A New Model Updating Method for Quadratic Eigenvalue Problems

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Abstract

Finite element model updating of quadratic eigenvalue problems (QEPs) is proposed by Friswell, Inman and Pilkey 1998, to incorporate the measured model data into the finite element model to produce an adjusted finite element model on the damping and stiffness that closely match the experimental modal data. In this paper, we mainly develop an efficient numerical method for the finite element model updating of QEPs which needs only $O(nk^2)$ flops and is stored in a sparse technique, where n is the size of coefficient matrices of the QEP and k is the number of the measured eigenpairs.

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1 Introduction

Given $n \times n$ real matrices M, C and K, the task of finding scalars $\lambda \in \mathbb{C}$ and nonzero vectors $\mathbf{x} \in \mathbb{C}^n$ satisfying

$$Q(\lambda)\mathbf{x} \equiv (\lambda^2 M + \lambda C + K)\mathbf{x} = \mathbf{0} \tag{1.1}$$

is known as the quadratic eigenvalue problem (QEP), which corresponds to solving the homogeneous second-order differential equation (see e.g., [10])

$$M\ddot{\mathbf{q}}(t) + C\dot{\mathbf{q}}(t) + K\mathbf{q}(t) = 0. \tag{1.2}$$

The scalar λ and the associated vector $\mathbf{x}$ in (1.1) are called, respectively, eigenvalues and eigenvectors of the quadratic pencil $Q(\lambda)$. It is known that the QEP has $2n$ finite eigenvalues, provided the leading matrix M is nonsingular. Recently the QEP has received much attention because its information has repeatedly arisen in many different discipline, including applied mechanics, electrical oscillation, vibro-acoustics, fluid mechanics and signal processing. A nice survey paper for the QEP can be found in [14] by Tisseur and Meerbergen. Vibrating systems, such as automotives, bridges, highways, buildings are described by distributed parameters. However, due to lack of viable computational methods to handle distributed parameter systems, a finite element method is generally used to discretize such systems to an analytical model (finite element model), namely,

$$Q_a(\lambda) = \lambda^2 M_a + \lambda C_a + K_a, \tag{1.3}$$

where M_a, C_a and K_a represent the mass, damping and stiffness, respectively, that all are real $n \times n$ symmetric matrices with M_a being symmetric positive definite (Denoted by $M_a > 0$). See the book [9] by Friswell and Mottershead for details.

In the finite element model (1.3) for structural dynamics, the mass and stiffness are, in general, clearly defined by physical parameters. However, the damping for precise dissipative effects is not well understood because it is a purely dynamics property that can not be measured statically. A common simplification is to assume proportional or modal damping, but it seems to be sufficient where damping levels are lower than 10% of critical [8].

Finite element model updating has emerged in the 1990's as a significant subject to the design, construction, and maintenance of mechanical systems [9, 12]. Model updating, at its ambitious, is used to correct inaccurate analytical models by measured data, such as natural frequencies, damping ratios, mode shapes and frequency response function, which can usually be obtained by vibration test. In the past decade, Baruch/Bar-Itzak [1, 2], Bermann/Nagy [3, 4] and Wei [15, 17, 16] considered variant aspects of finite element model updating by using measured data for undamped structured systems (i.e. $C = C_a = 0$). In the works by Datta/Elhay/Ram/Sarkissian [5, 6, 7], studies are undertaken toward a feedback design problem for second-order control system. That consideration eventually leads to a partial eigenstructure assignment problem for the QEP. Recently, Friswell, Inman and Pilkey [8] proposed to incorporate the measured model data into the finite element model to produce an adjusted finite element model on the damping and stiffness with modal properties that closely match the experimental modal data. That is, with $M = M_a$ the penalty function

$$J = \frac{1}{2}\|N^{-1}(K - K_a)N^{-1}\|_F^2 + \frac{1}{2}\mu\|N^{-1}(C - C_a)N^{-1}\|_F^2, \tag{1.4a}$$

is minimized, subject to

$$M_a\Phi\Lambda^2 + C\Phi\Lambda + K\Phi = 0, \tag{1.4b}$$

$$C^\top = C,\ K^\top = K, \tag{1.4c}$$

where $N = M_a^{\frac{1}{2}}$, μ is a weighting parameter, C and K are the updated damping and stiffness matrices, respectively, $\Phi \in \mathbb{C}^{n\times k}$ and $\Lambda \in \mathbb{C}^{k\times k}$ are the measured eigenvector and eigenvalue matrices, respectively. In practice, k is much less than the matrix size n. The solutions K and C of (1.4) are given by [8, 13] with,

