TANNAKIAN DUALITY FOR ANDERSON-DRINFELD MOTIVES AND ALGEBRAIC INDEPENDENCE OF CARLITZ LOGARITHMS

MATTHEW A. PAPANIKOLAS

Abstract. We develop a theory of Tannakian Galois groups for t-motives and relate this to the theory of Frobenius semilinear difference equations. We show that the transcendence degree of the period matrix associated to a given t-motive is equal to the dimension of its Galois group. Using this result we prove that Carlitz logarithms of algebraic functions that are linearly independent over the rational function field are algebraically independent.

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1. Introduction

1.1. Periods of t-motives.

1.1.1. *Notation.* Let $\mathbb{F}_q$ be the field of q elements, where q is a power of a prime p. Let $k := \mathbb{F}_q(\theta)$, where θ is transcendental over $\mathbb{F}_q$, and define an absolute value $|\cdot|_\infty$ at the infinite place of k so that $|\theta|_\infty = q$. Let $k_\infty := k((1/\theta))$ be the ∞-adic completion of k, let $\overline{k_\infty}$ be an algebraic closure, let $\mathbb{K}$ be the ∞-adic completion of $\overline{k_\infty}$, and let $\overline{k}$ be the algebraic closure of k in $\mathbb{K}$.

1.1.2. *Anderson t-motives.* Let t be a variable over $\mathbb{F}_q$ that is independent from θ, and let $\overline{k}[t;\sigma]$ be the ring of polynomials in t and σ over $\overline{k}$ subject to the relations

$$ct = tc, \quad \sigma t = t\sigma, \quad \sigma c = c^{1/q}\sigma, \quad c \in \overline{k}.$$

An Anderson t-motive is a left $\overline{k}[t;\sigma]$-module M that is free and finitely generated as both a left $\overline{k}[t]$-module and as a left $\overline{k}[\sigma]$-module and that satisfies $(t-\theta)^n\mathsf{M} \subseteq \sigma\mathsf{M}$ for all n sufficiently large (see §3.4). Anderson t-motives were originally defined in [2], where they were called "dual t-motives."

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1.1.3. *Rigid analytic triviality.* We let $\mathbb{T} := \mathbb{K}\{t\}$ be the Tate algebra of power series in $\mathbb{K}[\![t]\!]$ that are convergent on the closed unit disk in $\mathbb{K}$, and let $\mathbb{L} \subseteq \mathbb{K}(\!(t)\!)$ be its fraction field. Let $\mathbb{E}$ be the subring of $\mathbb{T}$ consisting of power series that are everywhere convergent and whose coefficients lie in a finite extension of k_∞. Finally, for a Laurent series $f = \sum_i a_i t^i \in \mathbb{K}(\!(t)\!)$ and an integer $n \in \mathbb{Z}$, we set $f^{(n)} := \sum_i a_i^{q^n} t^i$.

If M is an Anderson t-motive and $\mathsf{m} \in \mathrm{Mat}_{r\times 1}(\mathsf{M})$ has entries comprising a $\overline{k}[t]$-basis of M, then there is a matrix $\Phi \in \mathrm{Mat}_r(\overline{k}[t])$ representing multiplication by σ on M so that

$$\sigma \mathsf{m} = \Phi \mathsf{m}$$

and $\det \Phi = c(t-\theta)^s$ for some $c \in \overline{k}^\times$ and $s \geq 1$. The Anderson t-motive is rigid analytically trivial (see Proposition 3.4.7) if there is a matrix $\Psi \in \mathrm{GL}_r(\mathbb{T})$ so that

$$\Psi^{(-1)} = \Phi\Psi.$$

It can be shown that the entries of Ψ are in fact in $\mathbb{E}$ (see Proposition 6.1.3).

1.1.4. *Connection with t-modules.* The category of rigid analytically trivial Anderson t-motives is equivalent to the category of uniformizable abelian t-modules defined over $\overline{k}$, as in [1]. For a given Anderson t-motive M and associated t-module E, there is an explicit connection

$$\text{periods of } E \quad \longleftrightarrow \quad \overline{k}\text{-linear combinations of entries of } \Psi(\theta)^{-1}.$$

The details of this relationship will be the subject of a future paper with Anderson, but examples are already seen in §3.3 for the Carlitz motive (see also S. K. Sinha [26, §5.2] for examples involving special values of the function field Γ-function).

1.1.5. *Remarks on t-motive terminology.* G. Anderson introduced t-motives in [1]. Later in [2] dual t-motives, which had several technical advantages, were introduced. The algebraic properties of these two types of t-motives are essentially the same, and the two categories are anti-equivalent to each other. In this paper we will follow the dual t-motive point of view only, and throughout we refer to them as Anderson t-motives. In the following paragraph we discuss a third type of t-motive, defined properly in §3.4, which are our primary objects of study.

1.1.6. *Tannakian category of t-motives.* In §3.4 we show that the category of rigid analytically trivial Anderson t-motives up to isogeny embeds as a full subcategory of a neutral Tannakian category $\mathcal{T}$ over $\mathbb{F}_q(t)$. Objects in $\mathcal{T}$ are called simply t-motives, and throughout the paper the term "t-motive" will refer exclusively to an object in $\mathcal{T}$. In particular, from this standpoint all t-motives are rigid analytically trivial.

By Tannakian duality, for each object M in $\mathcal{T}$, the Tannakian subcategory $\mathcal{T}_M$ generated by M satisfies an equivalence of categories

$$\mathcal{T}_M \approx \mathbf{Rep}(\Gamma_M, \mathbb{F}_q(t)),$$

where $\mathbf{Rep}(\Gamma_M, \mathbb{F}_q(t))$ is the category of finite dimensional representations over $\mathbb{F}_q(t)$ of some algebraic subgroup $\Gamma_M \subseteq \mathrm{GL}_r$ defined over $\mathbb{F}_q(t)$ (see §3.5). The group Γ_M is called the Galois group of M.

It should be noted that R. Pink [21] has defined a category $\mathcal{H}$ of mixed Hodge structures for function fields that is a neutral Tannakian category over $\mathbb{F}_q(t)$. He showed that the category of rigid analytically trivial Anderson t-motives that are also "mixed" embeds as a full subcategory of $\mathcal{H}$. It would be interesting to investigate the relationships among Pink's Hodge structures, the t-motives defined in this paper, and their associated Galois groups. In the end our category of t-motives is best suited for our transcendence applications, so we do not pursue further here the connections with Pink's work. See also D. Goss [13] for additional comparisons between t-motives and motives over $\mathbb{Q}$.

The following is the main theorem of this paper (restated later as Theorem 6.2.2).

Theorem 1.1.7. *Let M be a t-motive, and let Γ_M be its Galois group. Suppose that $\Phi \in \mathrm{GL}_r(\overline{k}(t)) \cap \mathrm{Mat}_r(\overline{k}[t])$ represents multiplication by σ on M and that $\det \Phi = c(t-\theta)^s$, $c \in \overline{k}^{\times}$. Let Ψ be a rigid analytic trivialization of Φ in $\mathrm{GL}_r(\mathbb{T}) \cap \mathrm{Mat}_r(\mathbb{E})$. Finally, let L be the subfield of $\overline{k_\infty}$ generated over $\overline{k}$ by the entries of $\Psi(\theta)$. Then*

$$\mathrm{tr.\,deg}_{\overline{k}}\, L = \dim \Gamma_M.$$

1.1.8. *Grothendieck's conjecture.* In light of §1.1.4, the statement of Theorem 1.1.7 can be thought of as a function field version of Grothendieck's conjecture on periods of algebraic varieties. For an abelian variety A over $\overline{\mathbb{Q}}$ of dimension d, let P be the period matrix of A that represents an isomorphism between $H^1(A(\mathbb{C}),\mathbb{Q})\otimes_{\mathbb{Q}}\mathbb{C}$ and $H^1_{\mathrm{DR}}(A/\mathbb{C})$, with basis defined over $\overline{\mathbb{Q}}$. Grothendieck's conjecture is that

$$\mathrm{tr.\,deg}_{\overline{\mathbb{Q}}}\, \overline{\mathbb{Q}}(P) = \dim \mathrm{MT}(A),$$

where $\mathrm{MT}(A)$ is the Mumford-Tate group of A and is an algebraic subgroup of $\mathrm{GL}_{2d} \times \mathbb{G}_{\mathrm{m}}$ over $\mathbb{Q}$. P. Deligne [11, Cor. I.6.4] has proved that the dimension of $\mathrm{MT}(A)$ is an upper bound for the transcendence degree. Conjecturally the Mumford-Tate group is isomorphic to the motivic Galois group of the motive $h_1(A) \oplus \mathbb{Q}(1)$ over $\mathbb{Q}$. More generally Grothendieck's period conjecture states that if X is a smooth variety over $\overline{\mathbb{Q}}$, then

$$\mathrm{tr.\,deg}_{\overline{\mathbb{Q}}}\, \overline{\mathbb{Q}}(P(X)) = \dim \Gamma_X^{\mathrm{mot}},$$

where $P(X)$ is the period matrix of X and Γ_X^{mot} is the motivic Galois group of X over $\mathbb{Q}$. It should be pointed out that by work of C. Bertolin [5] many standard transcendence conjectures over $\overline{\mathbb{Q}}$, such as Schanuel's conjecture, follow from expanded versions of Grothendieck's period conjecture.

1.2. **Algebraic independence of Carlitz logarithms.** One application of Theorem 1.1.7 is a characterization of algebraic relations over $\overline{k}$ of Carlitz logarithms of algebraic numbers.

