions on the tree. To expedite this process, it is advantageous to consider first the order zero case, and develop the required tensor apparatus for ℓ^p spaces, the order zero analogue of a holomorphic Besov space. Then we proceed to define $HB_{p,m}(\mathcal{T}_n)$ inductively for $m \geq 2$ and establish the appropriate positivity properties of their reproducing kernels, which will require a careful choice of the structural constants λ and θ in the construction of the Bergman tree $\mathcal{T}_n$, along with an additional modification of the centers of the Bergman balls. Finally, we establish for these spaces the Carleson measure theorem and the restriction theorem, and then complete the proof of the multiplier interpolation loop for

$1 < p < 2 + \frac{1}{n-1}$, that was previously left open.

2.1 The order zero and order one holomorphic Besov spaces

Recall that for $1 < p < \infty$ we defined the order 1 holomorphic Besov space $HB_{p,1}(\mathcal{T}_n)$ on $\mathcal{T}_n$ in Definition 2 to consist of all complex-valued sequences $f = \{f(\alpha)\}_{\alpha \in \mathcal{T}_n}$ such that

$$\begin{aligned} \|f\|^p_{HB_{p,1}(\mathcal{T}_n)} &= |f(o)|^p + \sum_{\alpha \in \mathcal{T}_n} \left| r^{-d(\alpha)} f'(\alpha) P_\alpha + r^{-\frac{d(\alpha)}{2}} f'(\alpha) Q_\alpha \right|^p \\ &+ \sum_{\alpha \in \mathcal{T}_n} |\mathcal{Q}_\alpha \mathcal{D}_\alpha f|^p < \infty, \end{aligned}$$

where we have written

$$r = e^{2\theta} \tag{5}$$

for convenience, so that,

$$1 - |\alpha|^2 \approx e^{-2\beta(0,\alpha)} \approx e^{-2\theta d(\alpha)} = r^{-d(\alpha)}. \tag{6}$$

In comparing this definition with that for the real Besov space $B_{p,1}(\mathcal{T}_n)$ on the Bergman tree $\mathcal{T}_n$,

$$\|f\|^p_{B_{p,1}(\mathcal{T}_n)} = |f(o)|^p + \sum_{\alpha \in \mathcal{T}_n} |\mathcal{D}_\alpha f|^p < \infty,$$

recall that the set of differences $\mathcal{D}_\alpha f$ can be written as the linear sum of the two pieces $\mathcal{P}_\alpha \mathcal{D}_\alpha f$ and $\mathcal{Q}_\alpha \mathcal{D}_\alpha f$; the first piece $\mathcal{P}_\alpha \mathcal{D}_\alpha f$ lying in the range of L_α (the holomorphic part), and the second piece $\mathcal{Q}_\alpha \mathcal{D}_\alpha f$ orthogonal to the range of L_α. The first piece $\mathcal{P}_\alpha \mathcal{D}_\alpha f$ can be further decomposed using the identities

$$\begin{aligned} L_\alpha f'(\alpha) &= \mathcal{P}_\alpha \mathcal{D}_\alpha f, \\ f'(\alpha) &= L_\alpha^{-1} \mathcal{P}_\alpha \mathcal{D}_\alpha f, \end{aligned}$$

where the second one follows since L_α is one-to-one and $\mathcal{P}_\alpha$ is orthogonal projection onto the range of L_α. We then decompose $f'(\alpha)$ as $f'(\alpha) P_\alpha + f'(\alpha) Q_\alpha$. Now L_α has the $N \times n$ matrix representation $\left[\alpha_k^j - \alpha_k\right]_{1 \le j \le N, 1 \le k \le n}$ where $\alpha^j = \left(\alpha_k^j\right)_{k=1}^n$ and $\alpha = (\alpha_k)_{k=1}^n$, and so resembles the nonisotropic linear operator

$$\mathbf{R}^{-d(\alpha)} \equiv r^{-d(\alpha)} P_\alpha + r^{-\frac{d(\alpha)}{2}} Q_\alpha,$$

whose action on a vector $v \in \mathcal{L}(\mathbb{C}^n, \mathbb{C})$ is given by $\mathbf{R}^{-d(\alpha)} v \equiv r^{-d(\alpha)} v P_\alpha + r^{-\frac{d(\alpha)}{2}} v Q_\alpha$. Thus we formally have that

$$\begin{aligned} \mathcal{D}_\alpha f &= L_\alpha f'(\alpha) + \mathcal{Q}_\alpha \mathcal{D}_\alpha f \\ &\sim \mathbf{R}^{-d(\alpha)} f'(\alpha) + \mathcal{Q}_\alpha \mathcal{D}_\alpha f. \end{aligned}$$

We actually proved that $|\mathcal{D}_\alpha f| \approx \left|\mathbf{R}^{-d(\alpha)} f'(\alpha)\right| + |\mathcal{Q}_\alpha \mathcal{D}_\alpha f|$ above.

Thus locally we measure the holomorphic parts $\mathcal{P}_\alpha \mathcal{D}_\alpha f$ of the differences $\mathcal{D}_\alpha f$ by the Bergman Riemannian metric, where the radial directions $f'(\alpha) P_\alpha$ are weighted by $r^{-d(\alpha)}$, and the tangential directions $f'(\alpha) Q_\alpha$ weighted by $\sqrt{r^{-d(\alpha)}} = r^{-\frac{d(\alpha)}{2}}$. This is analogous to the definition of the almost invariant holomorphic derivative D_a given in the ball by

$$D_a f(z) = -f'(z) \left\{ \left(1 - |a|^2\right) P_a + \left(1 - |a|^2\right)^{\frac{1}{2}} Q_a \right\}, \qquad a \in \mathbb{B}_n.$$

We then take the $\ell^p(\mathcal{T}_n)$ norm of these local measures. We measure the anti-holomorphic parts $\mathcal{Q}_\alpha \mathcal{D}_\alpha f$ of the differences $\mathcal{D} f(\alpha)$ by the $\ell^p(\mathcal{T})$-norm. We now consider reproducing kernels associated to these norms.

2.1.1 Reproducing kernels

The reproducing kernel k_α for the space $\ell^p(\mathcal{T}_n)$ is trivial: $k_\alpha(\gamma) = \chi_{\{\alpha\}}(\gamma)$, the delta function at α. We must work harder to obtain the reproducing kernel for $HB_{p,1}(\mathcal{T}_n)$. We first observe that we can recover a function $f \in HB_{p,1}(\mathcal{T}_n)$ from its differences

$$\begin{aligned} \mathcal{D}_\alpha f &= \mathcal{P}_\alpha \mathcal{D}_\alpha f + \mathcal{Q}_\alpha \mathcal{D}_\alpha f \\ &= L_\alpha f'(\alpha) + \mathcal{Q}_\alpha \mathcal{D}_\alpha f \end{aligned}$$

and hence also from its derivatives and antiholomorphic differences $\mathcal{Q}_\alpha \mathcal{D}_\alpha f$. In order to give an explicit formula, we write

$$\mathcal{P}_\alpha \mathcal{D}_\alpha f = \left\{ \mathcal{P}_\alpha \mathcal{D}_\alpha f\left(\alpha^j\right) \right\}_{j=1}^N \text{ and } \mathcal{Q}_\alpha \mathcal{D}_\alpha f = \left\{ \mathcal{Q}_\alpha \mathcal{D}_\alpha f\left(\alpha^j\right) \right\}_{j=1}^N,$$

so that $\triangle f(\alpha^j) = \mathcal{P}_\alpha \mathcal{D}_\alpha f(\alpha^j) + \mathcal{Q}_\alpha \mathcal{D}_\alpha f(\alpha^j)$, where $\alpha = A\alpha^j$. Thus $\mathcal{P}_\alpha \mathcal{D}_\alpha f(\alpha^j)$ is the j^{th} component of the N-vector $\mathcal{P}_\alpha \mathcal{D}_\alpha f = L_\alpha f'(\alpha)$, i.e.

