NCTS Student Seminar on Lie Algebra
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This is a set of notes written to introduce Lie algebras and prepare to explore combinatorial problems. It is used in the NCTS Student Seminar in Summer 2005. It is very elementary, requiring only a little linear algebra and group theory as background knowledge. Many details and arguments are missing on purpose, in order to encourage the students to deliver in-class presentations. It discusses the subject only minimally, and interested students can consult the following for more.

J. Humphreys, *Introduction to Lie Algebras and Representation Theory.*

A. W. Knapp, *Lie Groups Beyond an Introduction.*

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1 Introduction

Throughout this notes, we assume all vector spaces to be finite dimensional, over **R** or **C**. A Lie algebra is a vector space L with an anti-symmetric bilinear map

$$[\ ,\ ] : L \times L \longrightarrow L$$

which satisfies the Jacobi identity

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0.$$

In the most natural manner, we define Lie algebra homomorphism and isomorphism, Lie subalgebra, direct sum of two Lie algebras.

Let F be **R** or **C**. Let $M_n(F)$ be the $n \times n$ F-matrices. Define

$$[A,B] = AB - BA$$

for all $A, B \in M_n(F)$. Then we get a lot of examples of Lie algebras.

Example 1 $M_n(F)$.

Example 2 $\mathbf{sl}(n,F)$ are the matrices in $M_n(F)$ with trace 0.

Example 3 $\mathbf{u}(n)$ are the matrices A in $M_n(\mathbf{C})$ with $A+A^* = 0$ (skew-Hermitian).

Example 4 $\mathbf{o}(n) = \mathbf{u}(n) \cap M_n(\mathbf{R})$ (skew-symmetric).

Example 5 $\mathbf{su}(n) = \mathbf{u}(n) \cap \mathbf{sl}(n,\mathbf{C})$.

A Lie algebra is called abelian if $[X,Y] = 0$ for all X, Y. The center of a Lie algebra L is defined by

$$\{X \in L\ ;\ [X,Y] = 0 \text{ for all } Y \in L\}.$$

Find the center Z of $M_n(F)$, and check that $M_n(F)/Z \cong \mathbf{sl}(n,F)$. Show that $\mathbf{o}(3) \cong \mathbf{su}(2)$, and check that their center is 0.

2 Killing Form

Given $X \in L$, define the adjoint map

$$\mathrm{ad}_X : L \longrightarrow L \ \ , \ \ \mathrm{ad}_X(Y) = [X, Y].$$

The Killing form of a Lie algebra L is the symmetric bilinear form $(\ ,\ ) : L \times L \longrightarrow F$ defined by

$$(X, Y) = \mathrm{trace}(\mathrm{ad}_X \cdot \mathrm{ad}_Y).$$

Explain that the trace of a linear map is independent of the choice of basis, so the Killing form is well-defined.

The Killing form can often be hard to compute, because it involves a careful choice of basis to avoid messy computation. If L is a subalgebra of $M_n(F)$, it is often convenient to use the trace form

$$(A, B) = \mathrm{trace}(AB) \ \ ; \ \ A, B \in L$$

to substitute for the Killing form. If L is simple (we shall define this soon), the Killing form and the trace form differs by a nonzero scalar.

A Lie algebra is called semisimple if its Killing form is non-degenerate. Check that $\mathbf{sl}(2, \mathbf{C})$ is semisimple. In fact $\mathbf{sl}(n, F), \mathbf{su}(n), \mathbf{o}(n)$ are all semisimple. Check that the trace form of $\mathbf{sl}(n, F)$ is non-degenerate.

Let $\mathbf{g}$ be a real Lie algebra. Define $L = \mathbf{g} \otimes \mathbf{C} = \mathbf{g} + i\mathbf{g}$. Try to give L a natural complex Lie algebra structure. In this case we say that L is the complexification of $\mathbf{g}$, and that $\mathbf{g}$ is a real form of L. Non-isomorphic real Lie algebras can have the same complexification. Verify this statement by showing that the complexifications of $\mathbf{sl}(n, \mathbf{R})$ and $\mathbf{su}(n)$ are both $\mathbf{sl}(n, \mathbf{C})$. Why are $\mathbf{sl}(n, \mathbf{R})$ and $\mathbf{su}(n)$ non-isomorphic?