1.2.1. *Carlitz exponential.* The Carlitz exponential is the power series

$$\exp_C(z) := z + \sum_{i=1}^{\infty} \frac{z^{q^i}}{(\theta^{q^i}-\theta)(\theta^{q^i}-\theta^q)\cdots(\theta^{q^i}-\theta^{q^{i-1}})}.$$

As is well known (see [14, Ch. 3], [29, §2.5]), the function defined by $\exp_C$ converges everywhere on $\mathbb{K}$, is $\mathbb{F}_q$-linear, and has kernel $\mathbb{F}_q[\theta]\,\widetilde{\pi}$, where

$$\widetilde{\pi} := \theta \sqrt[q-1]{-\theta} \prod_{i=1}^{\infty} \left(1-\theta^{1-q^i}\right)^{-1} \in k_\infty(\sqrt[q-1]{-\theta})^{\times}.$$

The Carlitz exponential also satisfies the functional equation

$$\exp_C(\theta z) = \theta \exp_C(z) + \exp_C(z)^q, \quad z \in \mathbb{K}.$$

Moreover, this functional equation induces an exact sequence of $\mathbb{F}_q[t]$-modules,

$$0 \to \mathbb{F}_q[\theta]\,\widetilde{\pi} \to \mathbb{K} \to \mathfrak{C}(\mathbb{K}) \to 0,$$

where $\mathfrak{C}(\mathbb{K})$ is the $\mathbb{F}_q[t]$-module of $\mathbb{K}$-valued points on the Carlitz module $\mathfrak{C}$ (see §3.4.4) and where t acts by multiplication by θ on the first two terms. The number $\widetilde{\pi}$ is called the Carlitz period.

1.2.2. *Carlitz logarithm.* The Carlitz logarithm is the inverse of $\exp_C(z)$,

$$\log_C(z) := z + \sum_{i=1}^{\infty} \frac{z^{q^i}}{(\theta - \theta^q)(\theta - \theta^{q^2}) \cdots (\theta - \theta^{q^i})},$$

which as a function on $\mathbb{K}$ converges for all $z \in \mathbb{K}$ with $|z|_\infty < |\theta|_\infty^{q/(q-1)}$. The Carlitz logarithm is $\mathbb{F}_q$-linear and satisfies the functional equation

$$\theta \log_C(z) = \log_C(\theta z) + \log_C(z^q),$$

for all $z \in \mathbb{K}$ where all three terms converge.

1.2.3. *Linear forms in Carlitz logarithms.* We recall a theorem of J. Yu. Suppose $\lambda_1, \dots, \lambda_r \in \mathbb{K}$ satisfy $\exp_C(\lambda_i) \in \overline{k}$ for each $i = 1, \dots, r$. As in the previous section there are many potential k-linear relations among $\lambda_1, \dots, \lambda_r$. However, Yu proved that these are the only possible linear relations over $\overline{k}$ in the following function field analogue of Baker's theorem on linear forms in logarithms.

Theorem 1.2.4 (Yu [31, Thm. 4.3])**.** *Suppose $\lambda_1, \dots, \lambda_r \in \mathbb{K}$ satisfy $\exp_C(\lambda_i) \in \overline{k}$ for $i = 1, \dots, r$. If $\lambda_1, \dots, \lambda_r$ are linearly independent over k, then the numbers $1, \lambda_1, \dots, \lambda_r$ are linearly independent over $\overline{k}$.*

Yu's result is an application of his far reaching Theorem of the Sub-t-module [31, Thm. 0.1], which characterizes all $\overline{k}$-linear relations among logarithms of points in $\overline{k}$ on general t-modules. Transcendence results about the Carlitz periods and Carlitz logarithms go back to Carlitz and Wade in the 1940's. For detailed accounts of the history of transcendence results for Drinfeld modules, including Yu's theorem, see W. D. Brownawell [9] and D. S. Thakur [29, Ch. 10].

1.2.5. *Algebraic independence of Carlitz logarithms.* In characteristic 0, Baker's theorem on linear forms in natural logarithms of algebraic numbers is best known. In the situation of Carlitz logarithms we use Theorem 1.1.7 to prove the following theorem (restated later as Theorem 7.4.2).

Theorem 1.2.6. *Let $\lambda_1, \dots, \lambda_r \in \mathbb{K}$ satisfy $\exp_C(\lambda_i) \in \overline{k}$ for each $i = 1, \dots, r$. If $\lambda_1, \dots, \lambda_r$ are linearly independent over k, then they are algebraically independent over $\overline{k}$.*

It should be noted that, using Mahler's method, L. Denis [12] has proved the special case of this theorem where $\lambda_1, \dots, \lambda_r$ are restricted to values of $\log_C$ on elements of $\mathbb{F}_q(\theta^{1/e})$, $e \geq 1$, of degree in θ less than $q/(q-1)$.

1.3. Methods of proof.

1.3.1. *σ-semilinear difference equations.* The category of t-motives is a certain full subcategory in the category of left $\overline{k}(t)[\sigma, \sigma^{-1}]$-modules which are finite dimensional as $\overline{k}(t)$-vector spaces. To every t-motive M one can associate a matrix $\Phi \in \mathrm{GL}_r(\overline{k}(t))$ representing multiplication by σ and a rigid analytic trivialization $\Psi \in \mathrm{GL}_r(\mathbb{L})$ so that $\Psi^{(-1)} = \Phi\Psi$. Here recall that $\mathbb{L}$ is the fraction field of the Tate algebra $\mathbb{T}$. Thus the columns of Ψ satisfy a system of σ-semilinear difference equations in the sense of [23], and we develop the theory of such equations in this context in §4. In spirit this theory is close to the Galois theory of differential equations and difference equations in characteristic 0 [4], [6], [10], [19], [22], [23], [24].

In §4 we develop the Picard-Vessiot theory for certain kinds of difference equations for σ and construct their difference Galois groups (see Theorem 4.4.6). However, careful attention must be paid to the fact that the fixed field of σ in $\overline{k}(t)$ is $\mathbb{F}_q(t)$. The Galois theory of difference equations developed by M. van der Put and M. F. Singer [23] is quite useful here, but it does not completely apply because they fundamentally use at several occasions that the field of fixed elements under the difference automorphism is algebraically closed. On the one hand, because the fixed field of σ in $\mathbb{L}$ is also $\mathbb{F}_q(t)$, the Galois groups we construct are themselves defined over $\mathbb{F}_q(t)$. However, that $\mathbb{F}_q(t)$ is not algebraically closed nor even perfect presents several difficulties because in general the $\mathbb{F}_q(t)$-valued points of the Galois group need not be dense and the group itself need not be a priori smooth.

1.3.2. *t-motives and difference Galois groups.* Given a t-motive M of dimension r over $\overline{k}(t)$, the difference Galois group Γ_Ψ is a subgroup of GL_r over $\mathbb{F}_q(t)$. Let Σ_Ψ be the $\overline{k}(t)$-subalgebra of $\mathbb{L}$ generated by the entries of Ψ and $\det(\Psi)^{-1}$, and let Λ_Ψ be its fraction field. The field $\mathbb{L}$ is naturally a left $\overline{k}(t)[\sigma,\sigma^{-1}]$-module via the automorphism $\sigma : f \mapsto f^{(-1)}$, and Σ_Ψ and Λ_Ψ are both σ-invariant. Then

$$\Gamma_\Psi(\mathbb{F}_q(t)) \cong \mathrm{Aut}^\sigma(\Sigma_\Psi/\overline{k}(t)),$$

where the right-hand side is the group of automorphisms of Σ_Ψ over $\overline{k}(t)$ that commute with σ. Moreover, this identification is compatible with base extensions of $\mathbb{F}_q(t)$ (see Proposition 4.4.2).

We work out an explicit description of $\Gamma_\Psi(\overline{\mathbb{F}_q(t)})$ in §5.2, and, using crucially that $\mathbb{L}$ is a separable extension of $\mathbb{F}_q(t)$ and that $\overline{k}(t)$ is algebraically closed in $\mathbb{L}$, we show that Γ_Ψ has the following properties:

- $\dim \Gamma_\Psi = \mathrm{tr.\,deg}_{\overline{k}(t)}\, \Lambda_\Psi$, (Theorem 5.2.12(a));
- Γ_Ψ is smooth over $\mathbb{F}_q(t)$ (Theorem 5.2.12(b));
- The elements of Λ_Ψ fixed by $\Gamma_\Psi(\overline{\mathbb{F}_q(t)})$ are precisely $\overline{k}(t)$ (Theorem 5.3.2).

These properties are essential for proving in Theorem 5.4.10 that

$$\Gamma_\Psi \cong \Gamma_M,$$

where Γ_M is the Galois group associated to M by Tannakian duality.

1.3.3. *The proof of Theorem 1.1.7.* The primary vehicle for proving this theorem is a $\overline{k}$-linear independence criterion from [2, Thm. 3.1.1]. It is stated here in Theorem 6.1.1. We apply this criterion to the rigid analytic trivializations of tensor powers of M so as to compare the dimensions of the $\overline{k}$-span of monomials of the entries of $\Psi(\theta)$ of a given degree and the $\overline{k}(t)$-span of monomials in the entries of Ψ. Ultimately we show that

$$\mathrm{tr.\,deg}_{\overline{k}}\, L = \mathrm{tr.\,deg}_{\overline{k}(t)}\, \Lambda_\Psi,$$

the latter of which is the same as the dimension of Γ_M.

1.3.4. *Carlitz logarithms.* For $\alpha_1, \dots, \alpha_r \in \overline{k}^\times$ with $|\alpha_i|_\infty < |\theta|_\infty^{q/(q-1)}$ for $i = 1, \dots, r$, we define a t-motive X so that the field generated over $\overline{k}$ by the entries of its rigid analytic trivialization Ψ evaluated at $t = \theta$ is precisely

$$L = \overline{k}(\Psi(\theta)) = \overline{k}(\widetilde{\pi}, \log_C(\alpha_1), \dots, \log_C(\alpha_r)).$$

Moreover, we show that arbitrary logarithms are k-linear combinations of logarithms of this form in a precise way. We determine a set of defining equations of the Galois group Γ_X of X in Theorem 7.3.2 each of which is a linear polynomial over $\overline{k}(t)$. These linear relations each produce a k-linear relation on the logarithms and $\widetilde{\pi}$. We then use Theorem 1.1.7 to show that all algebraic relations must arise from these relations.

