A Structure-Preserving Doubling Algorithm for Nonsymmetric Algebraic Riccati Equation

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Abstract

Based on the ideas that are used to Algebraic Riccati equations, a doubling structure-preserving transformation is generalized to a more common matrix pencil, and its some basic properties are presented. Further, we presented a unified convergence theory for this algorithm for solving nonsymmetric algebra Riccati equation.

Keywords: nonsymmetric algebraic Riccati equation, minimal nonnegative solution, structure-preserving, doubling transformation.

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1 Introduction

In this paper we investigate the convergence of the structure-preserving doubling algorithm (SDA) for the computation of the minimal nonnegative solution to the following nonsymmetric Riccati equation (NARE):

$$XCX - XD - AX + B = 0. \tag{1.1}$$

where A, B, C, D are real matrices of sizes $m \times m, m \times n, n \times m, n \times n$, respectively. This equation occur in many important applications, for example, transport theory(see [2]), the Wiener-Hopf factorization of Markov chains (see [3]). Equation (1.1) has been studied by several authors (see [4, 5, 6, 7, 8]).

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A class of methods, referred to as the doubling algorithms, has attracted much interest in 70s and 80s (see [12] and references therein). These methods originate from the fixed-point iteration derived from the DARE:

$$X_{k+1} = A^T X_k (I + GX_k)^{-1} A + H.$$

Instead of generating the sequence X_k, doubling algorithms generate X_{2^k}. Doubling algorithms were largely forgotten in the past decade. recently, they have been revived for DAREs and CAREs, because their nice numerical behavior: quadratical convergence rate, low cost computational cost per step, and good numerical stability (see [9, 10, 11, 13]).

In this paper, by employing the same technique as in [13], we derive a SDA algorithm for solving the NARE. The SDA algorithms for solving CAREs and DAREs can be viewed as applying some special structure-preserving transformation to the associated symplectic pencils repeatedly. Therefore, we first generalize the structure-preserving transformation of a symplectic pencil to a more common matrix pencil, which is referred to as the doubling transformation, and then prove this kind of transformations still have the feature that is structure preserving, eigenspace preserving, and eigenvalue doubling. Finally, based on the nice properties of the doubling transformations we develop a unified convergence theory for the SDA algorithm by using only the knowledge from elementary theory.

To get a nice theory for (1.1), we need to add some conditions on the matrices A, B, C, and D.

For any matrices $A, B \in \mathbb{R}^{m\times n}$, we write $A \geq B (A > B)$ if $a_{ij} \geq b_{ij} (a_{ij} > b_{ij})$ for all i, j. We can then define positive matrices, nonnegative matrices, etc. A real square matrix A is called a Z-matrix if all its off-diagonal elements are nonpositive. It is clear that any Z-matrix A can be written as $sI - B$ with $B \geq 0$. A Z-matrix A is called an M-matrix if $s \geq \rho(B)$, where $\rho(\cdot)$ is the spectral radius. It is called a singular M-matrix if $s = \rho(B)$; it is called a nonsingular M-matrix if $s > \rho(B)$. Note that only nonsingular M-matrix defined here are called M-matrices in [7]. The slight change of definitions is made here for future convenience. The spectrum of a square matrix A will be denoted by $\sigma(A)$. The open left half-plane, the open right half-plane, the closed left half-plane, and the closed right half-plane will be denoted by $\mathbb{C}_<, \mathbb{C}_>, \mathbb{C}_\leq, \mathbb{C}_\geq$, respectively.

The paper is organized as follows. In section 2, we generalize the structure-preserving transformation of a symplectic pencil to a more common matrix pencil, and develop its properties. Besides, we give some basic results. In section 3, we do the convergence analysis of the SDA algorithm for solving the NARE based on the theory establized in section 2. In section 4, we report some numerical results.

2 Doubling transformation and some basic results

In [10, 13], the authors introduce a structure-preserving transformation of a symplectic pencil. First, we generalize this transformation to a more common matrix pencil, and develop its some basic properties.