A real linear mapping $\phi : V \longrightarrow V$ extends naturally to a complex linear mapping $\Phi : V \otimes \mathbf{C} \longrightarrow V \otimes \mathbf{C}$. Check that ϕ and Φ have the same trace. Conclude that $\mathbf{g}$ is semisimple if and only if $\mathbf{g} \otimes \mathbf{C}$ is semisimple.

3 Root Space Decomposition

The Cartan subalgebra is a subalgebra $\mathbf{h} \subset L$ such that

(1) $\mathbf{h}$ is maximal abelian ,

(2) $\mathrm{ad}_X : L \longrightarrow L$ is diagonalizable for all $X \in \mathbf{h}$.

Check that the diagonal matrices form a Cartan subalgebra of $M_n(\mathbf{C})$.

Write $\mathbf{h} \subset L$ to denote a Cartan subalgebra. Let $X, Y \in \mathbf{h}$. Then $\mathrm{ad}_X, \mathrm{ad}_Y$ are both diagonalizable. The Jacobi identity implies that ad_X commutes with ad_Y, and so they have the same eigenspaces. So we can write

$$L = \sum_{\alpha \in \Delta} L_\alpha, \tag{3.1}$$

where each L_α is a simultaneous eigenspace for all of $\{\mathrm{ad}_X \ ; \ X \in \mathbf{h}\}$, and Δ is some index set for the eigenspaces L_α.

Let us describe Δ systematically. Let E be an eigenspace of ad_X. If $Y \in E$, then $\mathrm{ad}_X Y = \alpha Y$ for some $\alpha \in \mathbf{C}$. But as we vary $X \in \mathbf{h}$, we get $\alpha : \mathbf{h} \longrightarrow \mathbf{C}$. From the bilinear property of the Lie bracket, α depends linearly on $\mathbf{h}$, namely $\alpha \in \mathbf{h}^*$. Therefore,

$$E = L_\alpha = \{Y \in L \ ; \ \mathrm{ad}_X Y = \alpha(X) Y \text{ for all } X \in \mathbf{h}\} \tag{3.2}$$

with $\alpha \in \mathbf{h}^*$. So we let $\Delta \subset \mathbf{h}^*$, and define (3.1) and (3.2).

Check that $\mathbf{h} = L_0$ in the above notation. We require that each $L_\alpha \neq 0$, so Δ is a finite set because L is finite dimensional. It is a convention to exclude 0 from Δ. We modify (3.1) and write

$$L = \mathbf{h} + \sum_{\Delta} L_\alpha, \tag{3.3}$$

where $0 \notin \Delta \subset \mathbf{h}^*$ is a finite set, and $L_\alpha \neq 0$ is defined in (3.2). This is called the root space decomposition. Here Δ is called a root system, its elements are called roots, and L_α is called a root space.

An important exercise to learn the root space decomposition is to work out the case of $M_n(\mathbf{C})$. Try it. Then modify it for $\mathbf{sl}(n, \mathbf{C})$. Helpful notation: Let E_{jk} be the $n \times n$ matrix with entry 1 at the (j, k)-entry and 0 elsewhere. Check that $E_{rs}E_{tu} = \delta_{st}E_{ru}$. Use $\mathbf{C}(E_{rs})$ to construct the root spaces.

4 Root System

Let $\alpha, \beta \in \Delta$. Suppose that $\alpha + \beta \in \Delta \cup \{0\}$. Use the Jacobi identity to show that $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$. Suppose that L is semisimple. Show that the restriction of the Killing form to $\mathbf{h}$ is non-degenerate. Also, show that if α is a root, then $-\alpha$ is also a root. So the elements of Δ come in pairs. In fact Δ satisfies more properties. We next formulate the root system abstractly.

Let E be an inner product space, and $\Delta \subset E$ a finite set which spans E. Our motivation of course comes from $E = \mathbf{h}^*$, with inner product coming from the dual of the Killing form. Each nonzero $\alpha \in E$ defines a reflection r_α by $\alpha \mapsto -\alpha$ and fixing the hyperplane orthogonal to α. We say that Δ is a root system if it satisfies the following.

(1) If $\alpha \in \Delta$, the only multiples of α in Δ are $\pm\alpha$.
(2) If $\alpha \in \Delta$, the reflection r_α leaves Δ invariant.
(3) If $\alpha, \beta \in \Delta$, then $(\alpha, \beta) \in \mathbf{Z}$.

