# NON-ABELIAN LOCAL INVARIANT CYCLES

YEN-LUNG TSAI AND EUGENE Z. XIA

Abstract. Let $f$ be a degeneration of Kähler manifolds. The local invariant cycle theorem states that for a smooth fiber of the degeneration, any cohomology class, invariant under the monodromy action, rises from a global cohomology class. Instead of the classical cohomology, one may consider the non-abelian cohomology. This note demonstrates that the analogous non-abelian version of the local invariant cycle theorem does not hold if the first non-abelian cohomology is the moduli space (universal categorical quotient) of the representations of the fundamental group.

A degeneration of Kähler manifolds is a proper map $f$ from a Kähler manifold $X$ onto the unit disk $\Delta$ such that $f$ is of maximum rank for all $s \in \Delta$ except at the point $s = 0$. Let $\Delta^* = \Delta - \{0\}$. We call $X_t = f^{-1}(X_t)$ a smooth fiber or generic fiber when $t \in \Delta^*$ and $X_0 = f^{-1}(0)$ the singular or degenerated fiber. We assume the singularity in $X_0$ is of normal crossing.

Fix $t \in \Delta^*$ and a base point $x \in X_t$ once and for all. There is a monodromy action (see, for example, [3])

$$\pi_1(\Delta^*) \times \mathrm{H}^n(X_t, \mathbb{C}) \to \mathrm{H}^n(X_t, \mathbb{C}).$$

The local invariant cycle theorem states that a cohomology class in $\mathrm{H}^n(X_t, \mathbb{C})$, fixed by the monodromy action, is a restriction of a cohomology class in $\mathrm{H}^n(X, \mathbb{C})$ [2, 9, 10]. Fix a generator $T \in \pi_1(\Delta^*) \cong \mathbb{Z}$. Then $T$ determines the monodromy action and gives rise to the isomorphisms

$$T^* : \mathrm{H}^n(X_t, \mathbb{C}) \to \mathrm{H}^n(X_t, \mathbb{C}), \quad T_* : \pi_1(X_t, x) \to \pi_1(X_t, y).$$

These isomorphisms are actually induced from a Picard-Lefschetz diffeomorphism which we shall also denote by

$$T : X_t \to X_t,$$

where $y = T(x)$.

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The map $f$ induces a strong deformation retract $X \to X_0$, hence, also an isomorphism $\mathrm{H}^n(X_0, \mathbb{C}) \to \mathrm{H}^n(X, \mathbb{C})$ [2, 3, 6]. Define

$$c : X_t \hookrightarrow X \to X_0,$$

where the first map is the inclusion and the second the strong deformation retract. Then $c$ induces homomorphisms:

$$c^* : \mathrm{H}^n(X_0, \mathbb{C}) \to \mathrm{H}^n(X_t, \mathbb{C}), \quad c_* : \pi_1(X_t, x) \to \pi_1(X_0, c(x)).$$

The local invariant cycle theorem states that [2, 9, 10]

$$\mathrm{im}(c^*) = \{\alpha \in \mathrm{H}^n(X_t, \mathbb{C}) \mid T^*(\alpha) = \alpha\}. \tag{1}$$

To generalize the above statement to the non-abelian context, we choose the universal categorical quotient

$$\mathsf{H}^1(X_s, G) = \mathrm{Hom}(\pi_1(X_s, x), G)//G, \quad s \in \Delta$$

to be the first non-abelian cohomology of $X_s$, where $G$ is an algebraic group acting on $\mathrm{Hom}(\pi_1(X_s, x), G)$ by conjugation [4, 5]. For any $\rho \in \mathrm{Hom}(\pi_1(X_t, x), G)$, denote by $[\rho]$ its image in $\mathsf{H}^1(X_t, G)$ by the universal morphism.

The map $T_*$ induces a morphism

$$T^\# : \mathrm{Hom}(\pi_1(X_t, T(x)), G) \to \mathrm{Hom}(\pi_1(X_t, x), G)$$

defined by

$$T^\#(\rho)(A) = \rho \circ T_*^{-1}(A)$$

for all $A \in \pi_1(X_t, x)$. Similarly, $c$ induces a map

$$c^\# : \mathrm{Hom}(\pi_1(X_0, c(x)), G) \to \mathrm{Hom}(\pi_1(X_t, x), G)$$

defined by

$$c^\#(\rho)(A) = \rho \circ c_*(A).$$

The maps $T^\#$ and $c^\#$ descend to morphisms on their respective universal categorical quotients:

$$T^\# : \mathsf{H}^1(X_t, G) \to \mathsf{H}^1(X_t, G), \quad c^\# : \mathsf{H}^1(X_0, G) \to \mathsf{H}^1(X_t, G).$$

In this setting, the analogous statement of the local invariant cycle theorem is

$$\mathrm{im}(c^\#) = \{[\rho] \in \mathsf{H}^1(X_t, G) \mid T^\#([\rho]) = [\rho]\}. \tag{2}$$

Notice that when $G = \mathbb{C}$ (the addition group), $\mathsf{H}^1(X_s, G)$ is the regular first cohomology $\mathrm{H}^1(X_s, \mathbb{C})$ of $X_s$ for $s \in \Delta$ and (2) reduces to the classical local invariant cycle theorem (1).