Let $M - \lambda L \in \mathbb{R}^{(m+n)\times(m+n)}$ be a matrix pencil, and have the following structure

$$M = \begin{pmatrix} E & 0 \\ -H & I \end{pmatrix}, \qquad L = \begin{pmatrix} I & -G \\ 0 & F \end{pmatrix}, \tag{2.1}$$

where E, H, G, F are real matrices of sizes $n \times n, m \times n, n \times m, m \times m$, respectively. If $m = n, E = F^T, H^T = H, G^T = G$, then $M - \lambda L$ is a symplectic pencil.
Define

$$\mathcal{N}(M, L) = \{[M_*, L_*] : M_*, L_* \in \mathbb{R}^{(m+n)\times(m+n)}, rank[M_*, L_*] = m + n, [M_*, L_*] \begin{bmatrix} L \\ -M \end{bmatrix} \}.$$

Since rank$\begin{bmatrix} L \\ -M \end{bmatrix} \leq 2n$, it follows that $\mathcal{N}(M, L) \neq \emptyset$. For any given $[M_*, L_*] \in \mathcal{N}(M, L)$, define

$$\widehat{M} = M_* M, \qquad \widehat{L} = L_* L.$$

The transformation

$$M - \lambda L \longrightarrow \widehat{M} - \lambda \widehat{L}$$

is called a doubling transformation.

An important feature of this kind of transformations is that it is structure preserving, eigenspace preserving, and eigenvalue doubling, as shown in the following theorem.

Theorem 2.1. *Assume that the pencil $\widehat{M} - \lambda \widehat{L}$ is a doubling transformation of a pencil $M - \lambda L$ as defined as (2.1). Then we have*

(i) If $M \begin{bmatrix} U \\ V \end{bmatrix} = L \begin{bmatrix} U \\ V \end{bmatrix} S$, where $U \in \mathbb{R}^{n\times n}, V \in \mathbb{R}^{m\times n}$, and $S \in \mathbb{R}^{n\times n}$, then $\widehat{M} \begin{bmatrix} U \\ V \end{bmatrix} = \widehat{L} \begin{bmatrix} U \\ V \end{bmatrix} S^2$.

(ii) If the pencil $M - \lambda L$ has the Kronecker canonical form

$$WMZ = \begin{bmatrix} J_r & 0 \\ 0 & I_{2n-r} \end{bmatrix}, \qquad WLZ = \begin{bmatrix} I_r & 0 \\ 0 & N_{2n-r} \end{bmatrix},$$

where W, Z are nonsingular, J_r is a Jordan matrix corresponding to the finite eigenvalue of $M - \lambda L$, and N_{2n-r} a nilpotent Jordan matrix corresponding to the infinite eigenvalues of $M - \lambda L$, then there exists a nonsingular matrix $\widehat{W}$ such that

$$\widehat{W}\widehat{M}Z = \begin{bmatrix} {J_r}^2 & 0 \\ 0 & I_{2n-r} \end{bmatrix}, \qquad \widehat{W}\widehat{L}Z = \begin{bmatrix} I_r & 0 \\ 0 & N_{2n-r}^2 \end{bmatrix}.$$

(iii) There a matrix $[M_*, L_*] \in \mathcal{N}(M, L)$ can be constructed such that its corresponding doubling transformation $\widehat{M} - \lambda\widehat{L}$ is still the same structure as $M - \lambda L$.

Proof. The proof of propositions *(i)* and *(ii)* is exactly the same as that of (b), (c) of Theorem 2.1 (see [13]), respectively.