$$\mathcal{P}_{A\beta} \mathcal{D}_{A\beta} f(\beta) = f'(A\beta) \cdot (\beta - A\beta),$$

and we have the tree version of the Taylor expansion of order 1 at the point $A\beta$;

$$f(\beta) = f(A\beta) + f'(A\beta) \cdot (\beta - A\beta) + \mathcal{Q}_{A\beta} \mathcal{D}_{A\beta} f(\beta).$$

An explicit formula for $f(\alpha)$, when $\alpha \in \mathcal{T}_n$ has geodesic $[o, \alpha] = \{o, \alpha_1, \alpha_2, ..., \alpha_m\}$ (note the different use of the terminology α_j here), is now given by

$$\begin{aligned} f(\alpha) = f(\alpha_m) &= \sum_{k=1}^m \triangle f(\alpha_k) + f(o) \qquad (7) \\ &= f(o) + \sum_{k=0}^m \mathcal{P}_{\alpha_{k-1}} \mathcal{D}_{\alpha_{k-1}} f(\alpha_k) + \sum_{k=0}^m \mathcal{Q}_{\alpha_{k-1}} \mathcal{D}_{\alpha_{k-1}} f(\alpha_k) \\ &= f(o) + \sum_{k=0}^m f'(\alpha_{k-1}) \cdot (\alpha_k - \alpha_{k-1}) + \sum_{k=0}^m \mathcal{Q}_{\alpha_{k-1}} \mathcal{D}_{\alpha_{k-1}} f(\alpha_k). \end{aligned}$$

We can rewrite this as

$$f(\alpha) = f(o) + \sum_{\gamma<\alpha} f'(\gamma)\cdot(\gamma_\alpha - \gamma) + \sum_{\gamma<\alpha} \mathcal{Q}_\gamma \mathcal{D}_\gamma f(\gamma_\alpha), \tag{8}$$

where γ_α denotes the child of γ lying on the geodesic $[o,\alpha]$, so that $\gamma_\alpha = \alpha_k$ if $\gamma = \alpha_{k-1}$. Note that an immediate consequence of (8) is the inequality

$$|f(\alpha)| \leq C \|f\|_{HB_{p,1}(\mathcal{T}_n)} d(\alpha)^{\frac{1}{p'}}. \tag{9}$$

We have the following Proposition.

Proposition 5 *Let $1 < p < \infty$. Then the dual space of $HB_{p,1}(\mathcal{T}_n)$ can be identified with $HB_{p',1}(\mathcal{T}_n)$ under the pairing $\langle\langle\cdot,\cdot\rangle\rangle_1$ given in*

$$\begin{aligned} \langle\langle f,g\rangle\rangle_1 =& f(o)\overline{g(o)} \\ &+ \sum_{\alpha\in\mathcal{T}_n} \left\{ r^{-2d(\alpha)} f'(\alpha) P_\alpha \cdot \overline{g'(\alpha) P_\alpha} + r^{-d(\alpha)} f'(\alpha) Q_\alpha \cdot \overline{g'(\alpha) Q_\alpha} \right\} \\ &+ \sum_{\alpha\in\mathcal{T}_n} \mathcal{Q}_\alpha \mathcal{D}_\alpha f \cdot \overline{\mathcal{Q}_\alpha \mathcal{D}_\alpha g}. \end{aligned} \tag{10}$$

and the reproducing kernel $k_\alpha^{(1)}(\gamma)$ for this pairing is the unique function $k_\alpha^{(1)}$ satisfying

$$k_\alpha^{(1)\prime}(\gamma) = \begin{cases} r^{2d(\gamma)}\overline{P_\gamma(\gamma_\alpha - \gamma)} + r^{d(\gamma)}\overline{Q_\gamma(\gamma_\alpha - \gamma)} & if \gamma = \alpha_{k-1} \\ 0 & if \gamma \notin [o,\alpha] \end{cases}, \tag{11}$$

$$\mathcal{Q}_\gamma \mathcal{D} k_\alpha^{(1)}(\gamma) = \begin{cases} \mathcal{Q}_\gamma \mathbf{e}_{\gamma_\alpha}, & if \gamma \in [o,\alpha) \\ 0 & if \gamma \notin [o,\alpha) \end{cases}, \tag{12}$$

and

$$k_\alpha^{(1)}(o) = 1 \tag{13}$$

and given explicitly by

$$\begin{aligned} & k_\alpha^{(1)}(\beta) \\ =& k_\alpha^{(1)}(o) + \sum_{\gamma<\beta} k_\alpha^{(1)\prime}(\gamma)\cdot\left(\gamma_\beta - \gamma\right) + \sum_{\gamma<\beta} \mathcal{Q}_\gamma \mathcal{D}_\gamma k_\alpha^{(1)}\left(\gamma_\beta\right) \\ =& 1 + \sum_{\gamma<\beta} \left\{ r^{2d(\gamma)}\overline{P_\gamma(\gamma_\alpha - \gamma)} + r^{d(\gamma)}\overline{Q_\gamma(\gamma_\alpha - \gamma)} \right\}\cdot\left(\gamma_\beta - \gamma\right) + \sum_{\gamma<\beta} \mathcal{Q}_\gamma \mathbf{e}_{\gamma_\alpha}\left(\gamma_\beta\right) \end{aligned} \tag{14}$$

where $\mathcal{Q}_\gamma \mathbf{e}_{\gamma_\alpha}\left(\gamma_\beta\right)$ denotes the γ_β-coordinate of the N-vector $\mathcal{Q}_\gamma \mathbf{e}_{\gamma_\beta}$. If $\beta \neq \alpha$ is at distance exactly one from the geodesic $[o,\alpha)$, then $A\beta \in [o,\alpha)$ and the formula for $k_\alpha^{(1)}(\beta)$ is identical to that above except that the final term in the sum $\sum_{\gamma<\beta} \mathcal{Q}_\gamma \mathbf{e}_{\gamma_\alpha}\left(\gamma_\beta\right)$ is now $\mathcal{Q}_{A\beta}\mathbf{e}_{A\beta_\alpha}(\beta)$ instead of $\mathcal{Q}_{A\beta}\mathbf{e}_{A\beta_\alpha}(A\beta_\alpha)$. The function $k_\alpha^{(1)}$ is then determined for all remaining β by the requirement that $k_\alpha^{(1)}$ be constant on all successor sets $S(\gamma)$ with vertex γ at distance exactly one from the geodesic $[o,\alpha)$.