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2. Notation and preliminaries

2.1. **Table of symbols.**

$R[\![x]\!]$	= power series in x with coefficients in a ring R
$R(\!(x)\!)$	= Laurent series in x with coefficients in a ring R
$R\langle\!\langle x\rangle\!\rangle$	= generalized power series ring in x with coefficients in a ring R
$\mathbb{F}_q$	= finite field with $q = p^m$ elements
k	= $\mathbb{F}_q(\theta)$ = rational functions in the variable θ over $\mathbb{F}_q$
k_∞	= $\mathbb{F}_q(\!(1/\theta)\!)$ = ∞-adic completion of k
$\overline{k_\infty}$	= algebraic closure of k_∞
ζ_θ	= a fixed $(q-1)$-th root of $-\theta$ in $\overline{k_\infty}$
$\mathbb{K}$	= completion of $\overline{k_\infty}$
$\overline{k}$	= algebraic closure of k in $\mathbb{K}$
$\mathbb{T}$	= $\mathbb{K}\{t\}$ = ring of restricted power series; series in $\mathbb{K}[\![t]\!]$ that converge on the closed unit disk $\lvert t\rvert_\infty \le 1$
$\mathbb{L}$	= fraction field of $\mathbb{T}$
$M^\vee$	= dual vector space of a vector space M
$\mathbf{Vec}(F)$	= category of finite dimensional vector spaces over a field F
$\mathbf{Rep}(\Gamma, F)$	= for a field F the category of finite dimensional F-representations of an affine group scheme Γ over F

2.2. **Preliminaries.**

2.2.1. *Norms.* We let $|\cdot|_\infty$ denote a fixed ∞-adic norm on $\mathbb{K}$. For a matrix $E \in \mathrm{Mat}_{r\times s}(\mathbb{K})$, we set $|E|_\infty = \sup |E_{ij}|_\infty$. For matrices E and F, we observe that $|E+F|_\infty \le \max(|E|_\infty, |F|_\infty)$ and $|EF|_\infty \le |E|_\infty \cdot |F|_\infty$.

2.2.2. *Generalized power series.* Let F be a field of characteristic p. For a formal series $f := \sum_{i\in\mathbb{Q}} a_i t^i$ with $a_i \in F$, we let $\mathrm{Supp}(f) := \{i \in \mathbb{Q} \mid a_i \neq 0\}$. We let $F\langle\!\langle t\rangle\!\rangle$ be the set of such series for which $\mathrm{Supp}(f)$ is a well-ordered subset of $\mathbb{Q}$. This condition implies that $F\langle\!\langle t\rangle\!\rangle$ is a field under the natural addition and multiplication of these series so that $t^i t^j = t^{i+j}$ (see P. Ribenboim [25, §2]). If F is algebraically closed, then $F\langle\!\langle t\rangle\!\rangle$ is algebraically closed [25, §5]. If F is a perfect field, then $F\langle\!\langle t\rangle\!\rangle$ is also perfect.

It should be noted that, when F is algebraically closed, $F\langle\!\langle t\rangle\!\rangle$ is not the algebraic closure of the Laurent series field $F(\!(t)\!)$. For an explicit description of the field $\overline{F(\!(t)\!)} \subseteq F\langle\!\langle t\rangle\!\rangle$, the reader is directed to K. Kedlaya [16].

By considering the inclusions

$$\mathbb{F}_q(t) \subseteq \overline{k}(t) \subseteq \mathbb{K}(\!(t)\!) \subseteq \mathbb{K}\langle\!\langle t\rangle\!\rangle,$$

we fix once and for all the inclusions of algebraically closed fields

$$\overline{\mathbb{F}_q(t)} \subseteq \overline{k(t)} \subseteq \overline{\mathbb{K}(\!(t)\!)} \subseteq \mathbb{K}\langle\!\langle t\rangle\!\rangle.$$

These containments will be the rule throughout the paper unless otherwise stated.

2.2.3. *Entire functions.* A power series $f = \sum_{i=0}^{\infty} a_i t^i \in \mathbb{K}[\![t]\!]$ that satisfies

$$\lim_{i\to\infty} \sqrt[i]{|a_i|_\infty} = 0$$

and

$$[k_\infty(a_0, a_1, a_2, \ldots) : k_\infty] < \infty,$$

is an *entire power series*. As a function of t, such a power series f converges on all of $\mathbb{K}$, and, when restricted to $\overline{k_\infty}$, f takes values in $\overline{k_\infty}$. The ring of entire power series is denoted $\mathbb{E}$.

2.2.4. *Restricted Laurent series.* A power series $\sum_{i=0}^{\infty} a_i t^i \in \mathbb{K}[\![t]\!]$ that satisfies

$$\lim_{i\to\infty} |a_i|_\infty = 0,$$

is called a *restricted power series*. As functions of t, these power series converge on the closed unit disk in $\mathbb{K}$. The restricted power series form a subring $\mathbb{T} = \mathbb{K}\{t\}$ of $\mathbb{K}[\![t]\!]$, and $\mathbb{E}$ is a subring of $\mathbb{T}$. The fraction field of $\mathbb{T}$, denoted $\mathbb{L}$, is the field of *restricted Laurent series.*

Now at each point $a \in \mathbb{K}$ with $|a|_\infty \leq 1$, a function $f \in \mathbb{L}$ has a well-defined order of vanishing $\mathrm{ord}_a(f)$, and for all but finitely many $|a|_\infty \leq 1$, we have $\mathrm{ord}_a(f) = 0$. Also each $f \in \mathbb{L}$ has a unique factorization

$$f = \lambda \Biggl[\prod_{|a|_\infty \leq 1} (t-a)^{\mathrm{ord}_a(f)} \Biggr] \Biggl[1 + \sum_{i=1}^{\infty} b_i t^i \Biggr], \tag{2.2.4.1}$$

where $0 \neq \lambda \in \mathbb{K}$, $\sup |b_i|_\infty < 1$, and $|b_i|_\infty \to 0$ (see [1, Lem. 2.9.1]). The series $1 + \sum b_i t^i$ is a unit in $\mathbb{T}$, and it follows that $\mathbb{T}$ is a principal ideal domain with maximal ideals generated by each $t - a$, $|a|_\infty \leq 1$ (see [1, Lem. 2.9.2]).

For $f = \sum_{i=0}^{\infty} a_i t^i \in \mathbb{T}$, we define its norm $\|f\|$ to be

$$\|f\| := \sup_i |a_i|_\infty = \max_i |a_i|_\infty.$$

If $f \in \mathbb{T}$ is written as in (2.2.4.1), then $\|f\| = |\lambda|_\infty$. The norm $\|\cdot\|$ is an ultrametric norm on $\mathbb{T}$ and satisfies

$$\|cf\| = |c|_\infty \, \|f\|, \qquad \forall\, c \in \mathbb{K}, f \in \mathbb{T},$$
$$\|fg\| = \|f\| \cdot \|g\|, \qquad \forall\, f, g \in \mathbb{T}.$$

We then extend $\|\cdot\|$ to a norm on $\mathbb{L}$ multiplicatively by setting

$$\left\| \frac{g}{h} \right\| := \frac{\|g\|}{\|h\|}, \quad \forall\, g, h \in \mathbb{T}, h \neq 0.$$

In this way $\mathbb{L}$ is a complete field with ultrametric absolute value $\|\cdot\|$, and this absolute value extends the one on $\mathbb{K}$.

2.2.5. *Twisting.* We define an automorphism $\sigma : \mathbb{K}\langle\!\langle t \rangle\!\rangle \to \mathbb{K}\langle\!\langle t \rangle\!\rangle$ by setting

$$\sigma\biggl(\sum_{i \in \mathbb{Q}} a_i t^i \biggr) := \sum_{i \in \mathbb{Q}} a_i^{1/q} t^i.$$

If $f \in \mathbb{K}\langle\!\langle t \rangle\!\rangle$ and $n \in \mathbb{Z}$, the *n-fold twist* of f is defined to be

$$f^{(n)} := \sigma^{-n}(f).$$

The automorphism σ of $\mathbb{K}\langle\!\langle t \rangle\!\rangle$ induces automorphisms of several subrings, notably $\overline{k}[t]$, $\overline{k}(t)$, $\mathbb{E}$, $\mathbb{T}$, $\mathbb{L}$, $\mathbb{K}[\![t]\!]$, $\mathbb{K}(\!(t)\!)$. Moreover, σ also leaves $\overline{\mathbb{F}_q(t)}$, $\overline{k(t)}$, $\overline{\mathbb{L}}$, and $\overline{\mathbb{K}(\!(t)\!)}$ invariant.

If F is a subring of $\mathbb{K}\langle\langle t\rangle\rangle$ that is invariant under σ and $n \geq 1$, we set

$$F^{\sigma^n} := \{f \in F \mid \sigma^n(f) = f\}$$

to be the elements of F fixed by σ. It is clear that F^{σ^n} is a subring of F and that $F^{\sigma^m} \subseteq F^{\sigma^n}$ if $m \mid n$. For example,

$$\mathbb{K}\langle\langle t\rangle\rangle^{\sigma} = \mathbb{F}_q\langle\langle t\rangle\rangle, \qquad \overline{k(t)}^{\sigma} = \overline{\mathbb{F}_q(t)}^{\sigma} = \overline{\mathbb{F}_q(t)} \cap \mathbb{F}_q\langle\langle t\rangle\rangle,$$
$$\mathbb{K}((t))^{\sigma} = \mathbb{F}_q((t)), \qquad \overline{k}(t)^{\sigma} = \mathbb{F}_q(t).$$

The only item that requires any explanation here is the description of $\overline{k(t)}^{\sigma}$. For $\alpha \in \overline{k(t)}^{\sigma}$, let $x^m + b_{m-1}x^{m-1} + \cdots + b_0 \in \overline{k}(t)[x]$ be the minimal polynomial of α over $\overline{k}(t)$. Since $\sigma(\alpha) = \alpha$, we have that α is also a root of $x^m + \sigma(b_{m-1})x^{m-1} + \cdots + \sigma(b_0)$. Taking the difference of these two relations, we see that $\sigma(b_i) = b_i$ for each i, and so the minimal polynomial of α has coefficients in $\overline{k}(t)^{\sigma} = \mathbb{F}_q(t)$. Likewise,

$$\textstyle\bigcup_{n=1}^{\infty} \mathbb{K}\langle\langle t\rangle\rangle^{\sigma^n} = \overline{\mathbb{F}}_q\langle\langle t\rangle\rangle, \qquad \bigcup_{n=1}^{\infty} \overline{k(t)}^{\sigma^n} = \bigcup_{n=1}^{\infty} \overline{\mathbb{F}_q(t)}^{\sigma^n} = \overline{\mathbb{F}_q(t)}$$
$$\textstyle\bigcup_{n=1}^{\infty} \mathbb{K}((t))^{\sigma^n} = \overline{\mathbb{F}}_q((t)), \qquad \bigcup_{n=1}^{\infty} \overline{k}(t)^{\sigma^n} = \overline{\mathbb{F}}_q(t).$$

The top right string of equalities arise because

$$\textstyle\bigcup_{n=1}^{\infty} \overline{\mathbb{F}_q(t)}^{\sigma^n} = \overline{\mathbb{F}_q(t)} \cap \bigcup_{n=1}^{\infty} \mathbb{F}_{q^n}\langle\langle t\rangle\rangle = \overline{\mathbb{F}_q(t)} \cap \overline{\mathbb{F}}_q\langle\langle t\rangle\rangle = \overline{\mathbb{F}_q(t)},$$

where here the last equality here follows from the fact that $\overline{\mathbb{F}}_q\langle\langle t\rangle\rangle$ is algebraically closed.