(iii) The proof is similar to (a) of Theorem 2.2 (see [13],the more detail see [11]). Here, we get

$$M_* = \begin{bmatrix} (I - GH)^{-1} & 0 \\ -F(I - HG)^{-1}H & I \end{bmatrix}, \qquad L_* = \begin{bmatrix} I & -E(I - GH)^{-1}G \\ 0 & F(I - HG)^{-1} \end{bmatrix}, \tag{2.2}$$

such that

$$M_*L = L_*M.$$

We then compute M_*M and L_*L to produce

$$\widehat{M} = M_*M = \begin{bmatrix} \widehat{E} & 0 \\ -\widehat{H} & I \end{bmatrix}, \qquad \widehat{L} = L_*L = \begin{bmatrix} I & -\widehat{G} \\ 0 & \widehat{F} \end{bmatrix}$$

where

$$\widehat{E} = E(I - GH)^{-1}E, \tag{2.3}$$
$$\widehat{H} = H + F(I - HG)^{-1}HE, \tag{2.4}$$
$$\widehat{G} = G + E(I - GH)^{-1}GF, \tag{2.5}$$
$$\widehat{F} = F(I - HG)^{-1}F. \tag{2.6}$$

It is clear that the resulting pencil $\widehat{M} - \lambda\widehat{L}$ has the same structure as the pencil $M - \lambda L$.

Remark 2.1. The proof of *(iii)* in Theorem 2.1 provided us with the well defined computation formula for constructing the special structure preserving doubling transformation, which is the base for us to derive the SDA algorithms for solving the nonsymmetric algebraic Riccati equation.

Before deriving the SDA algorithm for solving the NARE, we first give some basic results. For the convenience of discuss, let us define two $(m+n) \times (m+n)$ matrices $\mathcal{H}$ and $\mathcal{K}$ as follows:

$$\mathcal{H} = \begin{pmatrix} D & C \\ -B & -A \end{pmatrix}, \qquad \mathcal{K} = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix}.$$

Lemma 2.1. *[1] Given a Z-matrix $A \in \mathbb{R}^{n\times n}$. Then the following statements are equivalent:*

(a) A is a nonsingular M-matrix;

(b) $A^{-1} \geq 0$;

(c) $Av > 0$ holds for some vector $v > 0$;

(d) $\sigma(A) \subset \mathbb{C}_>$.

Lemma 2.2. *Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular M-matrix. If the elements of $B \in \mathbb{R}^{n\times n}$ satisfy the relations*

$$b_{ii} \geq a_{ii}, \quad a_{ij} \leq b_{ij} \leq 0, \quad i \neq j, \quad 1 \leq i, j \leq n.$$

then B is also a nonsingular M-matrix.

Lemma 2.3. *If $A, B \in \mathbb{R}^{n\times n}$ are nonsingular M-matrices satisfy $A \leq B$, then $A^{-1} \geq B^{-1}$.*

Lemma 2.4. *Let $A \geq 0, B \geq 0$, and satisfy $Ae > 0, Be > 0$, then $AB \geq 0$, and $(AB)e > 0$, where $e = (1, 1, \cdots, 1)^T$.*

Lemma 2.5. *If $\mathcal{K}$ is a nonsingular M-matrix, then its Schur complements $W = A - BD^{-1}C, V = D - CA^{-1}B$ are still nonsingular M-matrices.*

Proof. Since $\mathcal{K}$ is nonsingular M-matrix, we have $\mathcal{K}^{-1} \geq 0$. By directly computation, we get

$$\begin{aligned} \mathcal{K}^{-1} &= \begin{pmatrix} D & -C \\ -B & A \end{pmatrix}^{-1} = \begin{pmatrix} D & -C \\ 0 & W \end{pmatrix}^{-1} \begin{pmatrix} I & 0 \\ -BD^{-1} & I \end{pmatrix}^{-1} \\ &= \begin{pmatrix} D^{-1} & D^{-1}CW^{-1} \\ 0 & W^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -BD^{-1} & I \end{pmatrix} \\ &= \begin{pmatrix} V^{-1} & D^{-1}CW^{-1} \\ W^{-1}BD^{-1} & W^{-1} \end{pmatrix}. \end{aligned}$$

It is clear that $W^{-1} \geq 0, V^{-1} \geq 0$, thus the matrices W, V are nonsingular M-matrices.

Let the nonsymmetric algebraic Riccati equation

$$XBX - XA - DX + C = 0 \tag{2.7}$$

be defined as the dual equation of (1.1).