$$K = K_a - 2M_a \mathrm{Re}(\Gamma_\Lambda \Phi^\top + \Phi \Gamma_\Lambda^\top) M_a \tag{1.5}$$

and

$$C = C_a - \frac{2}{\mu} M_a \mathrm{Re}(\Gamma_\Lambda \Lambda \Phi^\top + \Phi \Lambda \Gamma_\Lambda^\top) M_a, \tag{1.6}$$

where $\Gamma_\Lambda \in \mathbb{C}^{n\times k}$ solves the $2nk$ linear equations

$$\begin{aligned} 2M_a \mathrm{Re}(\Gamma_\Lambda \Phi^\top + \Phi \Gamma_\Lambda^\top) M_a \Phi + \frac{2}{\mu} M_a \mathrm{Re}(\Gamma_\Lambda \Lambda \Phi^\top + \Phi \Lambda \Gamma_\Lambda^\top) M_a \Phi \Lambda \\ = M_a \Phi \Lambda^2 + C_a \Phi \Lambda + K_a \Phi. \end{aligned} \tag{1.7}$$

Generally, the size n in a finite element model (1.3) is quite large. It is impractical to solve a complex matrix Γ_Λ for a large and dense $2nk \times 2nk$ linear system in (1.7). The purpose of this paper is to develop an efficient algorithm for the computation of the solutions C and K for (1.4) which is required only $O(nk^2)$ flops and is stored is a sparse technique.

2 Solving a PD-IQEP.

To match the partial measured data of the spectrum information of a QEP, we consider to solve the partially described inverse quadratic eigenvalue problem (PD-IQEP):
Let $(\Lambda, \Phi) \in \mathbb{R}^{k\times k} \times \mathbb{R}^{n\times k}$ $(k \le n)$ *be a given pair of matrices, where*

$$\Lambda = \mathrm{diag}(\lambda_1^{[2]}, \ldots, \lambda_\ell^{[2]}, \lambda_{2\ell+1}, \ldots, \lambda_k) \tag{2.1a}$$

with $\lambda_j^{[2]} = \begin{bmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{bmatrix}$, $\beta_j \neq 0$, *for* $j = 1, \ldots, \ell$, *and*

$$\Phi = [\boldsymbol{\varphi}_{1R}, \boldsymbol{\varphi}_{1I}, \ldots, \boldsymbol{\varphi}_{\ell R}, \boldsymbol{\varphi}_{\ell I}, \boldsymbol{\varphi}_{2\ell+1}, \ldots, \boldsymbol{\varphi}_k]. \tag{2.1b}$$

Suppose Λ has only simple eigenvalues and Φ is of full column rank. Find a general form of symmetric matrices M, C and K with M being symmetric positive definite that satisfy the equation

$$M\Phi\Lambda^2 + C\Phi\Lambda + K\Phi = 0. \tag{2.2}$$

Let Φ have the QR-decomposition

$$\Phi = Q \begin{bmatrix} R \\ 0 \end{bmatrix} \equiv [Q_1, Q_2] \begin{bmatrix} R \\ 0 \end{bmatrix}, \tag{2.3}$$

where $Q \in \mathbb{R}^{n\times n}$ is orthogonal with $Q_1 \in \mathbb{R}^{n\times k}$ and $R \in \mathbb{R}^{k\times k}$ is nonsingular. Partition $Q^\top MQ$, $Q^\top CQ$ and $Q^\top KQ$ by

$$Q^\top MQ = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix},\ Q^\top CQ = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix},\ Q^\top KQ = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}, \tag{2.4}$$

where M_{11}, C_{11} and $K_{11} \in \mathbb{R}^{k\times k}$, and M_{21}, C_{21} and $K_{21} \in \mathbb{R}^{(n-k)\times k}$.

A general solution of symmetric M, C, K with M being symmetric positive definite is given by the theorem in [11].