The corresponding formula for $\triangle\ k_\alpha^1$, the difference operator applied to the reproducing kernel k_α^1 for the abstract Besov space $B_{p,1}(\mathcal{T}_n)$, consists entirely of nonnegative entries, a feature that plays prominently in deriving the Carleson embedding property from multiplier interpolation using Böe's "curious lemma". The terms $k_\alpha^{(1)}(o)$, $k_\alpha^{(1)\prime}(\gamma)$ and $\mathcal{Q}_\gamma \mathcal{D}_\gamma k_\alpha^{(1)}$ arising in the above formula do not consist entirely of nonnegative entries, but the following two properties will serve as a suitable substitute:

$$\begin{aligned} &\left| r^{-d(\gamma)} k_\alpha^{(1)\prime}(\gamma) P_\gamma + r^{-\frac{d(\gamma)}{2}} k_\alpha^{(1)\prime}(\gamma) Q_\gamma \right| \\ &\quad \approx r^{-d(\gamma)} \operatorname{Re}\left\{ \overline{\gamma} \cdot r^{2d(\gamma)} P_\gamma (\gamma_\alpha - \gamma) + \overline{\gamma} \cdot r^{d(\gamma)} Q_\gamma (\gamma_\alpha - \gamma) \right\} \\ &\quad \approx r^{-d(\gamma)} \operatorname{Re}\left(\gamma \cdot k_\alpha^{(1)\prime}(\gamma) \right) \approx 1; \\ &\left| \mathcal{Q}_\gamma \mathcal{D}_\gamma k_\alpha^{(1)} \right| \leq 1. \end{aligned} \tag{15}$$

Analogues of these properties will be used later to complete the proof of equivalence of multiplier interpolation with the separation and tree conditions when $1 < p < 2 + \frac{1}{n-1}$.

An example We close this subsubsection by computing $k_\alpha^{(1)}$ for the simple case of the Bergman tree $\mathcal{T}_1$ in terms of the geometric embedding of $\mathcal{T}_1$ in the unit disk, and then verifying (15) in this case. The branching number for the tree $\mathcal{T}_1$ is 2. Fix $\alpha \in \mathcal{T}_1$ with geodesic $[o, \alpha] = \{o, \alpha_1, \alpha_2, ..., \alpha_m = \alpha\}$ as above, and let $\gamma = \alpha_{k-1}$ with children $\gamma_1 = \alpha_k$ and γ_2. Then

$$\begin{aligned} \triangle\ \gamma_1 &= \gamma_1 - \gamma, \\ \triangle\ \gamma_2 &= \gamma_2 - \gamma, \\ \mathcal{E}(\gamma) &= \{\triangle\ \gamma_1, \triangle\ \gamma_2\}, \end{aligned}$$

and for $v \in \mathbb{C}$,

$$L_\gamma v = \{v \triangle \gamma_1, v \triangle \gamma_2\}.$$

Thus

$$\begin{aligned} M_\gamma &= \{v \triangle \gamma_1, v \triangle \gamma_2 : v \in \mathbb{C}\} = \mathbb{C}\{\triangle\ \gamma_1, \triangle\ \gamma_2\}, \\ M_\gamma^\perp &= \mathbb{C}\{-\overline{\triangle\ \gamma_2}, \overline{\triangle\ \gamma_1}\}, \end{aligned}$$

since if $V = v\{\triangle\ \gamma_1, \triangle\ \gamma_2\} \in M_\gamma$ and $W = w\{-\overline{\triangle\ \gamma_2}, \overline{\triangle\ \gamma_1}\} \in M_\gamma^\perp$, then

$$\langle V, W \rangle = v\overline{w}\{(\triangle\ \gamma_1)(-\triangle\ \gamma_2) + (\triangle\ \gamma_2)(\triangle\ \gamma_1)\} = 0.$$

We now have

$$\begin{aligned} \mathcal{Q}_\gamma \mathcal{D}_\gamma k_\alpha^{(1)} &= \frac{\triangle\ \gamma_2}{|\triangle\ \gamma_1|^2 + |\triangle\ \gamma_2|^2} \{-\overline{\triangle\ \gamma_1}, \overline{\triangle\ \gamma_2}\} \\ &= \left\{ \frac{|\triangle\ \gamma_2|^2}{|\triangle\ \gamma_1|^2 + |\triangle\ \gamma_2|^2}, \frac{-\overline{\triangle\ \gamma_1}\ \triangle\ \gamma_2}{|\triangle\ \gamma_1|^2 + |\triangle\ \gamma_2|^2} \right\}. \end{aligned}$$

Since the projection P_γ is the identity in dimension $n=1$, we also have

$$k_\alpha^{(1)\prime}(\gamma) = r^{2d(\gamma)}\overline{(\gamma_\alpha - \gamma)},$$

and since it is geometrically evident that $\operatorname{Re}\gamma \cdot \overline{\gamma_\alpha - \gamma} \approx |\gamma|\,|\gamma_\alpha - \gamma|$, we now have

$$\begin{aligned} \operatorname{Re}\left(\gamma \cdot k_\alpha^{(1)\prime}(\gamma)\right) &= r^{2d(\gamma)} \operatorname{Re}\gamma \cdot \overline{\gamma_\alpha - \gamma} \approx r^{d(\gamma)} \approx \left|k_\alpha^{(1)\prime}(\gamma)\right|, \\ |\mathcal{Q}_\gamma \mathcal{D}_\gamma k_\alpha|^2 &\leq \frac{|\triangle\, \gamma_2|^4 + |\triangle\, \gamma_1|^2 |\triangle\, \gamma_2|^2}{\left(|\triangle\, \gamma_1|^2 |\triangle\, \gamma_2|^2\right)^2} \leq 1, \end{aligned}$$

which is (15) for the Bergman tree $\mathcal{T}_1$.

Using (14), we give an explicit formula for $k_\alpha(\beta)$, $\alpha, \beta \in \mathcal{T}_1$. When $\beta \in [o, \alpha]$ we have

$$k_\alpha^{(1)}(\beta) = 1 + \sum_{\gamma<\beta} r^{2d(\gamma)}\overline{(\gamma_\alpha - \gamma)}\left(\gamma_\beta - \gamma\right) + \sum_{\gamma<\beta} \frac{\left|\gamma_\beta^\perp - \gamma\right|^2}{\left|\gamma_\beta - \gamma\right|^2 + \left|\gamma_\beta^\perp - \gamma\right|^2},$$

where $\gamma_\beta^\perp$ is the child of γ not lying in $[o, \beta]$. The formula for $k_\alpha^{(1)}\left(\beta_\alpha^\perp\right)$, where $\beta \in [o, \alpha)$ and $\beta_\alpha^\perp$ is the child of β not lying in $[o, \alpha]$, is given by

$$\begin{aligned} k_\alpha^{(1)}\left(\beta_\alpha^\perp\right) &= 1 + \sum_{o<\gamma\leq\beta} r^{2d(\gamma)}\overline{(\gamma_\alpha - \gamma)}\left(\gamma_{\beta_\alpha^\perp} - \gamma\right) \\ &\quad + \sum_{o<\gamma<\beta} \frac{\left|\gamma_\beta^\perp - \gamma\right|^2}{\left|\gamma_\beta - \gamma\right|^2 + \left|\gamma_\beta^\perp - \gamma\right|^2} - \frac{\overline{(\beta_\alpha - \beta)}\left(\beta_\alpha^\perp - \beta\right)}{|\beta_\alpha - \beta|^2 + \left|\beta_\alpha^\perp - \beta\right|^2}. \end{aligned}$$

The values of $k_\alpha^{(1)}$ remain constant on the successor sets $S(\alpha)$ and $S\left(\beta_\alpha^\perp\right)$ for $\beta \in [o, \alpha)$, and this completes the evaluation of $k_\alpha^{(1)}(\beta)$ for all $\beta \in \mathcal{T}_1$.