Theorem 2.2. *[14] If $\mathcal{K}$ is a nonsingular M-matrix, then (1.1) and (2.7) have minimal nonnegative solutions S_1 and S_2 satisfying the following equality:*

$$\begin{pmatrix} D & -C \\ B & -A \end{pmatrix} \begin{pmatrix} I & S_2 \\ S_1 & I \end{pmatrix} = \begin{pmatrix} I & S_2 \\ S_1 & I \end{pmatrix} \begin{pmatrix} G_1 & 0 \\ 0 & -G_2 \end{pmatrix}, \tag{2.8}$$

with G_1 and G_2 are nonsingular M-matrices.

Remark 2.1. *If $\mathcal{K}$ is a nonsingular M-matrix, S_1 and S_2 are the only matrices satisfying (2.8) with G_1 and G_2 being M-matrices.*

In the subsequent discuss, we always assume that $\mathcal{K}$ is a nonsingular M-matrix without special comment.

3 SDA Algorithm for Solving NARE

Assume that $X \geq 0$ be the minimal nonnegative solution of (1.1), from (2.8) that (1.1) can be rewritten as

$$\mathcal{H}\begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} R, \tag{3.1}$$

where $R = D - CX$.

The matrix R is a nonsingular M-matrix. Using Cayley transformation with some appropriate $\gamma > 0$, we can transform (3.1) into the following form

$$(\mathcal{H} - \gamma I)\begin{bmatrix} I \\ X \end{bmatrix} = (\mathcal{H} + \gamma I)\begin{bmatrix} I \\ X \end{bmatrix} S, \tag{3.2}$$

where $S = (R + \gamma I)^{-1}(R - \gamma I)$.

For simplification of presentation, we give some notations defined by

$$D_\gamma = D + \gamma I, \quad A_\gamma = A + \gamma I, \qquad W_\gamma = A_\gamma - BD_\gamma^{-1}C, \quad V_\gamma = D_\gamma - CA_\gamma^{-1}B,$$

where W_γ, V_γ are the Schur complements.

Let

$$L_1 = \begin{bmatrix} D_\gamma^{-1} & 0 \\ -BD_\gamma^{-1} & I \end{bmatrix}, \quad L_2 = \begin{bmatrix} I & 0 \\ 0 & -W_\gamma^{-1} \end{bmatrix}, \quad L_3 = \begin{bmatrix} I & D_\gamma^{-1}C \\ 0 & I \end{bmatrix}$$

Then, direct calculation gives to

$$L = L_3L_2L_1(\mathcal{H} + \gamma I) = \begin{bmatrix} I & -2\gamma D_\gamma^{-1}CW_\gamma^{-1} \\ 0 & I - 2\gamma W_\gamma^{-1} \end{bmatrix} = \begin{bmatrix} I & -G \\ 0 & F \end{bmatrix},$$

$$M = L_3L_2L_1(\mathcal{H} - \gamma I) = \begin{bmatrix} I - 2\gamma V_\gamma^{-1} & 0 \\ -2\gamma W_\gamma^{-1}BD_\gamma^{-1} & I \end{bmatrix} = \begin{bmatrix} E & 0 \\ -H & I \end{bmatrix},$$

where $G = 2\gamma D_\gamma^{-1}CW_\gamma^{-1}, \quad F = I - 2\gamma W_\gamma^{-1}, \quad E = I - 2\gamma W_\gamma^{-1}, \quad H = 2\gamma W_\gamma^{-1}BD_\gamma^{-1}$. Then (3.2) can be transformed into the following form

$$M\begin{bmatrix} I \\ X \end{bmatrix} = L\begin{bmatrix} I \\ X \end{bmatrix} S, \tag{3.3}$$

and the pencil $M - \lambda L$ is the same structure as the pencil (2.1). Therefore, applying the doubling transformation defined by (2.3)-(2.6) repeatedly gives rise to the following structure-preserving doubling algorithm:

Algorithm SDA.