Theorem 2.1. [11] *For a given matrix pair (Λ, Φ) as in (2.1), the general solution of* PD-IQEP *is given by*

$$M_{11} : \textit{arbitrary fixed symmetric positive definite matrix}, \tag{2.5a}$$

$$C_{11} = -\left(M_{11}S + S^\top M_{11} + R^{-\top}DR^{-1}\right), \tag{2.5b}$$

$$K_{11} = S^\top M_{11}S + R^{-\top}D\Lambda R^{-1}, \tag{2.5c}$$

$$K_{21} = K_{12}^\top = -\left(M_{21}S^2 + C_{21}S\right), \tag{2.5d}$$

where $S = R\Lambda R^{-1}$ and

$$D = \operatorname{diag}\left(\begin{bmatrix} \xi_1 & \eta_1 \\ \eta_1 & -\xi_1 \end{bmatrix}, \ldots, \begin{bmatrix} \xi_\ell & \eta_\ell \\ \eta_\ell & -\xi_\ell \end{bmatrix},\ \xi_{2\ell+1}, \ldots,\ \xi_k\right) \tag{2.6}$$

in which ξ_i and η_i are arbitrary real numbers. Furthermore, $M_{21} = M_{12}^\top$, $C_{21} = C_{12}^\top$, $C_{22} = C_{22}^\top$ and $K_{22} = K_{22}^\top$ are chosen arbitrary, and $M_{22} = M_{22}^\top$ is chosen so that $M_{22} - M_{21}M_{11}^{-1}M_{12}$ is symmetric positive definite.

3 Solving the optimization problem (1.4).

In this section we shall develop an efficient algorithm for solving the optimization problem described in (1.4). We first solve two optimization problems. Let $(\Lambda, \Phi) \in \mathbb{R}^{k\times k} \times \mathbb{R}^{n\times k}$ be given in (2.1), D and R be given in (2.6) and (2.3), respectively. Denote

$$R^{-1} = [\mathbf{r}_1, \ldots, \mathbf{r}_k] = \begin{bmatrix} r_{11} & \ldots & r_{1k} \\ & \ddots & \vdots \\ 0 & & r_{kk} \end{bmatrix}. \tag{3.1}$$

Optimization Problem I. *Given $A = [\mathbf{a}_1, \ldots, \mathbf{a}_k]$, $B = [\mathbf{b}_1, \ldots, \mathbf{b}_k] \in \mathbb{R}^{k\times k}$ and let*

$$\mathbf{x} = (\xi_1,\ \eta_1, \ldots,\ \xi_\ell,\ \eta_\ell,\ \xi_{2\ell+1}, \ldots,\ \xi_k)^\top \tag{3.2}$$

correspond to the matrix D in (2.6). Minimize

$$\begin{aligned} f(\mathbf{x}) &= \mu\|A + R^{-\top}DR^{-1}\|_F^2 + \|B - R^{-\top}\Lambda^\top DR^{-1}\|_F^2 \\ &= \sum_{j=1}^{k} f_j(\mathbf{x}) \end{aligned} \tag{3.3a}$$

for $\mathbf{x}$, where

$$f_j(\mathbf{x}) = \mu\|\mathbf{a}_j + R^{-\top}D\mathbf{r}_j\|_2^2 + \|\mathbf{b}_j - R^{-\top}\Lambda^\top D\mathbf{r}_j\|_2^2, \quad j = 1, \ldots, k. \tag{3.3b}$$

Note that $\begin{bmatrix} \xi & \eta \\ \eta & -\xi \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u & v \\ -v & u \end{bmatrix}\begin{bmatrix} \xi \\ \eta \end{bmatrix}$. The vector $D\mathbf{r}_j$ in (3.3b) can be rewritten by

$$D\mathbf{r}_j = \Gamma_j\mathbf{x}, \quad j = 1, \ldots, k, \tag{3.4}$$

where

(i) $j = 2s,\ s \leqslant \ell,$

$$\Gamma_j = \operatorname{diag}\left(\begin{bmatrix} r_{1j} & r_{2j} \\ -r_{2j} & r_{1j} \end{bmatrix}, \dots, \begin{bmatrix} r_{2s-1,j} & r_{2s,j} \\ -r_{2s,j} & r_{2s-1,j} \end{bmatrix}, 0, \dots, 0 \right) \in \mathbb{R}^{k\times k},$$