Remark 3 *An important property of the reproducing kernel $k_\alpha^{(1)}$ is that its first order differences are supported within distance 1 of the geodesic $[o, \alpha]$.*

2.1.2 Tensor-valued functions

In order to extend our definitions to tensor-valued functions on the Bergman tree, it is convenient to consider first the simplest case of order zero.

Definition 6 *Let $1 < p < \infty$. For a $\mathbb{C}$-valued function $f(\alpha)$ defined for $\alpha \in \mathcal{T}_n$, define*

$$\|f\|_{HB_{p,0}(\mathcal{T}_n)} = \|f\|_{\ell^p(\mathcal{T}_n)} = \left(\sum_{\alpha\in\mathcal{T}_n} |f(\alpha)|^p\right)^{\frac{1}{p}}.$$

Define an operator $\mathbf{R}^{-d}$ *on* $\mathbb{C}^n$*-valued functions* $\mathbf{v}$ *on the tree* $\mathcal{T}_n$ *by*

$$\left(\mathbf{R}^{-d}\mathbf{v}\right)(\alpha) = r^{-d(\alpha)}\mathbf{v}(\alpha) P_\alpha + r^{-\frac{d(\alpha)}{2}}\mathbf{v}(\alpha) Q_\alpha.$$

For vectors $\mathbf{v} \in \mathbb{C}^n$ *, let*

$$|\mathbf{v}|_\alpha = \left|\left(\mathbf{R}^{-d}\mathbf{v}\right)(\alpha)\right| = \left|e^{-2d(\alpha)\theta}\mathbf{v}P_\alpha + e^{-d(\alpha)\theta}\mathbf{v}Q_\alpha\right| \approx \sqrt{(B(\alpha)\mathbf{v})\cdot\overline{\mathbf{v}}},$$

and for a $\mathbb{C}^n$*-valued function* $\mathbf{v}(\alpha)$ *defined for* $\alpha \in \mathcal{T}_n$*, define*

$$\|\mathbf{v}\|_{HB^{(1)}_{p,0}(\mathcal{T}_n)} = \left\||\mathbf{v}|_\alpha\right\|_{\ell^p(\mathcal{T}_n)} = \left(\sum_{\alpha\in\mathcal{T}_n} |\mathbf{v}|^p_\alpha\right)^{\frac{1}{p}}.$$

Remark 4 *Recall from (4) that we have* $HB_{p,1}(\mathcal{T}_n) = B_{p,1}(\mathcal{T}_n)$*. Let* $\mathcal{QD}f(\alpha) = \mathcal{Q}_\alpha\mathcal{D}_\alpha f$*. Then using Definitions 6 and 2, we have the following observation that will provide the basis for extending the definition of* $HB_{p,0}(\mathcal{T}_n)$ *to* $HB_{p,m}(\mathcal{T}_n)$ *for larger* m*:*

$$\begin{aligned}\|f\|^p_{HB_{p,1}(\mathcal{T}_n)} &= |f(o)|^p + |f'(o)|^p + \sum_{\alpha\in\mathcal{T}_n} |f'(\alpha)|^p_\alpha + \sum_{\alpha\in\mathcal{T}_n} |\mathcal{Q}_\alpha\mathcal{D}_\alpha f|^p \\ &= |f(o)|^p + \|f'\|^p_{HB^{(1)}_{p,0}(\mathcal{T}_n)} + \|\mathcal{QD}f\|^p_{\ell^p(\mathcal{T}_n)}.\end{aligned}$$

In Definition 6 above, we defined the order zero holomorphic Besov space $HB^{(1)}_{p,0}(\mathcal{T}_n)$ on $\mathcal{T}_n$ for $\mathbb{C}^n$-valued functions $\mathbf{v}(\alpha)$ using the nonisotropic norm

$$|\mathbf{v}|_\alpha = \left|r^{-d(\alpha)}\mathbf{v}P_\alpha + r^{-\frac{d(\alpha)}{2}}\mathbf{v}Q_\alpha\right|,$$

and where $\mathbf{v}(\alpha)$ was interpreted as a covariant tensor of order 1 acting on the "tangent space" $\mathbb{C}^n$ at α. We now wish to extend this definition to order zero holomorphic Besov spaces $HB^{(t)}_{p,0}(\mathcal{T}_n)$ of symmetric covariant tensors of order t, or t-tensors, on $\mathcal{T}_n$.

We consider the Hilbert space $\mathcal{E}^{(s,t)}_\alpha$ of symmetric (s,t)-tensors that are covariant of order s and contravariant of order t (see for example chapter 4 of Spivak [36]). Then $\mathcal{E}^{(t)}_\alpha = \mathcal{E}^{(t,0)}_\alpha$ and we will stop referring to tensors as covariant or contravariant. We define the tensor product of a (s_1,t_1)-tensor $\mathbf{B}$ and a (s_2,t_2)-tensor $\mathbf{A}$ to be the (s_1+s_2, t_1+t_2)-tensor $\mathbf{B}\otimes\mathbf{A}$ in the usual way,

$$\begin{aligned}&\mathbf{B}\otimes\mathbf{A}\left[v^1,...,v^{s_1},w^1,...,w^{t_1},x^1,...,x^{s_2},y^1,...,y^{t_2}\right] \\ &= \mathbf{B}\left[v^1,...,v^{s_1},w^1,...,w^{t_1}\right]\times\mathbf{A}\left[x^1,...,x^{s_2},y^1,...,y^{t_2}\right],\end{aligned}$$

as well as the Euclidean contraction $\mathbf{B}\wedge\mathbf{A}$ of an (s,t)-tensor $\mathbf{B}$ and a t-tensor $\mathbf{A}$ (see immediately below for this definition). We will see later that reproducing kernels for Besov spaces of t-tensor-valued functions can be interpreted as (t,t)-tensor-valued functions on $\mathcal{T}_n$.