$$\begin{aligned}
&E_0 = E, H_0 = H, G_0 = G, F_0 = F,\\
&E_{k+1} = E_k(I - G_kH_k)^{-1}E_k,\\
&H_{k+1} = H_k + F_k(I - H_kG_k)^{-1}H_kE_k,\\
&G_{k+1} = G_k + E_k(I - G_kH_k)^{-1}G_kF_k,\\
&F_{k+1} = F_k(I - H_kG_k)^{-1}F_k.
\end{aligned}$$

Remark 3.1. *Of course, to ensure that this iteration is well defined, the matrices* $I - G_kH_k, I - H_kG_k$ *must be nonsingular for all* k*. Below we shall prove that this conditions can be guaranteed for appropriate* $\gamma > 0$*.*

Now we establish the convergence theory of Algorithm SDA based on Theorem 2.1. The main results are listed in the following theorem.

Theorem 3.1. *Let* $X, Y \geq 0$ *be the minimal nonnegative solutions of*

$$XCX - XD - AX + B = 0, \tag{3.4}$$

$$YBY - YA - DY + C = 0, \tag{3.5}$$

respectively, where A, B, C, D *are defined as (1.1). Let*

$$S = (R + \gamma I)^{-1}(R - \gamma I), \quad T = (\Lambda + \gamma I)^{-1}(\Lambda - \gamma I), \tag{3.6}$$

where $R = D - CX, \Lambda = A - BY$ *are nonsingular* M*-matrices. If* $\gamma > 0$ *makes the following conditions*

$$E_0 \leq 0, F_0 \leq 0, E_0e < 0, F_0e < 0, \tag{3.7}$$

$$S \leq 0, T \leq 0, Se < 0, Te < 0, \tag{3.8}$$

$$I - G_0H_0, I - H_0G_0 \quad \text{are nonsingular } M\text{-matrices}, \tag{3.9}$$

be satisfied, where $e = (1, 1, \cdots, 1)^T$*. Then the matrices* E_k, H_k, G_k*, and* F_k *generated by Algorithm SDA satisfy that*

(*i*) $E_k = (I - G_kX)S^{2^k}, k = 0, 1, 2, \cdots, \quad E_k \geq 0, k = 1, 2, \cdots;$

(*ii*) $F_k = (I - H_kY)T^{2^k}, k = 0, 1, 2, \cdots, \quad F_k \geq 0, k = 1, 2, \cdots;$

(*iii*) $I - H_kG_k, I - G_kH_k$ *are nonsingular* M*-matrices;*

(*iv*) $0 \leq H_k \leq H_{k+1} \leq X$ *and* $X - H_k = F_kS^{2^k};$ (3.10)

(*v*) $0 \leq G_k \leq G_{k+1} \leq Y$ *and* $Y - G_k = E_kT^{2^k};$ (3.11)

Proof. (Apply Mathematical Induction.) Denote

$$M_k = \begin{bmatrix} E_k & 0 \\ -H_k & I \end{bmatrix}, \qquad L_k = \begin{bmatrix} I & -G_k \\ 0 & F_k \end{bmatrix}$$

. For $k = 1$, since $-E_0 \geq 0, -F_0 \geq 0, -E_0 e > 0, -F_0 > 0$, and$H_0 \geq 0, G_0 \geq 0, I - H_0 G_0, I - G_0 H_0$ are nonsingular matrices, it follows that E_1, H_1, G_1, F_1 are all well defined, by Lemma 2.1,2.4, we have

$$\begin{aligned}
&E_1 = E_0(I - G_0H_0)^{-1}E_0 = (-E_0)(I - G_0H_0)^{-1}(-E_0) \geq 0, E_1 e > 0, \\
&F_1 = F_0(I - H_0G_0)^{-1}F_0 = (-F_0)(I - H_0G_0)^{-1}(-F_0) \geq 0, F_1 e > 0, \\
&0 \leq H_0 \leq H_1, \\
&0 \leq G_0 \leq G_1.
\end{aligned}$$