(ii) $j = 2s+1,\ s < \ell,$

$$\Gamma_j = \operatorname{diag}\left(\begin{bmatrix} r_{1j} & r_{2j} \\ -r_{2j} & r_{1j} \end{bmatrix}, \dots, \begin{bmatrix} r_{2s-1,j} & r_{2s,j} \\ -r_{2s,j} & r_{2s-1,j} \end{bmatrix}, r_{2s+1,j}, r_{2s+1,j}, 0, \dots, 0 \right) \in \mathbb{R}^{k\times k},$$

(iii) $j > 2\ell,$

$$\Gamma_j = \operatorname{diag}\left(\begin{bmatrix} r_{1j} & r_{2j} \\ -r_{2j} & r_{1j} \end{bmatrix}, \dots, \begin{bmatrix} r_{2\ell-1,j} & r_{2\ell,j} \\ -r_{2\ell,j} & r_{2\ell-1,j} \end{bmatrix}, r_{2\ell+1,j}, \dots, r_{j,j}, 0, \dots, 0 \right) \in \mathbb{R}^{k\times k}.$$

Substituting (3.4) into (3.3b) we compute

$$\begin{aligned} \nabla f_j(\mathbf{x}) &= \left(\frac{\partial f_j}{\partial x_1}, \dots, \frac{\partial f_j}{\partial x_k} \right)^\top \\ &= 2\mu \left(R^{-\top}\Gamma_j\right)^\top \left(\mathbf{a}_j + R^{-\top}\Gamma_j\mathbf{x}\right) - 2\left(R^{-\top}\Lambda^\top\Gamma_j\right)^\top \left(\mathbf{b}_j - R^{-\top}\Lambda^\top\Gamma_j\mathbf{x}\right). \end{aligned} \tag{3.5}$$

Consequently,

$$\begin{aligned} \nabla f(\mathbf{x}) &= \sum_{j=1}^{k} \nabla f_j(\mathbf{x}) \\ &= 2\sum_{j=1}^{k} \left[\mu \left(R^{-\top}\Gamma_j\right)^\top \mathbf{a}_j + \mu\Gamma_j^\top \left(R^\top R\right)^{-1}\Gamma_j\mathbf{x} - \Gamma_j^\top \Lambda R^{-1}\mathbf{b}_j + \Gamma_j^\top\Lambda\left(R^\top R\right)^{-1}\Lambda^\top\Gamma_j\mathbf{x} \right]. \end{aligned} \tag{3.6}$$

Setting $\nabla f(\mathbf{x}) = \mathbf{0}$ we derive the linear equation for $\mathbf{x}$

$$G\mathbf{x} = \mathbf{b}, \tag{3.7}$$

where

$$G=\sum_{j=1}^{k}\left[\mu\Gamma_j^\top\left(R^\top R\right)^{-1}\Gamma_j+\Gamma_j^\top\Lambda\left(R^\top R\right)^{-1}\Lambda^\top\Gamma_j\right] \tag{3.8a}$$

and

$$\mathbf{b}=\sum_{j=1}^{k}\left(\Gamma_j^\top\Lambda R^{-1}\mathbf{b}_j-\mu\Gamma_j^\top R^{-1}\mathbf{a}_j\right). \tag{3.8b}$$

Since the function $f(\mathbf{x})$ in (3.3a) must have an optimum, the linear system of (3.7) is consistent, and therefore, $\mathbf{x}$ is solvable.

Optimization Problem II. *Given E, $F\in\mathbb{R}^{(n-k)\times k}$ and $S=R\Lambda R^{-1}$. Minimize*

$$g(X)=\mu\|E-X\|_F^2+\|F+XS\|_F^2 \tag{3.9}$$

for $X\in\mathbb{R}^{(n-k)\times k}$.