We define the *α-contraction* $\mathbf{B} \wedge_\alpha \mathbf{A}$ of an (s,t)-tensor

$$\mathbf{B} = \sum_{1 \leq i_1 \leq ... \leq i_t \leq n} b^{i_1,...,i_s}_{j_1,...,j_t} e^{i_1} \otimes ... \otimes e^{i_s} \otimes e_{j_1} \otimes ... \otimes e_{j_t}$$

and a t-tensor

$$\mathbf{A} = \sum_{1 \leq i_1 \leq ... \leq i_t \leq n} a^{i_1,i_2,...,i_t} e^{i_1} \otimes ... \otimes e^{i_t}$$

to be the s-tensor given by

$$\begin{aligned} \mathbf{B} \wedge_\alpha \mathbf{A} &= \sum_{1 \leq i_1 \leq ... \leq i_t \leq n} b^{i_1,...,i_s}_{j_1,...,j_t} a^{j_1,...,j_t} \langle e_{j_1}, e_{j_1} \rangle_\alpha \times ... \times \langle e_{j_t}, e_{j_t} \rangle_\alpha e^{i_1} \otimes ... \otimes e^{i_s} \\ &= \sum_i b^i_j a^j \eta_j (\alpha) e^{i_1} \otimes ... \otimes e^{i_s}, \end{aligned}$$

where by the summation convention, we also sum over the repeated upper and lower indices $j_1, ..., j_t$. The Euclidean contraction $\mathbf{B} \wedge \mathbf{A}$ is given by $\sum_i b^i_j a^j e^{i_1} \otimes ... \otimes e^{i_s}$ without the η_j. Thus $\mathbf{B} \wedge \mathbf{A} [v_1, ..., v_s]$ is the contraction (trace if $t = 1$) of the linear map λ given by

$$\lambda (w_1, ..., w_t, u_1, ..., u_t) = \mathbf{B} \otimes \mathbf{A} (v_1, ..., v_s, w_1, ..., w_t, u_1, ..., u_t)$$

(see page 4-27 in [36]). Note that if we interpret v^j as a $(0,1)$-tensor (contravariant of order 1), then we have

$$\mathbf{A} \left[v^1, ..., v^t \right] = \mathbf{A} \wedge \left(v^1 \otimes ... \otimes v^t \right). \tag{16}$$

Definition 7 *We define the "inner product" $\langle \mathbf{A}, \mathbf{B} \rangle^{(t)}_\alpha$ of a t-tensor $\mathbf{A}$ and a (t,t)-tensor $\mathbf{B}$ to be the t-tensor given by*

$$\langle \mathbf{A}, \mathbf{B} \rangle^{(t)}_\alpha = \overline{\mathbf{B}} \wedge_\alpha \mathbf{A}, \tag{17}$$

so that

$$\langle \mathbf{A}, \mathbf{B} \rangle^{(t)}_\alpha = \sum_i \overline{b^i_j} a^j \eta_j (\alpha) e^{i_1} \otimes ... \otimes e^{i_t}. \tag{18}$$

We also define an "inner product" $\langle \langle \mathbf{A}, \mathbf{B} \rangle \rangle^{(t)}_0$, for a t-tensor-valued function $\mathbf{A}$ and a (t,t)-tensor-valued function $\mathbf{B}$ on the tree $\mathcal{T}_n$, by

$$\langle \langle \mathbf{A}, \mathbf{B} \rangle \rangle^{(t)}_0 = \sum_{\gamma \in \mathcal{T}_n} \langle \mathbf{A} (\gamma), \mathbf{B} (\gamma) \rangle^{(t)}_\gamma. \tag{19}$$

Order zero spaces of t-tensors We now define the order zero holomorphic Besov spaces $HB_{p,0}^{(t)}(\mathcal{T}_n)$ of t-tensors on $\mathcal{T}_n$.

Definition 8 *For $1 < p < \infty$ and $t \in \mathbb{N}$, let $HB_{p,0}^{(t)}(\mathcal{T}_n)$ consist of all t-tensor-valued functions $\mathbf{A}(\alpha)$ defined for $\alpha \in \mathcal{T}_n$ such that the norm*

$$\|\mathbf{A}\|_{HB_{p,0}^{(t)}(\mathcal{T}_n)} = \||\mathbf{A}|_\alpha\|_{\ell^p(\mathcal{T}_n)} = \left(\sum_{\alpha \in \mathcal{T}_n} |\mathbf{A}(\alpha)|_\alpha^p\right)^{\frac{1}{p}}$$

is finite.

The inner product for the Hilbert space $HB_{2,0}^{(t)}(\mathcal{T}_n)$ is given by (**??**), and the dual space of $HB_{p,0}^{(t)}(\mathcal{T}_n)$ can be identified with $HB_{p',0}^{(t)}(\mathcal{T}_n)$ under the pairing $\langle\langle \cdot, \cdot \rangle\rangle_0^{(t)}$.

Lemma 9 *For $1 < p < \infty$ and $t \in \mathbb{N}$, we have*

$$\left|\langle\langle \mathbf{A}, \mathbf{B} \rangle\rangle_0^{(t)}\right| \leq \|\mathbf{A}\|_{HB_{p,0}^{(t)}(\mathcal{T}_n)} \|\mathbf{B}\|_{HB_{p',0}^{(t)}(\mathcal{T}_n)},$$
$$\|\mathbf{B}\|_{HB_{p',0}^{(t)}(\mathcal{T}_n)} = \sup_{\|\mathbf{A}\|_{HB_{p,0}^{(t)}(\mathcal{T}_n)} \leq 1} \left|\langle\langle \mathbf{A}, \mathbf{B} \rangle\rangle_0^{(t)}\right|.$$

Moreover, in the case $p = 2$, the function $\Lambda_\alpha^{v^1,...,v^t}$ defined by

$$\Lambda_\alpha^{v^1,...,v^t} \mathbf{A} = \mathbf{A}(\alpha)\left[v^1, ..., v^t\right]$$

is a continuous linear functional on $HB_{2,0}^{(t)}(\mathcal{T}_n)$ for $\alpha \in \mathcal{T}_n$ and every choice of $v^1, ..., v^t$. Thus there is a unique $\mathbf{k}_\alpha^{v^1,...,v^t}$ in the Hilbert space $HB_{2,0}^{(t)}(\mathcal{T}_n)$ such that

$$\left\langle\left\langle \mathbf{A}, \mathbf{k}_\alpha^{v^1,...,v^t} \right\rangle\right\rangle_0^{(t)} = \Lambda_{v^1,...,v^t} \mathbf{A} = \mathbf{A}(\alpha)\left[v^1, ..., v^t\right]. \tag{20}$$

By this uniqueness, the function that sends $v^1, ..., v^t$ to the t-tensor $\mathbf{k}_\alpha^{v^1,...,v^t}(\gamma)$ is multi-conjugate linear in $v^1, ..., v^t$, and so there is a unique (t,t)-tensor $\mathbf{k}_\alpha^{(0,t)}$ such that

$$\mathbf{k}_\alpha^{v^1,...,v^t}(\gamma)\left[w^1, ..., w^t\right] = \mathbf{k}_\alpha^{(0,t)}(\gamma)\left[w^1, ..., w^t, \overline{v^1}, ..., \overline{v^t}\right],$$

which by (16) is

$$\mathbf{k}_\alpha^{v^1,...,v^t}(\gamma) = \mathbf{k}_\alpha^{(0,t)}(\gamma) \wedge \overline{v^1 \otimes ... \otimes v^t}.$$

We refer to this (t,t)-tensor-valued function $\mathbf{k}_\alpha^{(0,t)}$ as the reproducing kernel for the holomorphic Besov space of t-tensors $HB_{p,0}^{(t)}(\mathcal{T}_n)$ relative to the pairing $\langle\langle \cdot, \cdot \rangle\rangle_0^{(t)}$.