If F is a matrix with entries in $\mathbb{K}\langle\langle t\rangle\rangle$, then $\sigma^{-n}(F) := F^{(n)}$ is defined by the rule $\bigl(F^{(n)}\bigr)_{ij} := \bigl(F_{ij}^{(n)}\bigr)$. If $F \in \mathrm{Mat}_{r\times s}(\mathbb{L})$, we set $\|F\| := \max_{i,j}\bigl\|F_{ij}\bigr\|$, in which case $\bigl\|F^{(n)}\bigr\| = \|F\|^{q^n}$.

Lemma 2.2.6. *For any $\alpha \in \mathbb{K}$, there is a positive integer s so that with respect to $|\cdot|_\infty$ on $\mathbb{K}$,*

$$\lim_{n\to\infty} \alpha^{(ns)} = \begin{cases} 0 & \text{if } |\alpha|_\infty < 1, \\ c \in \overline{\mathbb{F}}_q^{\times} & \text{if } |\alpha|_\infty = 1, \\ \infty & \text{if } |\alpha|_\infty > 1. \end{cases}$$

Proof. If $|\alpha|_\infty \neq 1$, then the result is clear. Otherwise, there is a unique $c \in \overline{\mathbb{F}}_q^{\times}$ so that

$$|\alpha - c|_\infty < 1.$$

(See [26, Lem. 2.4.4].) Then $c \in \mathbb{F}_{q^s}$ for some $s \geq 1$, and the result follows. $\square$

Lemma 2.2.7. *For any $f \in \mathbb{L}$ with $\|f\| \leq 1$, there is a positive integer s so that with respect to $\|\cdot\|$ on $\mathbb{L}$,*

$$\lim_{n\to\infty} f^{(ns)} \in \overline{\mathbb{F}}_q(t).$$

Also $\|f\| = 1$ if and only if $\lim_{n\to\infty} f^{(ns)} \neq 0$.

Proof. We use the factorization of f in (2.2.4.1). For each a, with $|a|_\infty \leq 1$, if $\mathrm{ord}_a(f) \neq 0$, then as in Lemma 2.2.6 choose $s_a \geq 1$ and $c_a \in \overline{\mathbb{F}}_q$ so that

$$\lim_{n\to\infty} a^{(ns_a)} = c_a.$$

Likewise, since $\|f\| \leq 1$, we have $|\lambda|_\infty \leq 1$, and so we can choose $s_\lambda \geq 1$ and $c_\lambda \in \overline{\mathbb{F}}_q$ with $\lambda^{(ns_\lambda)} \to c_\lambda$. Then we let s be the least common multiple of all the s_a's and s_λ.

From (2.2.4.1), with respect to $\|\cdot\|$,

$$\lim_{n\to\infty}\left[1+\sum_{i=1}^{\infty} b_i t^i\right]^{(ns)} = 1,$$

since $\sup |b_i|_\infty < 1$. Therefore,

$$\lim_{n\to\infty} f^{(ns)} = \lim_{n\to\infty} \lambda^{(ns)} \prod_{|a|_\infty \le 1} \bigl(t - a^{(ns)}\bigr)^{\mathrm{ord}_a(f)} = c_\lambda \prod_{|a|_\infty \le 1} (t - c_a)^{\mathrm{ord}_a(f)},$$

which is in $\overline{\mathbb{F}}_q(t)$. Furthermore, $\|f\| = 1$ if and only if $|\lambda|_\infty = 1$, which holds if and only if $c_\lambda \neq 0$. Thus $\|f\| = 1$ if and only if $\lim_{n\to\infty} f^{(ns)} \neq 0$. □

3. t-motives and Tannakian categories

Here we will define a category $\mathcal{T}$ of t-motives that is a neutral Tannakian category over $\mathbb{F}_q(t)$. For all definitions of tensor categories and Tannakian categories, we follow Deligne and J. S. Milne [11, §II]. Other useful references include [8], [10], [24, App. B].

As mentioned in §1.1.6, Tannakian categories for t-motives have been considered previously by Pink [21], though through a different construction. Parts of the theory of t-motives defined below have been considered by Y. Taguchi [27] and A. Tamagawa [28] in their study of the Tate conjecture for t-modules. Also our theory has similarities with the theory of σ-bundles defined by U. Hartl and Pink [15].

3.1. The rings $\overline{k}[t;\sigma]$ and $\overline{k}(t)[\sigma,\sigma^{-1}]$.

3.1.1. *Definition.* The ring $\overline{k}(t)[\sigma,\sigma^{-1}]$ is the noncommutative ring of Laurent polynomials in σ with coefficients in $\overline{k}(t)$, subject to the relation

$$\sigma f = f^{(-1)}\sigma$$

for all $f \in \overline{k}(t)$. Thus every element of $\overline{k}(t)[\sigma,\sigma^{-1}]$ has the form $\sum_{i=-m}^{m} f_i\sigma^i$, where $f_i \in \overline{k}(t)$, and the product of two elements $\sum f_i\sigma^i$, $\sum g_j\sigma^j \in \overline{k}(t)[\sigma,\sigma^{-1}]$ is determined by the rule

$$\Bigl(\sum_i f_i\sigma^i\Bigr)\Bigl(\sum_j g_j\sigma^j\Bigr) = \sum_{i,j} f_i g_j^{(-i)}\sigma^{i+j}.$$

3.1.2. *Ring-theoretic properties.* The polynomials in σ with coefficients in $\overline{k}[t]$ comprise the subring $\overline{k}[t;\sigma]$ of $\overline{k}(t)[\sigma,\sigma^{-1}]$. The ring $\overline{k}[\sigma]$ is the subring of polynomials with coefficients in $\overline{k}$. Both $\overline{k}[t;\sigma]$ and $\overline{k}(t)[\sigma,\sigma^{-1}]$ are domains. The center of $\overline{k}[t;\sigma]$ is $\mathbb{F}_q[t]$, and the center of $\overline{k}(t)[\sigma,\sigma^{-1}]$ is $\mathbb{F}_q(t)$. The fundamental properties of the ring $\overline{k}[t;\sigma]$ are covered in [2, §4].

3.2. **Pre-t-motives.** Here we define the category $\mathcal{P}$ of pre-t-motives and explore its basic properties. In particular we show in Theorem 3.2.13 that $\mathcal{P}$ is a rigid abelian $\mathbb{F}_q(t)$-linear tensor category.

3.2.1. *The category* $\mathcal{P}$. We let $\mathcal{P}$ be the category of left $\overline{k}(t)[\sigma,\sigma^{-1}]$-modules that are finite dimensional over $\overline{k}(t)$. Morphisms in $\mathcal{P}$ are left $\overline{k}(t)[\sigma,\sigma^{-1}]$-module homomorphisms. We call $\mathcal{P}$ the category of *pre-t-motives*, though it is worth noting that $\mathcal{P}$ is the category of difference modules with respect to the automorphism $\sigma : \overline{k}(t) \to \overline{k}(t)$ in the sense of [23].

3.2.2. *Preliminary properties of* $\mathcal{P}$. The category of pre-t-motives is an abelian category. For two objects P and Q in $\mathcal{P}$, it follows that $\mathrm{Hom}_{\mathcal{P}}(P,Q)$ is an $\mathbb{F}_q(t)$-vector space. A straightforward adaptation of the proof of [1, Thm. 2] shows that the map

$$\mathrm{Hom}_{\mathcal{P}}(P,Q) \otimes_{\mathbb{F}_q(t)} \overline{k}(t) \to \mathrm{Hom}_{\overline{k}(t)}(P,Q)$$

is injective. Thus $\mathrm{Hom}_{\mathcal{P}}(P,Q)$ is a finite dimensional $\mathbb{F}_q(t)$-vector space.

3.2.3. *Representations of pre-t-motives.* Given a $\overline{k}(t)$-vector space P and $p_1,\dots,p_r \in P$, we call the vector

$$\mathbf{p} = \begin{bmatrix} p_1 \\ \vdots \\ p_r \end{bmatrix} \in \mathrm{Mat}_{r\times 1}(P)$$

a *basis for* P if $p_1,\dots,p_r$ form an $\overline{k}(t)$-basis for P. If P is a pre-t-motive, then there is a unique matrix $\Phi = \Phi_{\mathbf{p}} \in \mathrm{GL}_r(\overline{k}(t))$ such that

$$\sigma \mathbf{p} = \Phi \mathbf{p}.$$

We say that Φ *represents multiplication by* σ *on* P. Moreover, the matrix $\Phi \in \mathrm{GL}_r(\overline{k}(t))$ uniquely determines the left $\overline{k}(t)[\sigma,\sigma^{-1}]$-module structure on P with respect to $\mathbf{p}$.