Let

$$\mathbf{x}=\operatorname{vec}(X),\ \mathbf{e}=\operatorname{vec}(E),\ \mathbf{f}=\operatorname{vec}(F). \tag{3.10}$$

Here " vec " denotes the vectorization of the given matrix columnwisely. By (3.10), the function $g(X)$ in (3.9) becomes

$$g(\mathbf{x})=g(X)=\mu\|\mathbf{e}-\mathbf{x}\|_2^2+\|\mathbf{f}+\left(S^\top\otimes I_{n-k}\right)\mathbf{x}\|_2^2. \tag{3.11}$$

Then

$$\nabla g(\mathbf{x})=2\left[-\mu(\mathbf{e}-\mathbf{x})+(S\otimes I_{n-k})\left(\mathbf{f}+\left(S^\top\otimes I_{n-k}\right)\mathbf{x}\right)\right]. \tag{3.12}$$

Here $\otimes$ denotes the Kronecker product of two matrices. The matrix form of (3.12) can be written by

$$\frac{\partial}{\partial X}g(X)=2\left[-\mu E+\mu X+FS^\top+XSS^\top\right]. \tag{3.13}$$

By setting $\frac{\partial}{\partial X}g(X)=\mathbf{0}$, we then solve

$$X=(\mu E-FS^\top)(\mu I+SS^\top)^{-1}. \tag{3.14}$$

We now solves the optimization problem (1.4). Redefine

$$C_a := M_a^{-\frac{1}{2}} C_a M_a^{-\frac{1}{2}}, \qquad K_a := M_a^{-\frac{1}{2}} K_a M_a^{-\frac{1}{2}}, \tag{3.15a}$$

$$C := M_a^{-\frac{1}{2}} C M_a^{-\frac{1}{2}}, \qquad K := M_a^{-\frac{1}{2}} K M_a^{-\frac{1}{2}}, \tag{3.15b}$$

$$\Phi := M_a^{\frac{1}{2}} \Phi. \tag{3.15c}$$

Let $Q = [Q_1, Q_2]$ be the orthogonal matrix such that $\Phi = Q[R^\top, 0]^\top$ with R nonsingular. Then the problem (1.4) becomes

$$\begin{aligned} \min &= \frac{1}{2}\mu \|C_a - C\|_F^2 + \frac{1}{2} \|K_a - K\|_F^2 \\ &= \frac{1}{2}\mu \left\| Q^\top C_a Q - \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \right\|_F^2 + \frac{1}{2} \left\| Q^\top K_a Q - \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \right\|_F^2, \end{aligned} \tag{3.16}$$

where $C_{ij} = Q_i^\top C Q_j$ and $K_{ij} = Q_i^\top K Q_j$, $i, j = 1, 2$, are undetermined. Note that $M = I_n$. Set

$$C_{22} = Q_2^\top C_a Q_2, \qquad K_{22} = Q_2^\top K_a Q_2. \tag{3.17}$$

Then from (2.5) the optimization problem (3.16) is equivalent to the following two optimization problems:

$$\begin{aligned} \min =& \mu \left\| Q_1^\top C_a Q_1 - C_{11} \right\|_F^2 + \left\| Q_1^\top K_a Q_1 - K_{11} \right\|_F^2 \\ =& \mu \left\| Q_1^\top C_a Q_1 + S + S^\top + R^{-\top} D R^{-1} \right\|_F^2 \\ &+ \left\| Q_1^\top K_a Q_1 - S^\top S - R^{-\top} D \Lambda R^{-1} \right\|_F^2 \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} \min =& \mu \left\| Q_2^\top C_a Q_1 - C_{21} \right\|_F^2 + \left\| Q_2^\top K_a Q_1 - K_{21} \right\|_F^2 \\ =& \mu \left\| Q_2^\top C_a Q_1 - C_{21} \right\|_F^2 + \left\| Q_2^\top K_a Q_1 + C_{21} S \right\|_F^2 . \end{aligned} \tag{3.19}$$

Hence, the optimal solutions C_{11} and K_{11} of (3.18) are solved by the **Optimization Problem I** by setting

$$A = Q_1^\top C_a Q_1 + S + S^\top, \quad B = Q_1^\top K_a Q_1 - S^\top S. \tag{3.20}$$

The optimal solutions C_{21} and K_{21} of (3.19) are solved by the **Optimization Problem II** by setting

$$E = Q_2^\top C_a Q_1 \quad \text{and} \quad F = Q_2^\top K_a Q_1. \tag{3.21}$$

Reset $M = M_a$,

$$C = M_a^{\frac{1}{2}} Q \begin{bmatrix} C_{11} & C_{21}^\top \\ C_{21} & C_{22} \end{bmatrix} Q^\top M_a^{\frac{1}{2}}, \quad K = M_a^{\frac{1}{2}} Q \begin{bmatrix} K_{11} & K_{21}^\top \\ K_{21} & K_{22} \end{bmatrix} Q^\top M_a^{\frac{1}{2}} \tag{3.22}$$

which solve the optimization problem (1.4). The steps of computation for solving (1.4) are summarized in the following algorithm.