**Theorem 1.** *For each $g > 1$, there exists an $n$ and a degeneration $f : X \to \Delta$ with the generic fiber $X_t$ a Riemann surface of genus $g$ and such that*

$$\mathrm{im}(c^{\#}) \neq \{[\rho] \in \mathsf{H}^1(X_t, G) \mid T^{\#}([\rho]) = [\rho]\},$$

*where $G = \mathrm{SL}(n, \mathbb{C})$. More precisely, for such an $n$, there exists an irreducible representation $\rho \in \mathrm{Hom}(\pi_1(X_t, x), G)$ with $\mathrm{im}(\rho) \subset \mathrm{SU}(n)$ and $[\rho] \in \{[\rho] \in \mathsf{H}^1(X_t, G) \mid T^{\#}([\rho]) = [\rho]\}$ but $[\rho] \notin \mathrm{im}(c^{\#})$.*

*Proof.* For each $n$, let $G = \mathrm{SL}(n, \mathbb{C})$ and $K = \mathrm{SU}(n)$. Let $\Gamma$ be the mapping class group of $X_t$. Then each $\gamma \in \Gamma$ induces a map

$$\gamma : \mathrm{Hom}(\pi_1(X_t, \gamma(x)), G) \to \mathrm{Hom}(\pi_1(X_t, x), G) : \ \gamma(\rho)(A) = \rho \circ \gamma^{-1}(A).$$

These maps provide a $\Gamma$-action on $\mathsf{H}^1(X_t, G)$. In [1], Andersen proved that for $g > 1$, there exists infinitely many $n$ such that there exists irreducible $\rho \in \mathrm{Hom}(\pi_1(X_t), G)$ with $\mathrm{im}(\rho) \subset K$ and $\gamma([\rho]) = [\rho]$ for all $\gamma \in \Gamma$.

Now suppose our choices of $\rho$ and $n$ satisfy the hypothesis and conclusions of Andersen's theorem [1]. Then since $T : X_t \to X_t$ is a diffeomorphism, $T_* \in \Gamma$. Hence $T^{\#}([\rho]) = [\rho]$.

The fundamental group $\pi_1(X_t, x)$ has a presentation

$$\langle A_i, 1 \leq i \leq 2g, \mid \prod_{i=1}^{g} A_i A_{g+i} A_i^{-1} A_{g+i}^{-1} \rangle,$$

where the $A_i$'s correspond to simple loops beginning and ending at $x$. Since $\rho$ is an irreducible representation, $\rho(A_i) \neq I \in G$ for some $i$.

Now let $f : X \to \Delta$ be a degeneration with $A_i$ as its vanishing cycle. Then $c_*(A_i) = e \in \pi(X_0, c(x))$. Hence for every $[\alpha] \in \mathsf{H}^1(X_0, G)$,

$$c^{\#}(\alpha)(A_i) = \alpha(c_*(A_i)) = I \in G.$$

Since $\rho(A_i) \neq I$, $[\rho] \neq [\alpha]$ for all $[\alpha] \in \mathrm{im}(c^{\#})$. $\square$

We refer to [1] for the detailed and explicit descriptions of these irreducible representation classes fixed by the $\Gamma$-action. Here we simply mention that all such representations have finite images in $\mathrm{SU}(n)$. It will be interesting to see whether there exist examples $[\rho]$, satisfying the conclusion of Theorem 1, with $\mathrm{im}(\rho)$ being Zariski dense in $G$. Unfortunately, such representations are unlikely to satisfy the overly stringent condition of being fixed by the $\Gamma$-action. In fact, it is likely that the $\Gamma$-orbit of $[\rho]$ is Zariski dense if $\mathrm{im}(\rho)$ is Zariski dense in $G$. This is the case when $G = \mathrm{SL}(2, \mathbb{C}), K = \mathrm{SU}(2)$ [7, 8].

## References

[1] Jørgen Ellegaard Andersen, *Fixed points of the mapping class group in the* $\mathrm{SU}(n)$ *moduli spaces*, Proc. Amer. Math. Soc. **125** (1997), no. 5, 1511–1515.

[2] C. H. Clemens, *Degeneration of Kähler manifolds*, Duke Math. J. **44** (1977), no. 2, 215–290.

[3] Vik. S. Kulikov and P. F. Kurchanov, *Complex algebraic varieties: periods of integrals and Hodge structures*, Algebraic geometry, III, Encyclopaedia Math. Sci., vol. 36, Springer, Berlin, 1998, pp. 1–217, 263–270.

[4] Alexander Lubotzky and Andy R. Magid, *Varieties of representations of finitely generated groups*, Mem. Amer. Math. Soc. **58** (1985), no. 336, xi+117.

[5] David Mumford, *Geometric invariant theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, Band 34, Springer-Verlag, Berlin, 1965.

[6] Ulf Persson, *On degenerations of algebraic surfaces*, Mem. Amer. Math. Soc. **11** (1977), no. 189, xv+144.

[7] Joseph P. Previte and Eugene Z. Xia, *Topological dynamics on moduli spaces. I*, Pacific J. Math. **193** (2000), no. 2, 397–417.

[8] ———, *Topological dynamics on moduli spaces. II*, Trans. Amer. Math. Soc. **354** (2002), no. 6, 2475–2494 (electronic).

[9] Wilfried Schmid, *Variation of Hodge structure: the singularities of the period mapping*, Invent. Math. **22** (1973), 211–319.

[10] Joseph Steenbrink, *Limits of Hodge structures*, Invent. Math. **31** (1975/76), no. 3, 229–257.

NATIONAL CENTER FOR THEORETICAL SCIENCES, MATHEMATICS DIVISION, NATIONAL TSING HUA UNIVERSITY, HSINCHU 300, TAIWAN
*E-mail address*: `yenlung@math.cts.nthu.edu.tw`

DEPARTMENT OF MATHEMATICS, NATIONAL CHENG KUNG UNIVERSITY, TAINAN 701, TAIWAN
*E-mail address*: `ezxia@ncku.edu.tw`