Now suppose that $\phi : P \to Q$ is a morphism of pre-t-motives and that $\mathbf{p} \in \mathrm{Mat}_{r\times 1}(P)$ and $\mathbf{q} \in \mathrm{Mat}_{s\times 1}(Q)$ are bases for P and Q respectively. If $B \in \mathrm{Mat}_{r\times s}(\overline{k}(t))$ represents ϕ as a map of $\overline{k}(t)$-vector spaces such that

$$\phi(\mathbf{f}\cdot\mathbf{p}) = \mathbf{f}\cdot B \cdot \mathbf{q}, \quad \mathbf{f} \in \mathrm{Mat}_{1\times r}(\overline{k}(t)),$$

then

$$B^{(-1)}\Phi_{\mathbf{q}} = \Phi_{\mathbf{p}} B.$$

In particular, if $\mathbf{q}$ is simply another basis of P, and $B \in \mathrm{GL}_r(\overline{k}(t))$ is the change of basis matrix, then $\Phi_{\mathbf{p}} = B^{(-1)}\Phi_{\mathbf{q}}B^{-1}$.

3.2.4. *Tensor products of pre-t-motives.* Let P and Q be pre-t-motives. Then the $\overline{k}(t)$-vector space $P \otimes_{\overline{k}(t)} Q$ is made into a $\overline{k}(t)[\sigma,\sigma^{-1}]$-module by defining

$$\sigma(m\otimes n) := (\sigma m) \otimes (\sigma n).$$

It is clear that then multiplication by σ is bijective on $P \otimes_{\overline{k}(t)} Q$ and that $P \otimes_{\overline{k}(t)} Q$ is a pre-t-motive. Likewise we define arbitrary finite tensor products of pre-t-motives with diagonal σ-action. For a fixed pre-t-motive P and $n \geq 1$, we set $P^{\otimes n} := \bigotimes_{i=1}^n P$ to be the n-th tensor power of P.

3.2.5. *Representations of tensor products.* Let $\mathbf{p} = [p_1,\dots,p_r]^{\mathrm{tr}}$ and $\mathbf{q} = [q_1,\dots,q_s]^{\mathrm{tr}}$ be $\overline{k}(t)$-bases for pre-t-motives P and Q respectively. Then, with respect to the basis

$$\mathbf{p}\otimes\mathbf{q} := [p_1\otimes q_1, p_1\otimes q_2, \dots, p_r \otimes q_s]^{\mathrm{tr}},$$

on $P\otimes Q$, the Kronecker product,

$$\Phi_{\mathbf{p}\otimes\mathbf{q}} = \Phi_{\mathbf{p}} \otimes \Phi_{\mathbf{q}} := \begin{bmatrix} \Phi_{\mathbf{p},11}\Phi_{\mathbf{q}} & \cdots & \Phi_{\mathbf{p},1r}\Phi_{\mathbf{q}} \\ \vdots & & \vdots \\ \Phi_{\mathbf{p},r1}\Phi_{\mathbf{q}} & \cdots & \Phi_{\mathbf{p},rr}\Phi_{\mathbf{q}} \end{bmatrix},$$

represents multiplication by σ on $P\otimes Q$. Similarly these conventions extend to arbitrary finite tensor products of pre-t-motives.

3.2.6. *The Carlitz motive.* We define the *Carlitz motive* to be the pre-t-motive C whose underlying $\overline{k}(t)$-vector space is $\overline{k}(t)$ itself and on which σ acts by

$$\sigma f := (t-\theta) f^{(-1)}, \quad f \in C.$$

For $n \geq 1$, the underlying $\overline{k}(t)$-vector space of $C^{\otimes n}$ is also $\overline{k}(t)$, and multiplication by σ on $C^{\otimes n}$ is given by

$$\sigma f = (t-\theta)^n f^{(-1)}, \quad f \in C^{\otimes n}.$$

See also [3].

3.2.7. *Internal Hom.* Let P and Q be pre-t-motives, and set

$$R := \mathrm{Hom}_{\overline{k}(t)}(P, Q).$$

Then R is a $\overline{k}(t)$-vector space. We define a $\overline{k}(t)[\sigma, \sigma^{-1}]$-module structure on R by setting

$$\sigma \cdot \rho := \sigma \circ \rho \circ \sigma^{-1}, \quad \rho \in R.$$

It is straightforward to check that $\sigma \cdot \rho : P \to Q$ is $\overline{k}(t)$-linear, and so $\sigma : R \to R$, and that this action of σ extends naturally to a left $\overline{k}(t)[\sigma, \sigma^{-1}]$-module structure on R. We write $\mathrm{Hom}(P, Q)$ for the $\overline{k}(t)[\sigma, \sigma^{-1}]$-module R just defined. It is also a pre-t-motive.

3.2.8. *Identity object.* Let $\mathbf{1} := \overline{k}(t)$ together with a σ-action defined by

$$\sigma f = f^{(-1)}, \quad f \in \mathbf{1}.$$

Then $\mathbf{1}$ is a pre-t-motive. Moreover, for any pre-t-motive P, the natural isomorphisms,

$$P \otimes \mathbf{1} \cong \mathbf{1} \otimes P \cong P,$$

are isomorphisms of pre-t-motives. Thus $\mathbf{1}$ is an identity object with respect to tensor products in $\mathcal{P}$.

Lemma 3.2.9. $\mathrm{End}_{\mathcal{P}}(\mathbf{1}) \cong \mathbb{F}_q(t)$.

Proof. Suppose $\phi : \mathbf{1} \to \mathbf{1}$ is a morphism in $\mathcal{P}$. As a map of $\overline{k}(t)$-vector spaces, there is some $a \in \overline{k}(t)$ so that $\phi(f) = af$ for all $f \in \overline{k}(t)$. Since ϕ is also $\overline{k}(t)[\sigma, \sigma^{-1}]$-linear, we must have $\sigma a = a\sigma$, from which it follows that a is in the center of $\overline{k}(t)[\sigma, \sigma^{-1}]$. Thus $a \in \mathbb{F}_q(t)$. □

3.2.10. *Duals.* Let P be a pre-t-motive. Then set

$$P^\vee := \mathrm{Hom}(P, \mathbf{1}).$$

The pre-t-motive $P^\vee$ is called the *dual of* P. As a $\overline{k}(t)$-vector space, $P^\vee$ is the dual vector space of P. If $\mathbf{p}$ forms a basis for P, let $\mathbf{p}^\vee$ be the dual basis. We find easily that

$$\Phi_{\mathbf{p}^\vee} = \left(\Phi_{\mathbf{p}}^{-1}\right)^{\mathrm{tr}}.$$

If $\phi : P \to Q$ is a morphism of pre-t-motives, then the dual morphism of $\overline{k}(t)$-vector spaces,

$$\phi^\vee : Q^\vee \to P^\vee,$$

is also $\overline{k}(t)[\sigma, \sigma^{-1}]$-linear. These constructions are functorial in P and Q, and thus $P \mapsto P^\vee : \mathcal{P} \to \mathcal{P}$ defines a contravariant $\mathbb{F}_q(t)$-linear functor.

3.2.11. *Dual of the Carlitz motive.* Using the definition of the Carlitz motive in §3.2.6, we see that $C^\vee$ is isomorphic to $\overline{k}(t)$ as a $\overline{k}(t)$-vector space and that

$$\sigma f = \frac{1}{t-\theta} \cdot f^{(-1)}, \quad f \in C^\vee \ (= \overline{k}(t)).$$

Furthermore, we see that

$$C^\vee \otimes C \cong \mathbf{1}$$

and that C is an invertible object in $\mathcal{P}$. Thus the functor

$$P \mapsto P \otimes C : \mathcal{P} \to \mathcal{P}$$

is an equivalence of categories. We define for $n \in \mathbb{Z}$,

$$C(n) := \begin{cases} C^{\otimes n} & \text{if } n > 0, \\ \mathbf{1} & \text{if } n = 0, \\ (C^\vee)^{\otimes -n} & \text{if } n < 0. \end{cases}$$

3.2.12. *Rigid abelian tensor category.* In the language of [11, §II.1], it is easily shown that the category of pre-t-motives is an abelian $\mathbb{F}_q(t)$-linear tensor category. We omit the details, but we observe that

- each $\mathrm{Hom}_{\mathcal{P}}(P,Q)$ is a finite dimensional vector space over $\mathbb{F}_q(t)$;
- $\otimes$ is compatibly associative and commutative;
- $\otimes$ is $\mathbb{F}_q(t)$-bilinear;
- $\mathbf{1}$ is an identity object with respect to tensor products.

Furthermore, it is straightforward to check that

- the pre-t-motive $\mathrm{Hom}(P,Q)$ defines an internal Hom in $\mathcal{P}$ that is compatible with tensor products;
- for each pre-t-motive P, there is a natural isomorphism $P \cong P^{\vee\vee}$.

Therefore, $\mathcal{P}$ is also rigid. We record this information in the following theorem.

Theorem 3.2.13. *The category $\mathcal{P}$ of pre-t-motives is a rigid abelian $\mathbb{F}_q(t)$-linear tensor category.*

3.3. **Rigid analytic triviality.**

3.3.1. *The category $\mathcal{R}$.* Let P be a pre-t-motive. We set

$$P^\dagger := \mathbb{L} \otimes_{\overline{k}(t)} P,$$

and give $P^\dagger$ a left $\overline{k}(t)[\sigma,\sigma^{-1}]$-module structure by setting

$$\sigma(f \otimes m) := f^{(-1)} \otimes \sigma m.$$

Let

$$P^{\mathrm{B}} := (P^\dagger)^\sigma = \{\mu \in P^\dagger \mid \sigma\mu = \mu\}.$$

Then P^{B} is an $\mathbb{F}_q(t)$-vector space, and $P \mapsto P^{\mathrm{B}}$ is a covariant functor from $\mathcal{P}$ to the category of $\mathbb{F}_q(t)$-vector spaces. (The "B" in P^{B} stands for "Betti.") It is straightforward to check that $P \mapsto P^{\mathrm{B}}$ is left exact.

We say that P is *rigid analytically trivial* if the natural map

$$\mathbb{L} \otimes_{\mathbb{F}_q(t)} P^{\mathrm{B}} \to P^\dagger$$

is an isomorphism. If $P \cong Q$ as pre-t-motives and P is rigid analytically trivial, then so is Q. We let $\mathcal{R}$ denote the strictly full subcategory of $\mathcal{P}$ whose objects are the rigid analytically trivial pre-t-motives. Clearly the zero object is rigid analytically trivial, and so $\mathcal{R}$ is nonempty. We shall see momentarily that $\mathbf{1}$ and C are also rigid analytically trivial.