The reproducing kernel $\mathbf{k}_\alpha^{(0,t)}$ is in fact given by the (t,t)-tensor

$$\sum_i \eta_i(\alpha)^{-1} e^{i_1} \otimes ... \otimes e^{i_t} \otimes e_{i_1} \otimes ... \otimes e_{i_t}$$

times the delta function at α, i.e.

$$\begin{aligned} \mathbf{k}_\alpha^{(0,t)}(\gamma) &= \chi_{\{\alpha\}}(\gamma) \sum_i \eta_i(\alpha)^{-1} \left\{e^{i_1} \otimes ... \otimes e^{i_t}\right\} \otimes \left\{e_{i_1} \otimes ... \otimes e_{i_t}\right\} \qquad (21) \\ &= \chi_{\{\alpha\}}(\gamma) \sum_i \left\{\mathbf{R}^{d(\alpha)} e^{i_1} \otimes ... \otimes \mathbf{R}^{d(\alpha)} e^{i_t}\right\} \otimes \left\{\mathbf{R}^{d(\alpha)} e_{i_1} \otimes ... \otimes \mathbf{R}^{d(\alpha)} e_{i_t}\right\}. \end{aligned}$$

Order one spaces of t-tensors We next turn to defining the order one holomorphic Besov spaces $HB_{p,1}^{(t)}(\mathcal{T}_n)$ of t-tensors on $\mathcal{T}_n$. In Remark 4, we have already defined the scalar case $HB_{p,1}(\mathcal{T}_n)$ using the norm (to the p^{th} power)

$$\|f\|_{HB_{p,1}(\mathcal{T}_n)}^p = |f(o)|^p + \|f'\|_{HB_{p,0}^{(1)}(\mathcal{T}_n)}^p + \|\mathcal{QD}f\|_{\ell^p(\mathcal{T}_n)}^p.$$

In order to replace f with a tensor, we first need to define the complex derivative $\mathbf{A}'(\alpha)$ of a t-tensor-valued function $\mathbf{A}(\alpha)$ on the tree $\mathcal{T}_n$. The derivative $\mathbf{A}'$ will be a $(t+1)$-tensor-valued function on the tree defined in the same spirit as f'.

Definition 10 *For $1 < p < \infty$ and $0 \le t \le M-1$, let $HB_{p,1}^{(t)}(\mathcal{T}_n)$ consist of all t-tensor-valued functions $\mathbf{A}$ on the tree $\mathcal{T}_n$ such that the norm (to the p^{th} power)*

$$\|\mathbf{A}\|_{HB_{p,1}^{(t)}(\mathcal{T}_n)}^p = |\mathbf{A}(o)|^p + \|\mathbf{A}'\|_{HB_{p,0}^{(t+1)}(\mathcal{T}_n)}^p + \left\|\mathcal{Q}^{(t)}\mathcal{D}\mathbf{A}\right\|_{\ell^p(\mathcal{T}_n)}^p$$

is finite.

As in Proposition 5, we can obtain the duality of $HB_{p,1}^{(t)}(\mathcal{T}_n)$ and $HB_{p',1}^{(t)}(\mathcal{T}_n)$ relative to the inner product $\langle\langle\cdot,\cdot\rangle\rangle_1^{(t)}$ for the Hilbert space $HB_{2,1}^{(t)}(\mathcal{T}_n)$:

$$\begin{aligned} \langle\langle\mathbf{A},\mathbf{B}\rangle\rangle_1^{(t)} &= \mathbf{A}(o)\overline{\mathbf{B}(o)} + \sum_{\alpha\in\mathcal{T}_n} \langle\mathbf{A}',\mathbf{B}'\rangle_\alpha^{(t+1)} + \sum_{\alpha\in\mathcal{T}_n} \mathcal{Q}_\alpha\mathcal{D}_\alpha\mathbf{A}\cdot\overline{\mathcal{Q}_\alpha\mathcal{D}_\alpha\mathbf{B}} \qquad (22) \\ &= \mathbf{A}(o)\overline{\mathbf{B}(o)} + \langle\langle\mathbf{A}',\mathbf{B}'\rangle\rangle_0^{(t+1)} + \sum_{\alpha\in\mathcal{T}_n} \mathcal{Q}_\alpha\mathcal{D}_\alpha\mathbf{A}\cdot\overline{\mathcal{Q}_\alpha\mathcal{D}_\alpha\mathbf{B}}. \end{aligned}$$

Lemma 11 *For $1 < p < \infty$ and $t \in \mathbb{N}$, we have*

$$\begin{aligned} \left|\langle\langle\mathbf{A},\mathbf{B}\rangle\rangle_1^{(t)}\right| &\le \|\mathbf{A}\|_{HB_{p,1}^{(t)}(\mathcal{T}_n)} \|\mathbf{B}\|_{HB_{p',1}^{(t)}(\mathcal{T}_n)}, \\ \|\mathbf{B}\|_{HB_{p',1}^{(t)}(\mathcal{T}_n)} &= \sup_{\|\mathbf{A}\|_{HB_{p,1}^{(t)}(\mathcal{T}_n)}\le 1} \left|\langle\langle\mathbf{A},\mathbf{B}\rangle\rangle_1^{(t)}\right|. \end{aligned}$$

Combining the arguments for the order one space $HB_{p,1}(\mathcal{T}_n)$ in Subsubsection 8.1.1 with the arguments above for the order zero space $HB_{p,0}^{(t)}(\mathcal{T}_n)$ of t-tensors, we can show there is a unique reproducing kernel $\mathbf{k}_\alpha^{(1,t)}$ for the holomorphic Besov space of t-tensors $HB_{p,1}^{(t)}(\mathcal{T}_n)$ relative to this pairing. Again, $\mathbf{k}_\alpha^{(1,t)}$ is a (t,t)-tensor-valued function on the tree satisfying

$$\mathbf{A}(\alpha) = \left\langle\left\langle \mathbf{A}, \mathbf{k}_\alpha^{(1,t)} \right\rangle\right\rangle_1^{(t)}, \qquad \alpha \in \mathcal{T}_n,$$

for $\mathbf{A} \in HB_{p,1}^{(t)}(\mathcal{T}_n)$, where just as in Definition 7, the notation $\langle\langle \mathbf{A}, \mathbf{B} \rangle\rangle_1^{(t)}$ represents a scalar if $\mathbf{B}$ is a t-tensor, and a t-tensor if $\mathbf{B}$ is a (t,t)-tensor.

2.2 The order m holomorphic Besov space

We cannot simply define the m^{th} order holomorphic Besov space $HB_{p,m}^{(t)}(\mathcal{T}_n)$ of t-tensor-valued functions inductively to consist of all f such that $f' \in HB_{p,m-1}^{(t+1)}(\mathcal{T}_n)$. Besides the question of how to handle the error term $\left\{\mathcal{Q}_\alpha^{(t)}(\mathcal{D}_\alpha \mathbf{A})\right\}_{\alpha \in \mathcal{T}_n}$, the restriction theorem from $B_p(\mathbb{B}_n)$ to $HB_{p,2}(\mathcal{T}_n)$ will fail because $\|f'\|_{HB_{p,1}^{(1)}(\mathcal{T}_n)}$ will not in general be controlled by $\|F\|_{B_p(\mathbb{B}_n)}$ when f is the restriction of $F \in B_p(\mathbb{B}_n)$ to the Bergman tree $\mathcal{T}_n$. The problem arises when we minimize the distance from $\mathcal{D}_\alpha f$ to M_α by letting $f'(\alpha) = L_\alpha^{-1} P_\alpha \mathcal{D}_\alpha f$. The resulting vector $f'(\alpha)$ is within order 2, but not within order 3, of the restriction $F'(\alpha)$ of F' to the tree. In order to circumvent this difficulty, we will simultaneously define the first, second and through to the m^{th} order derivatives $f'(\alpha)$, $f''(\alpha)$, ..., $f^{(m)}(\alpha)$ at α using a single orthogonal projection onto the range of an appropriate generalization of the operator L_α. We will also need to define holomorphic Besov spaces of t-tensor-valued functions as well, in order to implement an inductive definition. Recall that we defined a variant $L_\alpha^{(t)}$ of $L_\alpha \equiv L_\alpha^{(0)}$ in order to define the complex derivative of a t-tensor-valued function on the Bergman tree. We will now use the notation $L_\alpha^{(1,t)}$ to denote these operators, and use the notation $L_\alpha^{(m,t)}$ to define a corresponding linear operator that will allow us to simultaneously define first through m^{th} order complex derivatives of a t-tensor-valued function on the Bergman tree.