It is a fact that if L is complex semisimple with root space decomposition (3.3), then $\Delta \subset \mathbf{h}^*$ satisfies these properties. So the study of L can often be reduced to the study of Δ.

If $\Delta_i \subset E_i$ are root systems for $i = 1, 2$, then $\Delta_1 \times \Delta_2 \subset E_1 \times E_2$ is also a root system. A root system is called reducible if it is a non-trivial product of two root systems. Otherwise, it is called irreducible. A semisimple Lie algebra is called simple if it cannot be expressed as a direct sum of two Lie algebras. So a semisimple Lie algebra is simple if and only if its root system is irreducible. The simple Lie algebras, or accordingly the irreducible root systems, are the building blocks in the theory of semisimple Lie algebras. This is similar to the role of prime numbers among the natural numbers. They will be our main focus.

The classification of complex simple Lie algebras is done by E. Cartan in the end of the 19th century. They are gathered into 7 classes, denoted by A, B, C, D, E, F, G. Using the subscript to denote the dimension of $\mathbf{h}$ (or the span of Δ), they are $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$. The classes A, B, C, D have infinite number of members, and are called the classical algebras. The classes E, F, G have finite number of members, and are called the exceptional algebras. The root systems of the cases with indices 2 can be easily drawn on the paper. Look for them in the books.

Let W be the group generated by all the reflections r_α with $\alpha \in \Delta$. It is called the Weyl group. Since Δ is finite, its automorphism group is also finite. Since W is a subgroup of the automorphism group of Δ, W is a finite group.

As an exercise, find the Weyl group of the root system of A_2. More generally, show that the roots of $\mathbf{sl}(n, \mathbf{C})$ are given by $e_j - e_k \in \mathbf{h}^*$, where $e_j(E_{kk}) = \delta_{jk}$. Then show that the Weyl group of $\mathbf{sl}(n, \mathbf{C})$ is the symmetric group S_n.

Given a root system $\Delta \subset E$, a subset $\Pi \subset \Delta$ is called a simple system if every $\alpha \in \Delta$ can be uniquely written as

$$\alpha = \sum_{\beta \in \Pi} c_\beta \beta \tag{4.1}$$

where the coefficients c_β are all non-positive or non-negative. You can easily check that there are 6 choices of simple system for the root system of A_2. The elements of Π are called simple roots. If all $c_\beta \geq 0$ in (4.1), we say that α is a positive root. Otherwise we say that α is a negative root. Use the notation $\Delta^\pm$ for the positive and negative roots, so that $\Pi \subset \Delta^+ \subset \Delta$ and $\Delta = \Delta^+ \cup \Delta^-$.

5 Dynkin Diagram

We shall draw a graph $D = D(L)$ or $D = D(\Delta)$ to represent a complex semisimple Lie algebra L or root system Δ. It is called the Dynkin diagram. Pick a choice of simple system Π. The vertices of D are the elements of Π. Let $\alpha, \beta \in \Pi$. Write

$$\langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)}$$

for convenience, where $(\ ,\ )$ is the inner product of E. Check that this notation is helpful in the earlier computation of $r_\alpha(\beta)$. The vertices of α and β in D are joint by

$$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \tag{5.1}$$

edges. Note in particular that there is no edge joining them if and only if the roots are orthogonal. So the semisimple Lie algebra is simple if and only if its Dynkin diagram is connected. It suffices to study the connected Dynkin diagrams. As a fact, the quantity in (5.1) is 0,1,2 or 3. Suppose that the roots are not orthogonal, so that (5.1) is $1, 2, 3$. Since $\langle \alpha, \beta \rangle$ and $\langle \beta, \alpha \rangle$ are integers, it follows that α and β have the same length if and only if (5.1) is 1. In the case that it is 2 or 3, draw an arrow pointing from the long root to the short root. The resulting diagram is the Dynkin diagram. Look for the Dynkin diagrams of the complex simple Lie algebras in the book.

The beauty of the Dynkin diagram is that D is independent of the choice of $\mathbf{h}$ or Π. So L determines D. Conversely, non-isomorphic algebras have distinct diagrams. So $L_1 \cong L_2$ if and only if $D(L_1) \cong D(L_2)$. Check that the Dynkin diagram of $\mathbf{sl}(n, \mathbf{C})$ is a straight line with $n - 1$ vertices.