Lemma 3.3.2. *We have* $\mathbb{L}^\sigma = \mathbb{F}_q(t)$.

Proof. For $f \in \mathbb{L}^\sigma$ we have $f^{(-1)} = f$, and so by (2.2.4.1) the polar divisor D of f on the closed unit disk in $\mathbb{K}$ must also satisfy $D^{(-1)} = D$. Therefore D is the divisor of zeros of a polynomial c in $\mathbb{F}_q[t]$. Then $cf \in \mathbb{T}$, and $(cf)^{(-1)} = cf$, from which we have $cf \in \mathbb{T} \cap \mathbb{F}_q(\!(t)\!) = \mathbb{F}_q[t]$. □

Proposition 3.3.3. *The pre-t-motive* $\mathbf{1}$ *is rigid analytically trivial.*

Proof. It is clear that $\mathbf{1}^\dagger = \mathbb{L}$ with $\sigma f = f^{(-1)}$ for $f \in \mathbb{L}$. Therefore, by Lemma 3.3.2,

$$\mathbf{1}^{\mathrm{B}} = \mathbb{L}^\sigma = \mathbb{F}_q(t).$$

Thus $\mathbb{L} \otimes_{\mathbb{F}_q(t)} \mathbf{1}^{\mathrm{B}} \cong \mathbf{1}^\dagger$. □

3.3.4. *The power series* Ω. Consider the power series

$$\Omega = \Omega(t) := \zeta_\theta^{-q} \prod_{i=1}^{\infty} \Bigl(1 - t/\theta^{(i)}\Bigr) \in k_\infty(\zeta_\theta)[\![t]\!] \subseteq \mathbb{K}[\![t]\!].$$

It is not difficult to show that $\Omega(t)$ has an infinite radius of convergence, and so $\Omega \in \mathbb{E} \subseteq \mathbb{T}$. Since Ω has infinitely many zeros in $\mathbb{K}$, it follows that $\Omega \notin \mathbb{K}(t)$. Since Ω has no zeros inside the unit disk, it follows that $\Omega \in \mathbb{T}^\times$. It also satisfies the functional equation

$$\Omega^{(-1)} = (t - \theta)\Omega.$$

The number

$$\widetilde{\pi} = -\frac{1}{\Omega(\theta)} = \theta\zeta_\theta^q \prod_{i=1}^{\infty} \Bigl(1 - \theta^{1-q^i}\Bigr)^{-1} \in k_\infty(\zeta_\theta)$$

is the *Carlitz period*, which figures prominently in our transcendence considerations later on (see also [3, Cor. 5.2.8], [14, §3.2], [29, §2.5]).

Lemma 3.3.5. *Suppose* $f \in \mathbb{L}$ *satisfies* $(t-\theta)^n f^{(-1)} = f$ *for some* $n \in \mathbb{Z}$. *Then* $f = c/\Omega^n$ *for some* $c \in \mathbb{F}_q(t)$.

Proof. Let $c = f\Omega^n$. Then c satisfies $c^{(-1)} = c$, and so by Lemma 3.3.2, $c \in \mathbb{F}_q(t)$. □

Proposition 3.3.6. *The Carlitz motive* C *is rigid analytically trivial.*

Proof. We see that $C^\dagger = \mathbb{L}$ with $\sigma f = (t-\theta) f^{(-1)}$ for $f \in \mathbb{L}$. Therefore, by Lemma 3.3.5,

$$C^{\mathrm{B}} = \{ f \in \mathbb{L} \mid (t-\theta) f^{(-1)} = f \} = \frac{1}{\Omega} \cdot \mathbb{F}_q(t).$$

Therefore $\mathbb{L} \otimes_{\mathbb{F}_q(t)} C^{\mathrm{B}} \cong C^\dagger$. □

Lemma 3.3.7. *Let* P *be a pre-t-motive, and let* $\mu_1, \dots, \mu_m \in P^{\mathrm{B}}$. *If* $\mu_1, \dots, \mu_m$ *are linearly independent over* $\mathbb{F}_q(t)$, *then they are linearly independent over* $\mathbb{L}$ *in* $P^\dagger$.

Proof. Suppose that $m \geq 2$ is minimal such that $\mu_1, \dots, \mu_m$ are linearly independent over $\mathbb{F}_q(t)$ but that

$$\sum_{i=1}^{m} f_i \mu_i = 0, \quad f_i \in \mathbb{L},\ f_1 = 1.$$

Now,

$$\sigma \sum_{i=1}^{m} f_i \mu_i = \sum_{i=1}^{m} f_i^{(-1)} \mu_i = 0.$$

Therefore,

$$\sum_{i=2}^{m}(f_i - f_i^{(-1)})\mu_i = 0.$$

By the minimality of m and Lemma 3.3.2, we must have each $f_i \in \mathbb{F}_q(t)$. However, this violates the $\mathbb{F}_q(t)$-linear independence of $\mu_1, \dots, \mu_m$. ☐

Proposition 3.3.8. *If P is a pre-t-motive, then*

$$\dim_{\mathbb{F}_q(t)} P^{\mathrm{B}} \leq \dim_{\overline{k}(t)} P.$$

Equality holds if and only if P is rigid analytically trivial.

Proof. From Lemma 3.3.7, the map $\mathbb{L} \otimes_{\mathbb{F}_q(t)} P^{\mathrm{B}} \to P^{\dagger}$ is injective. The inequality in the statement of the proposition follows from the equality $\dim_{\overline{k}(t)} P = \dim_{\mathbb{L}} P^{\dagger}$. By the definition of rigid analytic triviality, equality holds if and only if the map above is also surjective. ☐

Proposition 3.3.9. *Suppose that P is a pre-t-motive and that Φ represents multiplication by σ on P with respect to the basis $\mathbf{p}$ of P.*

(a) *P is rigid analytically trivial if and only if there is a matrix $\Psi \in \mathrm{GL}_r(\mathbb{L})$ satisfying*

$$\Psi^{(-1)} = \Phi\Psi.$$

Such a matrix Ψ is called a rigid analytic trivialization of Φ *(cf.* [1, Thm. 5]*,* [2, Lem. 4.4.13]*).*

(b) *If Ψ is a rigid analytic trivialization of Φ, then the entries of $\Psi^{-1}\mathbf{p}$ form an $\mathbb{F}_q(t)$-basis for P^{B}.*

(c) *If P is rigid analytically trivial, $\Phi \in \mathrm{Mat}_r(\overline{k}[t])$, and $\det(\Phi) = d(t-\theta)^s$ for some $s \geq 0$ and $d \in \overline{k}^{\times}$, then there is a rigid analytic trivialization Ψ of Φ with $\Psi \in \mathrm{GL}_r(\mathbb{T})$.*

Proof. The proofs of parts (a) and (b) are essentially the same as the proof of [2, Lem. 4.4.13] with minor modifications. We provide a sketch for completeness. ((a) $\Leftarrow$; (b)): Certainly if we have such a Ψ, then the entries of $\Psi^{-1}\mathbf{p}$ are both an $\mathbb{L}$-basis of $P^{\dagger}$ and also an $\mathbb{F}_q(t)$-linearly independent set in P^{B}. By Proposition 3.3.8, the entries of $\Psi^{-1}\mathbf{p}$ must be an $\mathbb{F}_q(t)$-basis of P^{B}, and thus P is rigid analytically trivial. ((a) $\Rightarrow$): On the other hand, if P is rigid analytically trivial, then there is a matrix $\Theta \in \mathrm{GL}_r(\mathbb{L})$ so that the entries of $\Theta\mathbf{p}$ are both an $\mathbb{L}$-basis of $P^{\dagger}$ and an $\mathbb{F}_q(t)$-basis of P^{B}. Setting $\Psi := \Theta^{-1}$ gives the desired matrix.

For part (c), we first let P be the $\overline{k}[t]$-span of the entries of $\mathbf{p}$, and set

$$\mathsf{P}^{\dagger} := \mathbb{T} \otimes_{\overline{k}[t]} \mathsf{P}, \quad \mathsf{P}^{\mathrm{B}} := \{\mu \in \mathsf{P}^{\dagger} \mid \sigma\mu = \mu\}.$$

For $\mu \in P^{\mathrm{B}}$, write $\mu = \sum f_i p_i = \mathbf{f} \cdot \mathbf{p}$ with $\mathbf{f} \in \mathrm{Mat}_{1\times r}(\mathbb{L})$. We claim that for some $c \in \mathbb{F}_q[t]$, we have $c\mu \in \mathsf{P}^{\mathrm{B}}$. Let $\mathrm{den}(\mathbf{f}) \in \mathbb{K}[t]$ denote the monic least common multiple of the denominators of $\mathbf{f}$, which is well-defined by (2.2.4.1). Then since $\sigma\mu = \mu$, we have

$$\mathbf{f} \cdot \mathbf{p} = \sigma(\mathbf{f} \cdot \mathbf{p}) = \mathbf{f}^{(-1)} \cdot \Phi\mathbf{p}.$$

Therefore, $\mathrm{den}(\mathbf{f}) = \mathrm{den}(\mathbf{f}^{(-1)} \cdot \Phi)$. But $\Phi \in \mathrm{Mat}_r(\overline{k}[t])$, so $\mathrm{den}(\mathbf{f}^{(-1)} \cdot \Phi)$ divides $\mathrm{den}(\mathbf{f}^{(-1)})$. Degree considerations force $\mathrm{den}(\mathbf{f}) = \mathrm{den}(\mathbf{f}^{(-1)})$. Therefore take $c = \mathrm{den}(\mathbf{f}) \in \mathbb{F}_q[t]$. This proves the claim, and moreover we have shown that

$$P^{\mathrm{B}} \cong \mathbb{F}_q(t) \otimes_{\mathbb{F}_q[t]} \mathsf{P}^{\mathrm{B}}.$$

Furthermore, it follows that as $\mathbb{L}$-vector spaces,

$$P^\dagger \cong \mathbb{L} \otimes_{\mathbb{T}} \mathsf{P}^\dagger \cong \mathbb{L} \otimes_{\mathbb{F}_q[t]} \mathsf{P}^{\mathrm{B}}.$$