Algorithm 3.1. Given $Q_a(\lambda) = \lambda^2 M_a + \lambda C_a + K_a$ and $(\Lambda, \Phi) \in \mathbb{R}^{k\times k} \times \mathbb{R}^{n\times k}$ as in (2.1). The optimal solutions C and K of (1.4) are computed by

step 1. Set $C_a := M_a^{-\frac{1}{2}} C_a M_a^{-\frac{1}{2}}, \ K_a := M_a^{-\frac{1}{2}} K_a M_a^{-\frac{1}{2}}, \ \Phi := M_a^{\frac{1}{2}} \Phi$;

step 2. Compute the QR-factorization of Φ :

$$\Phi = [Q_1, Q_2] \begin{bmatrix} R \\ 0 \end{bmatrix}, \quad \text{and } S = R\Lambda R^{-1};$$

step 3. Compute $C_{22} = Q_2^\top C_a Q_2, \ K_{22} = Q_2^\top K_a Q_2$;

step 4. Solve $G\mathbf{x} = \mathbf{b}$ for $\mathbf{x} = [\xi_1, \eta_1, \cdots, \xi_\ell, \eta_\ell, \xi_{2\ell+1}, \cdots, \xi_k]^\top$,

where

$$G = \sum_{j=1}^{k} \Gamma_j^\top \left[\mu \left(R^\top R\right)^{-1} + \Lambda \left(R^\top R\right)^{-1} \Lambda^\top\right] \Gamma_j,$$

$$\mathbf{b} = \sum_{j=1}^{k} \Gamma_j^\top \left(\Lambda R^{-1}\mathbf{v}_j - \mu R^{-1}\mathbf{u}_j\right),$$

$$\Gamma_j = \mathrm{diag}\left(\begin{bmatrix} r_{1,j} & r_{2,j} \\ -r_{2,j} & r_{1,j} \end{bmatrix}, \cdots, \begin{bmatrix} r_{2\ell-1,j} & r_{2\ell,j} \\ -r_{2\ell,j} & r_{2\ell-1,j} \end{bmatrix}, r_{2\ell+1,j}, \cdots, r_{k,j} \right),$$

$$[\mathbf{u}_1, \cdots, \mathbf{u}_k] = Q_1^\top C_a Q_1 + S + S^\top,$$

$$[\mathbf{v}_1, \cdots, \mathbf{v}_k] = Q_1^\top K_a Q_1 - S^\top S,$$

$$(r_{1,j}, \cdots, r_{k,j})^\top = R^{-1}\mathbf{e}_j;$$

step 5. Compute

$$C_{11} = -\left(S + S^\top + R^{-\top} D R^{-1}\right),$$

$$K_{11} = S^\top S + R^{-\top} D \Lambda R^{-1},$$

where $D = \mathrm{diag}\left(\begin{bmatrix} \xi_1 & \eta_1 \\ \eta_1 & -\xi_1 \end{bmatrix}, \cdots, \begin{bmatrix} \xi_\ell & \eta_\ell \\ \eta_\ell & -\xi_\ell \end{bmatrix}, \xi_{2\ell+1}, \cdots, \xi_k \right)$,

and compute

$$C_{21} = Q_2^\top \left(\mu C_a Q_1 - K_a Q_1 S^\top\right) (\mu I + SS^\top)^{-1},$$

$$K_{21} = -C_{21} S;$$

step 6. $C = M_a^{\frac{1}{2}} Q \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} Q^\top M_a^{\frac{1}{2}},\ K = M_a^{\frac{1}{2}} Q \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} Q^\top M_a^{\frac{1}{2}},$

where $Q = [Q_1, Q_2]$.