Definition 12 *For $1 < p < \infty$ and $0 \leq m + t \leq M$, let $HB_{p,m}^{(t)}(\mathcal{T}_n)$ consist of all t-tensor-valued functions $\mathbf{A}$ on the tree $\mathcal{T}_n$ such that the norm (to the p^{th} power)*

$$\|\mathbf{A}\|_{HB_{p,m}^{(t)}(\mathcal{T}_n)}^p = |\mathbf{A}(o)|^p + \sum_{\ell=1}^m \left\|D_m^\ell \mathbf{A}\right\|_{HB_{p,m-\ell}^{(t+\ell)}(\mathcal{T}_n)}^p + \left\|\mathcal{Q}^{(m,t)} \mathcal{D} \mathbf{A}\right\|_{\ell^p(\mathcal{T}_n)}^p$$

is finite. We write simply $HB_{p,m}(\mathcal{T}_n)$ for the scalar case $t = 0$.

2.2.1 Higher order reproducing kernels for tensors and the positivity property

In this subsubsection we establish the key positivity property of reproducing kernels that will permit us to use the technique of Böe's "curious lemma". It is this property that yields the fruit of our labour in developing the theory of holomorphic Besov spaces on trees. Let $p=2$. Then the inner product corresponding to the Hilbert space norm $\|\mathbf{A}\|_{HB^{(t)}_{2,m}(\mathcal{T}_n)}$ defined on t-tensor-valued functions $\mathbf{A}$ on the tree $\mathcal{T}_n$ is defined by induction on m by

$$\langle\langle \mathbf{A},\mathbf{B}\rangle\rangle^{(t)}_m = \mathbf{A}(o)\cdot\overline{\mathbf{B}(o)} + \sum_{\ell=1}^{m} \left\langle\left\langle D^\ell_m \mathbf{A}, D^\ell_m \mathbf{B}\right\rangle\right\rangle^{(t+\ell)}_{m-\ell}$$
$$+\sum_{\alpha\in\mathcal{T}_n}\left\langle \mathcal{Q}^{(m,t)}_\alpha(\mathcal{D}_\alpha\mathbf{A}), \mathcal{Q}^{(m,t)}_\alpha(\mathcal{D}_\alpha\mathbf{B})\right\rangle_\alpha,$$

where the cases $m=0$ and $m=1$ are already defined. We have the following duality relation.

Proposition 13 *For $1<p<\infty$ and $t\in\mathbb{N}$, we have*

$$\left|\langle\langle \mathbf{A},\mathbf{B}\rangle\rangle^{(t)}_m\right| \le \|\mathbf{A}\|_{HB^{(t)}_{p,m}(\mathcal{T}_n)} \|\mathbf{B}\|_{HB^{(t)}_{p',m}(\mathcal{T}_n)},$$
$$\|\mathbf{B}\|_{HB^{(t)}_{p',m}(\mathcal{T}_n)} = \sup_{\|\mathbf{A}\|_{HB^{(t)}_{p,m}(\mathcal{T}_n)}\le 1}\left|\langle\langle \mathbf{A},\mathbf{B}\rangle\rangle^{(t)}_m\right|.$$

Denote by $\mathbf{k}^{(m,t)}_\alpha$ the reproducing kernel for $\alpha\in\mathcal{T}_n$ relative to this inner product, which exists by a modification of the argument in the case $m=0$ immediately following Lemma 9. We have the recursion formula

$$D^\ell_m\mathbf{k}^{(m,t)}_\alpha = \sum_{o<\beta\le\alpha}\frac{1}{\ell!}\mathbf{k}^{(m-\ell,t+\ell)}_{A\beta}\wedge\left(\otimes^\ell\overline{\beta-A\beta}\right),\qquad 1\le\ell\le m, \tag{23}$$

as well as

$$\mathbf{A}(o) = \mathbf{A}(o)\cdot\overline{\mathbf{k}^{(m,t)}_\alpha(o)},$$
$$\sum_{o<\beta\le\alpha}\mathcal{Q}^{(m,0)}_{A\beta}\mathcal{D}_{A\beta}\mathbf{A}(\beta) = \sum_{\alpha\in\mathcal{T}_n}\left\langle \mathcal{Q}^{(m,t)}_\alpha(\mathcal{D}_\alpha\mathbf{A}), \mathcal{Q}^{(m,t)}_\alpha\left(\mathcal{D}_\alpha\mathbf{k}^{(m,t)}_\alpha\right)\right\rangle_\alpha.$$

Note that the left side (23) is a tensor of order $2t+\ell$, while that of the right side has order $2(t+\ell)-\ell$, the same order. We now use the recursion formula in (23) to establish by induction the following positivity property for derivatives of the reproducing kernels $\mathbf{k}^{(m,t)}_\alpha$.

Lemma 14 *Let $0\le m+t\le M$. Then provided we choose λ small enough and θ large enough in the construction of the Bergman tree, we have for all $\alpha,\gamma\in\mathcal{T}_n$,*

$$r^{-md(\gamma)}\,\mathrm{Re}\left(D^m_m k^{(m,0)}_\alpha(\gamma)\wedge\{\otimes^m\gamma\}\right)\approx 1 \tag{24}$$
$$\left|D^\ell_m k^{(m,t)}_\alpha(\gamma)\right|_\gamma + \left|\mathcal{Q}^{(m,t)}_\gamma\left(\mathcal{D}_\gamma k^{(m,t)}_\alpha\right)\right|_\gamma \le \begin{cases} C & \text{for } \gamma\le\alpha \\ 0 & \text{otherwise}\end{cases},$$

where $D^m_m k^{(m,0)}_\alpha(\gamma)\wedge\{\otimes^m\gamma\} = D^m_m k^{(m,0)}_\alpha(\gamma)[\gamma,...,\gamma]$.

2.3 Carleson measures

Here we characterize Carleson measures on the holomorphic Besov space $HB_{p,m}(\mathcal{T}_n)$.