Let $\boldsymbol{\nu} = [\nu_1, \dots, \nu_r]^{\mathrm{tr}}$ be an $\mathbb{F}_q[t]$-basis for P^{B}. Then for some $\Theta \in \mathrm{GL}_r(\mathbb{L}) \cap \mathrm{Mat}_r(\mathbb{T})$, we have $\boldsymbol{\nu} = \Theta \mathbf{p}$. Since $\sigma \boldsymbol{\nu} = \boldsymbol{\nu}$, it follows that

$$\Theta^{(-1)} \Phi = \Theta.$$

By our initial hypotheses, $d(t-\theta)^s \det(\Theta)^{(-1)} = \det(\Theta)$. Choose $b \in \overline{k}^\times$ so that $d = b^{(-1)}/b$. Then from Lemmas 3.3.2 and 3.3.5 (and the fact that $\Theta \in \mathrm{Mat}_r(\mathbb{T})$), we see that

$$b \det(\Theta) = \frac{\gamma}{\Omega^s}, \quad \gamma \in \mathbb{F}_q[t].$$

We claim that $\gamma \in \mathbb{F}_q^\times$. If not, then $\det(\Theta) \equiv 0 \pmod{\gamma}$ in $\mathbb{T}$, and so there is a $\mathbf{f} = [f_1, \dots, f_r] \in \mathrm{Mat}_{1\times r}(\mathbb{T})$ so that

$$\mathbf{f} \cdot \Theta \equiv 0 \pmod{\gamma}.$$

Since $\mathbb{T}/\gamma\mathbb{T} \cong \mathbb{K}[t]/\gamma\mathbb{K}[t]$, without loss of generality we can assume that each f_i is a polynomial in $\mathbb{K}[t]$ of degree strictly less than the degree of γ, that $\|f_i\| \leq 1$ for all i, and that at least one f_i satisfies $\|f_i\| = 1$. Now define a norm $\|\cdot\|_\dagger$ on $P^\dagger$ by

$$\left\| \sum h_i p_i \right\|_\dagger := \sup \|h_i\|, \quad h_1, \dots, h_r \in \mathbb{L}.$$

Then $\|\cdot\|_\dagger$ defines a complete ultrametic norm on $P^\dagger$ that satisfies

$$\|h\mu\|_\dagger = \|h\| \cdot \|\mu\|_\dagger, \quad h \in \mathbb{L},\ \mu \in P^\dagger.$$

As such,

$$\mathsf{P}^\dagger = \{\mu \in P^\dagger \mid \|\mu\|_\dagger \leq 1\}.$$

Consider

$$\mathbf{f} \cdot \Theta^{(-1)} \Phi = \mathbf{f} \cdot \Theta \equiv 0 \pmod{\gamma}.$$

Since γ is relatively prime to $\det(\Phi)$, it follows that Φ is invertible modulo γ, and so

$$\mathbf{f} \cdot \Theta^{(-1)} \equiv 0 \pmod{\gamma}.$$

Repeating this argument we find that

$$\mathbf{f} \cdot \Theta^{(-n)} \equiv 0 \pmod{\gamma}, \quad \forall\, n \geq 0.$$

Now, by choice of $\mathbf{f}$,

$$\frac{1}{\gamma}\mathbf{f} \cdot \boldsymbol{\nu} = \frac{1}{\gamma}\mathbf{f} \cdot \Theta \cdot \mathbf{p} \in \mathsf{P}^\dagger,$$

and for each n, the above congruences for $\mathbf{f} \cdot \Theta^{(-n)}$ imply that

$$\frac{1}{\gamma}\mathbf{f}^{(n)} \cdot \boldsymbol{\nu} = \frac{1}{\gamma}\mathbf{f}^{(n)} \cdot \Theta \cdot \mathbf{p} \in \mathsf{P}^\dagger.$$

Now by Lemma 2.2.7, there is an $m > 0$ so that with respect to the $\|\cdot\|_\dagger$ metric,

$$\lim_{n\to\infty} \frac{1}{\gamma} \sum f_i^{(mn)} \nu_i = \frac{1}{\gamma} \sum c_i \nu_i \in \mathsf{P}^\dagger,$$

where $c_i \in \overline{\mathbb{F}}_q[t]$ and at least one $c_i \neq 0$, say $c_a \neq 0$. Now for some $l \geq 1$, we have every $c_i \in \mathbb{F}_{q^l}[t]$. Since the trace map $\mathbb{F}_{q^l} \to \mathbb{F}_q$ is not trivial, by dividing each c_i by a fixed element in $\mathbb{F}_{q^l}^\times$, we can assume that

$$c_a + c_a^{(-1)} + \cdots + c_a^{(1-l)} \neq 0.$$

Therefore,

$$\sum_{j=0}^{l-1} \sigma^j \biggl(\frac{1}{\gamma} \sum_{i=1}^{r} c_i \nu_i \biggr) = \frac{1}{\gamma} \sum_{i=1}^{r} \biggl(\sum_{j=0}^{l-1} c_i^{(-j)} \biggr) \nu_i \in \mathsf{P}^\dagger.$$

Thus we obtain

$$\mu := \frac{1}{\gamma} \sum d_i \nu_i \in \mathsf{P}^\dagger, \quad d_i \in \mathbb{F}_q[t],\ d_a \neq 0.$$

Easily we see that $\mu \in \mathsf{P}^{\mathrm{B}}$ and $\mu \neq 0$. Since $\deg f_i < \deg \gamma$ for each i, we have $\deg d_i < \deg \gamma$ for each i. In particular, γ does not divide d_a. Thus $\mu \in \mathsf{P}^{\mathrm{B}}$ but μ is not in the $\mathbb{F}_q[t]$-span of $\boldsymbol{\nu}$, which contradicts that $\boldsymbol{\nu}$ is an $\mathbb{F}_q[t]$-basis of P^{B}. Therefore, it follows that $\gamma \in \mathbb{F}_q^\times$, and since $\Omega \in \mathbb{T}^\times$, we have $\det(\Theta) \in \mathbb{T}^\times$. Taking $\Psi = \Theta^{-1}$ provides the desired rigid analytic trivialization. □

3.3.10. *Remark.* It is worth noting that multiplication by Θ induces the isomorphism of $\mathbb{L}$-vector spaces,

$$\mathbb{L} \otimes_{\mathbb{T}} \mathsf{P}^\dagger \cong \mathbb{L} \otimes_{\mathbb{T}} (\mathbb{T} \otimes_{\mathbb{F}_q[t]} \mathsf{P}^{\mathrm{B}}).$$

Since $\Theta \in \mathrm{GL}_r(\mathbb{T})$, this then implies

$$\mathsf{P}^\dagger \cong \mathbb{T} \otimes_{\mathbb{F}_q[t]} \mathsf{P}^{\mathrm{B}}$$

as $\mathbb{T}$-modules.

Proposition 3.3.11. *Let*

$$0 \to P \to Q \to R \to 0$$

be an exact sequence of pre-t-motives.

(a) *If Q is rigid analytically trivial, then both P and R are rigid analytically trivial.*
(b) *If P, Q, and R are rigid analytically trivial, then the sequence*

$$0 \to P^{\mathrm{B}} \to Q^{\mathrm{B}} \to R^{\mathrm{B}} \to 0$$

is an exact sequence of $\mathbb{F}_q(t)$-vector spaces.

Proof. The sequence $0 \to P^{\mathrm{B}} \to Q^{\mathrm{B}} \to R^{\mathrm{B}}$ is exact. Now suppose that Q is rigid analytically trivial. Let $\kappa : \mathbb{L} \otimes_{\mathbb{F}_q(t)} Q^{\mathrm{B}} \to \mathbb{L} \otimes_{\mathbb{F}_q(t)} R^{\mathrm{B}}$ be the natural map. Then we have a commutative diagram with exact rows,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{L} \otimes_{\mathbb{F}_q(t)} P^{\mathrm{B}} & \longrightarrow & \mathbb{L} \otimes_{\mathbb{F}_q(t)} Q^{\mathrm{B}} & \longrightarrow & \mathrm{im}(\kappa) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow \wr & & \downarrow & & \\
0 & \longrightarrow & P^\dagger & \longrightarrow & Q^\dagger & \longrightarrow & R^\dagger & \longrightarrow & 0,
\end{array}$$

where the central vertical map is an isomorphism by hypothesis, and the other two are injective by Lemma 3.3.7. The injectivity of all three maps then implies that each is an isomorphism. Thus we see immediately that P is rigid analytically trivial. Also we see that

$$\dim_{\mathbb{F}_q(t)} R^{\mathrm{B}} = \dim_{\mathbb{L}} \mathbb{L} \otimes_{\mathbb{F}_q(t)} R^{\mathrm{B}} \geq \dim_{\mathbb{L}} \mathrm{im}(\kappa) = \dim_{\mathbb{L}} R^\dagger = \dim_{\overline{k}(t)} R,$$

which by Proposition 3.3.8 must be a string of equalities. Therefore R is rigid analytically trivial, which completes part (a).

Now suppose that P, Q, and R are all rigid analytically trivial. Then

$$\dim_{\mathbb{F}_q(t)} Q^{\mathrm{B}} = \dim_{\overline{k}(t)} Q = \dim_{\overline{k}(t)} P + \dim_{\overline{k}(t)} R = \dim_{\mathbb{F}_q(t)} P^{\mathrm{B}} + \dim_{\mathbb{F}_q(t)} R^{\mathrm{B}},$$

which proves part (b). □

3.3.12. *Remark.* In particular, it follows from Proposition 3.3.11 that kernels and cokernels exist in $\mathcal{R}$, which implies that $\mathcal{R}$ is an abelian $\mathbb{F}_q(t)$-linear category. We also see that

$$P \to P^{\mathrm{B}} : \mathcal{R} \to \mathbf{Vec}(\mathbb{F}_q(t)),$$

where $\mathbf{Vec}(\mathbb{F}_q(t))$ is the category of finite dimensional vector spaces over $\mathbb{F}_q(t)$, is an exact $\mathbb{F}_q(t)$-linear functor.