Since (3.4)(3.5) imply that

$$\begin{aligned}
M_0 \begin{bmatrix} I \\ X \end{bmatrix} &= L_0 \begin{bmatrix} I \\ X \end{bmatrix} S, \\
M_0 \begin{bmatrix} Y \\ I \end{bmatrix} T &= L_0 \begin{bmatrix} Y \\ I \end{bmatrix},
\end{aligned}$$

where S, T are defined by (3.6), applying *(i)* of Theorem 2.1, we get

$$\begin{bmatrix} E_1 & 0 \\ -H_1 & I \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I & -G \\ 0 & F_1 \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} S^2, \tag{3.12}$$

$$\begin{bmatrix} E_1 & 0 \\ -H_1 & I \end{bmatrix} \begin{bmatrix} Y \\ I \end{bmatrix} T^2 = \begin{bmatrix} I & -G \\ 0 & F_1 \end{bmatrix} \begin{bmatrix} Y \\ I \end{bmatrix}, \tag{3.13}$$

Equating the blocks of (3.12) gives rise to

$$X - H_1 = F_1 X S^2, \tag{3.14}$$

$$E_1 = (I - G_1 X) S^2, \tag{3.15}$$

$$Y - G_1 = E_1 Y T^2, \tag{3.16}$$

$$F_1 = (I - H_1 Y) T^2. \tag{3.17}$$

Note that $S, T \leq 0, Se < 0, Te < 0$, implies that $S^2 \geq 0, T^2 \geq 0, S^2 e > 0, T^2 e > 0$, together with (3.14) and (3.16) we get

$$H_1 < X, \qquad G_1 < Y. \tag{3.18}$$

From (3.15) we can derive that $E_1 e = (I - G_1 X) S^2 e > 0$, combining with Lemma 2.1, we have $I - G_1 X$ is a nonsingular. Using (3.18), we get $I - G_1 X < I - G_1 H_1$, therefore,

$$I - G_1 H_1 \quad \text{is a nonsingular } M\text{-matrix.} \tag{3.19}$$

Similarly, $I - H_1G_1$ is a nonsingular matrix. Thus, we have proved that this Theorem is true for $k = 1$.

Next, assume that the Theorem is true for all positive integers less or equal to k. Consider the case of $k+1$. Since $I - H_kG_k, I - G_kH_k$ are nonsingular M-matrices, it follows that $E_{k+1}, H_{k+1}, G_{k+1}, F_{k+1}$ are all well defined, and satisfying

$$\begin{aligned} &E_{k+1} \geq 0, \quad E_{k+1}e > 0, \\ &F_{k+1} \geq 0, \quad F_{k+1}e > 0, \\ &0 \leq H_k \leq H_{k+1}, \\ &0 \leq G_k \leq G_{k+1}. \end{aligned}$$

On the other hand, since $M_{j+1} - \lambda L_{j+1}$ is a doubling transformation of $M_j - \lambda L_j$ for $j = 0, 1, \cdots, k$, by using *(i)* of Theorem 2.1 $k+1$ times, we get

$$M_{k+1} \begin{bmatrix} I \\ X \end{bmatrix} = L_{k+1} \begin{bmatrix} I \\ X \end{bmatrix} S^{2^{k+1}}. \tag{3.20}$$

Equating the blocks of (3.20) yields that

$$\begin{aligned} X - H_{k+1} &= F_{k+1} S^{2^{k+1}}, \\ E_{k+1} &= (I - G_{k+1}X) S^{2^{k+1}}, \end{aligned}$$

following the same lines as the proof of (3.18) and (3.19) it can be proved that

$$\begin{aligned} &X - H_{k+1} \leq 0, \\ &I - G_{k+1}H_{k+1} \quad \text{is a nonsingular } M\text{-matrix.} \end{aligned}$$

Similarly, we also get

$$\begin{aligned} &X - G_{k+1} \leq 0, \\ &I - H_{k+1}G_{k+1} \quad \text{is a nonsingular } M\text{-matrix.} \end{aligned}$$

This shows that the theorem is also true for integer $k+1$. By induction principle the theorem is true for all positive integers.