We shall try to draw a picture to represent a real semisimple $\mathbf{g}$. Since $\mathbf{g} \otimes \mathbf{C}$ is complex semisimple, let $D = D(\mathbf{g} \otimes \mathbf{C})$ denote its Dynkin diagram. The idea is to add extra information to D to reveal the real form $\mathbf{g}$. The resulting picture, namely D with extra information, is called a Vogan diagram of $\mathbf{g}$.

6 Cartan Involution and Cartan Subalgebra

Let (,) be a non-degenerate symmetric bilinear form on a vector space V. If V is complex, then all such structures are similar. Note that $(v,v) > 0$ implies that $(iv,iv) < 0$. But if V is real, then (,) determines a signature (p,q) with $p+q = \dim V$. There exist subspaces V_1 and V_2 which contain positive and negative vectors such that $\dim V_1 = p$ and $\dim V_2 = q$. The typical example is $\mathbf{R}^{p+q}$ with $(x,y) = \sum_1^p x_j y_j - \sum_{p+1}^{p+q} x_k y_k$. Define an automorphism

$$\theta : V \longrightarrow V \ \ , \ \ \theta = 1 \text{ on } V_2 \ \ , \ \ \theta = -1 \text{ on } V_1.$$

Then

$$\theta^2 = 1 \ \ , \ \ -(\cdot, \theta \cdot) \text{ is an inner product on } V.$$

Let $\mathbf{g}$ be a real semisimple Lie algebra, so that the Killing form (,) is a non-degenerate symmetric bilinear form on $\mathbf{g}$. It is then possible to make the above θ a Lie algebra homomorphism. It is a convention to use the notations $\mathbf{p}$ and $\mathbf{k}$ instead of V_1 and V_2. Check that

$$[\mathbf{k},\mathbf{k}] \subset \mathbf{k} \ \ , \ \ [\mathbf{k},\mathbf{p}] \subset \mathbf{p} \ \ , \ \ [\mathbf{p},\mathbf{p}] \subset \mathbf{k}, \tag{6.1}$$

so $\mathbf{k}$ is a Lie subalgebra of $\mathbf{g}$. A subalgebra is said to be compact if the Killing form is negative definite on it. Here $\mathbf{k}$ is maximally compact.

A Cartan involution is an involution $\theta : \mathbf{g} \longrightarrow \mathbf{g}$ which is a Lie algebra homomorphism, and $-(\cdot, \theta\cdot)$ is an inner product. A Cartan decomposition of $\mathbf{g}$ is a sum $\mathbf{g} = \mathbf{k} + \mathbf{p}$ such that $\mathbf{k}$ is maximally compact and they satisfy (6.1). Check that Cartan involution and Cartan decomposition lead to each other, by taking $\mathbf{k}$ and $\mathbf{p}$ to be the ± 1 eigenspaces of θ.

For $\mathbf{g} = \mathbf{sl}(n, F)$, check that $\theta(A) = -A^*$ defines a Cartan involution.

Let $\mathbf{g}$ be a real semisimple Lie algebra. We say that $\mathbf{h} \subset \mathbf{g}$ is a Cartan subalgebra if $\mathbf{h} \otimes \mathbf{C}$ is a Cartan subalgebra of $\mathbf{g} \otimes \mathbf{C}$. The Cartan subalgebra is called θ-stable if $\theta(\mathbf{h}) = \mathbf{h}$. In this case

$$\mathbf{h} = (\mathbf{h} \cap \mathbf{k}) + (\mathbf{h} \cap \mathbf{p}).$$

Write $\mathbf{t} = \mathbf{h} \cap \mathbf{k}$ and $\mathbf{a} = \mathbf{h} \cap \mathbf{p}$, so that $\mathbf{h} = \mathbf{t} + \mathbf{a}$. We call $\mathbf{t}$ and $\mathbf{a}$ the compact and noncompact parts of $\mathbf{h}$. Various Cartan subalgebras $\mathbf{t} + \mathbf{a} \subset \mathbf{g}$ may have different dimensions for $\mathbf{t}$ and $\mathbf{a}$. But distinct Cartan subalgebras have the same dimension $\dim \mathbf{t} + \dim \mathbf{a}$. It is called the rank of $\mathbf{g}$.