Remark 3.1. (i) *In a finite element model, the size of the analytical matrices M_a, C_a and K_a are very large and sparse. M_a is, in general, a diagonal or banded matrix,* *and therefore, it is easily invertible. In practice, the number of the measured data of eigenpairs is much less than the size of the finite element model, i.e., $k \ll n$. The orthogonal matrix $Q = [Q_1, Q_2]$ in step* 2 *of Algorithm* 3.1 *can be computed and stored in the form of a diagonal matrix plus a low rank updating by Householder transformations. Suppose the sparse matrix C_a or K_a times a vector needs $O(n)$ flops. Then, the computational cost of Algorithm 3.1 can be easily estimated by $O(nk^2)$ flops.*

(ii) *Using Algorithm 3.1 to solve the optimization problem* (1.4) *is different from using* (1.7) *to solve* (1.4). *The latter needs to solve a large, but possibly dense $nk \times nk$ linear system as in* (1.7) *which is impractical in a finite element model updating process when n is sufficiently large.*

4 Numerical results.

A set of pseudo simulation data was provided by The Boeing Company for testing purpose. The dimension of matrices M_a, C_a and K_a are 42×42.

Test I. We first test that the Algorithm 3.1 computes the optimal solution of the optimization problem (1.4). Choosing an eigenmatrix pair $(\Lambda_a, \Phi_a) \in \mathbb{R}^{14\times 14} \times \mathbb{R}^{42\times 14}$ of the analytical model $Q_a = \lambda^2 M_a + \lambda C_a + K_a$, i.e., $M_a\Phi_a\Lambda_a^2 + C_a\Phi_a\Lambda_a + K_a\Phi_a = 0$, the Algorithm 3.1 should theoretically give the optimal solution $C = C_a$ and $K = K_a$. The numerical result of the relative errors computed by Algorithm 3.1 are estimated by

$$\frac{\|C - C_a\|_{F_a}}{\|C_a\|_{F_a}} \simeq 10^{-7}, \quad \frac{\|K - K_a\|_{F_a}}{\|K_a\|_{F_a}} \simeq 10^{-10},$$

where $\|\cdot\|_{F_a} = \|M_a^{-\frac{1}{2}}(\cdot)M_a^{-\frac{1}{2}}\|_F$.

Test II. Now we are given the measured eigenvalues

$$\begin{aligned}\{\lambda_{mj}\}_{j=1}^{14} = \{ & -0.60939 \pm 37.365\boldsymbol{\iota}, -0.73496 \pm 36.707\boldsymbol{\iota}, -2.8832 \pm 31.992\boldsymbol{\iota}, \\ & -1.8907 \pm 61.437\boldsymbol{\iota}, -1.9112 \pm 54.181\boldsymbol{\iota}, -2.2785 \pm 39.639\boldsymbol{\iota}, \\ & -5.0387, -4.3416\}\end{aligned} \tag{4.1}$$

and the measured mode shapes $\mathbf{v}_j \in \mathbb{R}^s$, $j = 1, \ldots, 14$. The measured eigenvectors $\boldsymbol{\varphi}_j$ is estimated by

$$\boldsymbol{\varphi}_j = D_j \widetilde{D}_j^{\dagger} \mathbf{v}_j, \quad j = 1, \ldots 14, \tag{4.2}$$

where D_j is defined by $D_j = [\lambda_{mj}^2 M_a + \lambda_{mj} C_a + K_a]^{-1} B_a$ with $B_a \in \mathbb{R}^{n\times t}$ $(t < s)$, and the matrix $\widetilde{D}_j$ consistent of the first s rows of D_j and the superscript " $\dagger$ " denotes the pseudo inverse. We construct the eigenmatrix pair $(\Lambda, \Phi) \in \mathbb{R}^{14\times 14} \times \mathbb{R}^{42\times 14}$ as in (2.1) associated with (4.1) and (4.2). The Algorithm 3.1 computes the new updated matrices $M = M_a$, C and K with $\mu = 0.1$, 1.0 and 10, which minimizes the optimization problem (1.4). We define the relative residual by

$$\text{res} = \frac{\|M\Phi\Lambda^2 + C\Phi\Lambda + K\Phi\|_2}{\|M\Phi\Lambda^2\|_2 + \|C\Phi\Lambda\|_2 + \|K\Phi\|_2}. \tag{4.3}$$

The numerical results are shown in Table 4.1.

Table 4.1 relative residuals

μ	0.1	1.0	10
res	1.4725e-014	1.4826e-014	1.4859e-014

5 Conclusions.

We have developed an efficient numerical algorithm for finite element model updating of quadratic eigenvalue problems. This method can serve as a fast and reliable manner for updating the analytical model. It was shown to be insightful in a simple pseudo test suit provided by The Boring Company.

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