Theorem 15 *Let $1 < p < \infty$ and $1 \leq m \leq M$. Then there are λ and θ in the construction of the Bergman tree, sufficiently small and large respectively, such that μ is a $HB_{p,m}(\mathcal{T}_n)$-Carleson measure, i.e.*

$$\left(\sum_{\alpha \in \mathcal{T}_n} |f(\alpha)|^p \mu(\alpha) \right)^{\frac{1}{p}} \leq C \|f\|_{HB_{p,m}(\mathcal{T}_n)}, \tag{25}$$

if and only if the tree condition holds, i.e.

$$\sum_{\beta \geq \alpha} \left(\sum_{\gamma \geq \beta} \mu(\gamma) \right)^{p'} \leq C \sum_{\beta \geq \alpha} \mu(\beta) < \infty, \qquad \alpha \in \mathcal{T}_n. \tag{26}$$

Remark 5 *It is crucial that the Carleson measures are characterized by the tree condition which is* independent *of m.*

2.4 The holomorphic restriction map

In the special case where f arises as the restriction $f = TF = \{F(c_\alpha)\}_{\alpha \in \mathcal{T}_n}$ of a holomorphic function $F \in B_p(\mathbb{B}_n)$, $m > \frac{2n}{p}$, then $D_m^\ell f(\alpha) \approx F^{(\ell)}(c_\alpha)$ for $1 \leq \ell \leq m$, and using Taylor's formula we will see that $\mathcal{Q}_\alpha^{(m,0)} \mathcal{D}_\alpha f$ is controlled by $F^{(m+1)}$. Similarly, $D_m^\ell f(\alpha) = F^{(\ell)}(\alpha) + \left\{ D_m^\ell f(\alpha) - F^{(\ell)}(\alpha) \right\}$, and we will show that each term in this sum is also controlled by $F^{(m+1)}$. In this way we will obtain the following Besov space restriction theorem, as well as the corresponding multiplier space restriction theorem:

Theorem 16 *Let $m > \frac{2n}{p}$. Then provided θ is chosen large enough in the construction of the Bergman tree $\mathcal{T}_n$, the restriction map*

$$TF = \{F(\alpha)\}_{\alpha \in \mathcal{T}_n}, \qquad \textit{where } TF(\alpha) = F(c_\alpha),$$

is bounded from $B_p(\mathbb{B}_n)$ to $HB_{p,m}(\mathcal{T}_n)$, and if in addition $p < 2 + \frac{1}{n-1}$, then T is also bounded from $M_{B_p(\mathbb{B}_n)}$ to $M_{HB_{p,m}(\mathcal{T}_n)}$.

3 Completing the multiplier interpolation loop

We can now complete the proof of the loop of implications for $M_{B_p(\mathbb{B}_n)}$ interpolation on the ball for all $1 < p < 2 + \frac{1}{n-1}$. As we will see, the following three properties of $HB_p(\mathcal{T}_n)$ essentially suffice to prove that $M_{B_p(\mathbb{B}_n)}$ interpolation implies the tree condition:

1. The restriction map is bounded from $M_{B_p(\mathbb{B}_n)}$ to $M_{HB_p(\mathcal{T}_n)}$.

2. The positivity property (24) holds for the reproducing kernels $k_\alpha^{(m,0)}$ of $HB_{p,m}(\mathcal{T}_n) = HB_p(\mathcal{T}_n)$, $m > \frac{2n}{p}$.

3. Carleson measures for $HB_p(\mathcal{T}_n)$ are characterized by the tree condition.

Indeed, property 1 will show that $M_{B_p(\mathbb{B}_n)}$ interpolation on the ball implies $M_{HB_p(\mathcal{T}_n)}$ interpolation on the Bergman tree. Then property 2 will show that the atomic measure μ associated with the interpolation sequence is a Carleson measure for $HB_p(\mathcal{T}_n)$. Finally, property 3 will then show that μ satisfies the tree condition. This will complete the multiplier interpolation loop since we have already shown in Section 5, that if μ satisfies the tree condition, then $M_{B_p(\mathbb{B}_n)}$ interpolation holds on the ball.

Before giving the details, we point out that property 1 follows from Theorem 16 if $m > \frac{2n}{p}$ and the structural constant θ is large enough; property 2 follows from Lemma 14 if in addition the structural constant λ is small enough; and finally, property 3 follows from Theorem 15 if both λ is small enough and θ is large enough.

We now give the details. If $\{z_j\}_{j=1}^\infty \subset \mathbb{B}_n$ interpolates $M_{B_p(\mathbb{B}_n)}$, i.e.

$$\text{The map } f \to \{f(z_j)\}_{j=1}^\infty \text{ takes } M_{B_p(\mathbb{B}_n)} \text{ boundedly into and onto } \ell^\infty, \tag{27}$$

and if we construct the Bergman tree $\mathcal{T}_n$ so that $\{c_\alpha\}_{\alpha\in\mathcal{T}_n}$ contains $\{z_j\}_{j=1}^\infty$, say with $z_j = c_{\alpha_j}$, then it follows easily from Theorem 16 that $\{\alpha_j\}_{j=1}^\infty$ interpolates $M_{HB_p(\mathcal{T}_n)}$, i.e.

$$\text{The map } f \to \{f(\alpha_j)\}_{j=1}^\infty \text{ takes } M_{HB_p(\mathcal{T}_n)} \text{ boundedly into and onto } \ell^\infty. \tag{28}$$

Indeed, to see that (28) holds, suppose that $\left\{\xi_j\right\}_{j=1}^\infty \in \ell^\infty$. Using (27) we can find $\varphi \in M_{B_p(\mathbb{B}_n)}$ satisfying

$$\begin{aligned} \varphi(z_j) &= \xi_j, \qquad 1 \le j < \infty, \\ \|\varphi\|_{M_{B_p(\mathbb{B}_n)}} &\le C \left\| \left\{\xi_j\right\}_{j=1}^\infty \right\|_\infty . \end{aligned}$$

Now define f on the tree $\mathcal{T}_n$ by

$$f(\alpha) = \varphi(c_\alpha), \qquad \alpha \in \mathcal{T}_n.$$

Then we have

$$f(\alpha_j) = \varphi\left(c_{\alpha_j}\right) = \varphi(z_j) = \xi_j$$

and Theorem 16 shows that

$$\|f\|_{M_{HB_p(\mathcal{T}_n)}} \le C \|\varphi\|_{M_{B_p(\mathbb{B}_n)}},$$

thus establishing (28).

We can now use soft arguments, together with the positivity property (24) of the reproducing kernels $k_\alpha^{(m,0)}$, with $m > \frac{2n}{p}$ in Lemma 14, to show that the measure

$$\mu = \sum_{j=1}^{\infty} \left\| k_{\alpha_j}^{(m,0)} \right\|_{HB_{p'}(\mathcal{T}_n)}^{-p} \delta_{\alpha_j}$$

is a $HB_p(\mathcal{T}_n)$-Carleson measure. Theorem 15 then shows that μ satisfies the tree condition. Finally then, to obtain that

$$\nu = \sum_{j=1}^{\infty} \left(\log \frac{1}{1 - |\alpha_j|^2} \right)^{1-p} \delta_{\alpha_j}$$

satisfies the tree condition, we use

$$\|k_\alpha\|_{HB_{p'}(\mathcal{T}_n)}^{p'} \approx \sum_{\gamma \in [o,\alpha]} 1 = d(\alpha) \approx \left(\log \frac{1}{1 - |\alpha|^2} \right),$$

by (24).

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