For example, in $\mathbf{g} = \mathbf{sl}(2, \mathbf{R})$, we can take the diagonal matrices or $\mathbf{so}(2)$ as the Cartan subalgebra. Using the trace form, check that in the first case $\dim \mathbf{t} = 0$ and $\dim \mathbf{a} = 1$; and in the second case $\dim \mathbf{t} = 1$ and $\dim \mathbf{a} = 0$.

A Cartan subalgebra is called maximally compact if the dimension of $\mathbf{t}$ is as large as possible. It is called compact if $\mathbf{h} = \mathbf{t}$. For example $\mathbf{so}(2)$ is a compact Cartan subalgebra of $\mathbf{sl}(2, \mathbf{R})$. For $\mathbf{sl}(3, \mathbf{R})$, which has rank 2, a maximally compact Cartan subalgebra is not compact because $\dim \mathbf{t} = \dim \mathbf{a} = 1$.

We can always find a θ-stable maximally compact Cartan subalgebra $\mathbf{h} = \mathbf{t} + \mathbf{a}$ of $\mathbf{g}$. Restrict the roots $\alpha \in (\mathbf{h} \otimes \mathbf{C})^*$ to $\mathbf{h}$. It is a fact that $\alpha(\mathbf{t}) \subset i\mathbf{R}$ and $\alpha(\mathbf{a}) \subset \mathbf{R}$. Since $\mathbf{h}$ is maximally compact, $\alpha(\mathbf{t}) \neq 0$. We say that α is imaginary or complex depending on whether $\alpha(\mathbf{h}) \subset i\mathbf{R}$ or not. So α is imaginary if and only if it annihilates $\mathbf{a}$.

7 Vogan Diagram

We shall describe the Vogan diagram of a real simple Lie algebra $\mathbf{g}$. Start with a Cartan involution θ, and a θ-stable maximally compact Cartan subalgebra $\mathbf{h} = \mathbf{t} + \mathbf{a}$. Extend θ to be complex linear on $\mathbf{g} \otimes \mathbf{C}$. We can let θ act on the roots such that $\theta(\mathbf{g} \otimes \mathbf{C})_\alpha = (\mathbf{g} \otimes \mathbf{C})_{\theta\alpha}$. Then $\theta\alpha = \alpha$ if and only if α is imaginary. We can further choose the simple system Π so that $\theta(\Pi) = \Pi$. This leads to an automorphism θ on the Dynkin diagram $D = D(\mathbf{g} \otimes \mathbf{C})$. So the imaginary simple roots are the vertices of D which are fixed by θ. The complex simple roots are the vertices on D which are not fixed by θ; they form 2-element orbits.

Let α be an imaginary simple root. Define the 2-dimensional space $\mathbf{g}_\alpha = \mathbf{g} \cap ((\mathbf{g} \otimes \mathbf{C})_\alpha + (\mathbf{g} \otimes \mathbf{C})_{-\alpha})$. Since θ fixes α, it follows that $\theta(\mathbf{g}_\alpha) = \mathbf{g}_\alpha$. So θ is 1 or -1 on $\mathbf{g}_\alpha$. Equivalently, $\mathbf{g}_\alpha \subset \mathbf{k}$ or $\mathbf{g}_\alpha \subset \mathbf{p}$. We say that α is compact or noncompact accordingly. So there are three types of simple roots: complex, imaginary compact, imaginary noncompact. Depending on whether the imaginary root is compact or noncompact, we color the vertex in D as white or black. The complex roots have no color.

A Vogan diagram is a Dynkin diagram with an involution θ, such that the vertices fixed by θ are either white or black. We have used a real simple Lie algebra to construct a Vogan diagram. Conversely, it is a fact that every Vogan diagram represents a real simple Lie algebra.

Distinct Vogan diagrams may represent the same $\mathbf{g}$, due to different choices of simple system Π. A theorem of Borel and de Siebenthal says that we can always choose Π such that there is only at most one noncompact simple root. In other words, $\mathbf{g}$ can be represented by a Vogan diagram with at most one black vertex.

The main theme of this seminar is to explore research problems on the combinatorial properties of the Vogan diagrams, especially their Lie algebraic meanings.