Remark 3.2. *In Theorem 3.1, the $\gamma > 0$ that satisfies the conditions (3.7)-(3.9) must be existed, we will prove this fact in the subsequent appendix.*

Noting that $R = D - CX, \Lambda = A - YB$ are nonsingular M-matrices, it follows that $\sigma(R), \sigma(\Lambda) \subset \mathbb{C}_{>0}$, thus $\rho(S) < 1, \rho(T) < 1$ for $\gamma > 0$. In addition, it is well know that $0 \leq A \leq B$ implies that $\|A\|_1 \leq \|B\|_1$. Consequently, from Theorem 3.1 we immediately get the following convergence result of Algorithm SDA.

Corollary 3.1. *Under the hypothesis of Theorem 3.1, then we have*

$$(i)\quad \|E_k\|_1 \le (1+\|YX\|_1)\|S^{2^k}\|_1 \longrightarrow 0, as \quad k \to 0;$$
$$(ii)\quad \|F_k\|_1 \le (1+\|XY\|_1)\|T^{2^k}\|_1 \longrightarrow 0, as \quad k \to 0;$$
$$(iii)\quad \|X-H_k\|_1 \le (\|X\|_1+\|XY\|_1\|X\|_1)\|S^{2^k}\|_1\|T^{2^k}\|_1 \longrightarrow 0, as \quad k \to 0;$$
$$(iv)\quad \|Y-G_k\|_1 \le (\|Y\|_1+\|YX\|_1\|Y\|_1)\|S^{2^k}\|_1\|T^{2^k}\|_1 \longrightarrow 0, as \quad k \to 0.$$

Remark 3.3. *From the above corollary, we can get the following results: The algorithm SDA is quadratically convergent, and* $\lim_{k\to\infty} H_k = X, \lim_{k\to\infty} G_k = Y$.

4 Numerical examples

For the Tables in the following examples, data for various methods are listed in rows with obvious heading. The heading "NW" is for the Newton's iteration method [7], and "SDA" stands for our SDA algorithm. "IT" denotes the number of iteration, "CPU" denotes the time, "RES" denotes the norm $\|\widetilde{X}C\widetilde{X} - \widetilde{X}D - A\widetilde{X} + B = 0\|_F$, where $\widetilde{X}$ is the approximate solution of (1.1).

All computation were performed using MATLAB/Version on Pentium. The machine precision is 2.22×10^{-16}.

Example 4.1. *We generate (and save) a random* 200×200 *matrix* R *with no zero elements using* $rand(200, 200)$ *in MATLAB 6.5. Let* $W = diag(Re) - R$*, where* $e = (1, 1, \cdots, 1) \in \mathbb{R}^{200}$*. So* W *is a singular* M*-matrix with no zero elements. We introduce a real parameter* κ *and let*

$$\kappa I + W = \begin{pmatrix} D & -C \\ -B & A \end{pmatrix},$$

where the matrices A, B, C, D *are all* 100×100*. The existence of a positive solution of (1.1) is guaranteed for* $\alpha \ge 0$*. In Table 1 we have recorded the numerical results for three values of* κ*.*

Example 4.2. *The matrices are taken as follows:*

$$A = D = \begin{pmatrix} 3 & -1 & & \\ & 3 & \ddots & \\ & & \ddots & -1 \\ -1 & & & 3 \end{pmatrix} \in \mathbb{R}^{n\times n}, \qquad B = C = I_n.$$

The numerical results see Table 2.

Table 1: Numerical results for Example 4.1

	$\kappa=0$		$\kappa=5$		$\kappa=10$	
method	NW	SDA	NW	SDA	NW	SDA
IT	13	13	5	5	4	4
ERR	1.11E-12	2.26E-13	3.10E-13	1.68E-13	3.67E-13	1.06E-13
CPU	3.75	0.39	1.61	0.17	1.34	0.16

Table 2: Numerical results for Example 4.2

	$n=64$		$n=128$		$\kappa=256$	
method	NW	SDA	NW	SDA	NW	SDA
IT	3	3	3	3	3	3
ERR	4.06E-14	3.04E-16	6.22E-14	3.04E-16	1.16E-13	3.04E-16
CPU	0.34	0.047	2.39	0.25	20.97	1.84

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