Proposition 3.3.13. *Let P and Q be rigid analytically trivial pre-t-motives. Then the natural map*

$$\mathrm{Hom}_{\mathcal{R}}(P,Q) \to \mathrm{Hom}_{\mathbb{F}_q(t)}(P^{\mathrm{B}}, Q^{\mathrm{B}})$$

is injective.

Proof. Suppose $\phi : P \to Q$ is a morphism in $\mathrm{Hom}_{\mathcal{R}}(P,Q)$. Then we have an exact sequence in $\mathcal{R}$,

$$0 \to \ker\phi \to P \xrightarrow{\phi} Q \to Q/\phi(P) \to 0,$$

which leads then to an exact sequence of $\mathbb{F}_q(t)$-vector spaces,

$$0 \to (\ker\phi)^{\mathrm{B}} \to P^{\mathrm{B}} \xrightarrow{\phi^{\mathrm{B}}} Q^{\mathrm{B}} \to (Q/\phi(P))^{\mathrm{B}} \to 0.$$

Since the dimension over $\overline{k}(t)$ of each term in the first sequence is the same as the dimension over $\mathbb{F}_q(t)$ of the corresponding term in the second sequence, we see that $\phi^{\mathrm{B}} = 0$ if and only if $\phi = 0$. □

Proposition 3.3.14. *If pre-t-motives P and Q are rigid analytically trivial, then*

(a) *$P \otimes Q$ is rigid analytically trivial, and the natural map*

$$P^{\mathrm{B}} \otimes_{\mathbb{F}_q(t)} Q^{\mathrm{B}} \to (P \otimes Q)^{\mathrm{B}}$$

is an isomorphism of $\mathbb{F}_q(t)$-vector spaces;

(b) *$P^\vee$ is rigid analytically trivial, and the natural map*

$$\left(P^{\mathrm{B}}\right)^\vee \to \left(P^\vee\right)^{\mathrm{B}}$$

is an isomorphism of $\mathbb{F}_q(t)$-vector spaces.

Proof. Here we make use of Proposition 3.3.9. We first note that

$$(P\otimes Q)^\dagger = \mathbb{L} \otimes_{\overline{k}(t)} (P \otimes_{\overline{k}(t)} Q) \cong (\mathbb{L} \otimes_{\overline{k}(t)} P) \otimes_{\mathbb{L}} (\mathbb{L} \otimes_{\overline{k}(t)} Q) = P^\dagger \otimes_{\mathbb{L}} Q^\dagger,$$

where the middle isomorphism is an isomorphism of $\mathbb{L}$-vector spaces that commutes with the action of σ. We observe that we can choose $\overline{k}(t)$-bases for P, Q, and $P\otimes Q$ so that multiplication by σ is represented by matrices Φ_P, Φ_Q, and $\Phi_{P\otimes Q}$ satisfying

$$\Phi_{P\otimes Q} = \Phi_P \otimes \Phi_Q.$$

By Proposition 3.3.9, we can choose Ψ_P, $\Psi_Q \in \mathrm{GL}_r(\mathbb{L})$ that are rigid analytic trivializations of Φ_P and Φ_Q. Then we note that $\Psi_{P\otimes Q} := \Psi_P \otimes \Psi_Q$ is a rigid analytic trivialization of $\Phi_{P\otimes Q}$. Now note that $\Phi_{P^\vee} := (\Phi_P^{-1})^{\mathrm{tr}}$ represents multiplication by σ with respect to the dual basis and that $\Psi_P^\vee := (\Psi_P^{-1})^{\mathrm{tr}}$ is a rigid analytic trivialization. The second parts of (a) and (b) are straightforward. □

3.3.15. *Remark.* In the proof of the preceding proposition we verified special cases of the following useful fact. Suppose

$$(P_1, \dots, P_m) \mapsto L(P_1, \dots, P_m)$$

is a functor $\mathbf{Vec}(\overline{k}(t)) \to \mathbf{Vec}(\overline{k}(t))$ constructed from taking a combination of direct sums, subquotients, tensor products, duals, internal Hom's, etc. If $P_1, \dots, P_m$, are pre-t-motives together with rigid analytic trivializations $\Psi_1, \dots, \Psi_m$, then

$$\Psi = L(\Psi_1, \dots, \Psi_m)$$

is a rigid analytic trivialization of $L(P_1, \dots, P_m)$.

Theorem 3.3.16. *The category $\mathcal{R}$ of rigid analytically trivial pre-t-motives is a neutral Tannakian category over $\mathbb{F}_q(t)$ with fiber functor*

$$P \mapsto P^{\mathrm{B}} : \mathcal{R} \to \mathbf{Vec}(\mathbb{F}_q(t)).$$

Proof. We have seen that

- $\mathbf{1}$ is in $\mathcal{R}$ (Proposition 3.3.3);
- $\mathcal{R}$ is an abelian category (Proposition 3.3.11 and §3.3.12);
- $\mathcal{R}$ is closed under tensor products and duals (Proposition 3.3.14).

Thus $\mathcal{R}$ is a rigid abelian $\mathbb{F}_q(t)$-linear tensor subcategory of $\mathcal{P}$ (see [11, Defs. II.1.14-15]). We have also shown that

- $\mathrm{End}_{\mathcal{R}}(\mathbf{1}) = \mathbb{F}_q(t)$ (Lemma 3.2.9);
- For each P in $\mathcal{R}$, the $\mathbb{F}_q(t)$-vector space P^{B} is finite dimensional (Proposition 3.3.8);
- $P \mapsto P^{\mathrm{B}}$ is $\mathbb{F}_q(t)$-linear and exact (Proposition 3.3.11 and §3.3.12);
- $P \mapsto P^{\mathrm{B}}$ is faithful (Proposition 3.3.13);
- $P \mapsto P^{\mathrm{B}}$ is a tensor functor (Proposition 3.3.14).

Thus $\mathcal{R}$ is a neutral Tannakian category over $\mathbb{F}_q(t)$ with fiber functor $P \mapsto P^{\mathrm{B}}$ (see [11, Def. II.2.19]). ☐

3.4. **Anderson t-motives.** Here we recall the definitions and essential properties of "dual t-motives" from [2]. So as not to confuse these objects with the duals of t-motives to be used later on, we call these objects *Anderson t-motives*, since they are simply the dual notion of the objects studied in [1].

3.4.1. *Definition.* An *Anderson t-motive* M is a left $\overline{k}[t;\sigma]$-module such that

- M is free and finitely generated over $\overline{k}[t]$;
- M is free and finitely generated over $\overline{k}[\sigma]$;
- $(t-\theta)^n \mathsf{M} \subseteq \sigma\mathsf{M}$ for all $n \gg 0$.

A morphism of Anderson t-motives is a left $\overline{k}[t;\sigma]$-module homomorphism. In this way Anderson t-motives form a category.

As in §3.2.3, if $\mathsf{m} \in \mathrm{Mat}_{r\times 1}(\mathsf{M})$ is a $\overline{k}[t]$-module basis for M, then there is a matrix $\Phi = \Phi_{\mathsf{m}} \in \mathrm{Mat}_{r\times 1}(\overline{k}[t])$ so that

$$\sigma\mathsf{m} = \Phi\mathsf{m}.$$

We say that Φ represents multiplication by σ on M. Then since a power of $t-\theta$ annihilates $\mathsf{M}/\sigma\mathsf{M}$, we have

$$\det\Phi = c(t-\theta)^s$$

for some $c \in \overline{k}^{\times}$, where s is the rank of M as a $\overline{k}[\sigma]$-module.

3.4.2. *Anderson t-motives to pre-t-motives.* Given an Anderson t-motive M we obtain a pre-t-motive M by setting

$$M := \overline{k}(t) \otimes_{\overline{k}[t]} \mathsf{M}$$

and defining

$$\sigma(f \otimes m) := f^{(-1)} \otimes \sigma m.$$

It is straightforward to check that M is a left $\overline{k}(t)[\sigma, \sigma^{-1}]$-module, and it is of course finite dimensional as a $\overline{k}(t)$-vector space. Moreover, $\mathsf{M} \mapsto M$ is a functor from the category of Anderson t-motives to the category of pre-t-motives.

3.4.3. *The Carlitz motive.* Let C be the Anderson t-motive whose underlying $\overline{k}[t]$-module is $\overline{k}[t]$ itself. Then the action of σ on C is defined by

$$\sigma(f) = (t-\theta)f^{(-1)}, \quad f \in \mathsf{C}.$$

It is not difficult to check that C is an Anderson t-motive, and that its image in $\mathcal{P}$ is the Carlitz motive. For any $n \geq 1$, we also have the n-th tensor power of C,

$$\mathsf{C}(n) := \mathsf{C} \otimes_{\overline{k}[t]} \cdots \otimes_{\overline{k}[t]} \mathsf{C},$$

with diagonal σ-action. It is an Anderson t-motive sent to $C(n)$ in $\mathcal{P}$.

3.4.4. *The Carlitz module.* The *Carlitz module* $\mathfrak{C}$ *over* $\overline{k}$ is defined to be the $\mathbb{F}_q$-algebra $\overline{k}$ together with an $\mathbb{F}_q[t]$-module structure defined by

$$\mathfrak{C}_t(x) := \theta x + x^q, \quad x \in \overline{k}.$$

That is the $\mathbb{F}_q$-algebra homomorphism $a \mapsto \mathfrak{C}_a : \mathbb{F}_q[t] \to \overline{k}[\sigma^{-1}]$ defined by $t \mapsto \theta + \sigma^{-1}$ induces an $\mathbb{F}_q[t]$-module structure on $\overline{k}$. See [14, Ch. 3] or [29, §2.5] for more details. To see the relationship with the Carlitz motive, we note that there is an isomorphism

$$\mathfrak{C}(\overline{k}) \cong \frac{\mathsf{C}}{(\sigma-1)\mathsf{C}}$$

of $\mathbb{F}_q[t]$-modules. Indeed if $x \in \overline{k}$, then

$$\begin{aligned} tx &= \theta x + (t-\theta)x \\ &= \theta x + \sigma(x^q) \\ &= \theta x + x^q + (\sigma - 1)x^q. \end{aligned}$$

Similarly $ax \equiv \mathfrak{C}_a(x) \pmod{\sigma - 1}$ for all $f \in \mathbb{F}_q[t]$. It is a simple matter to check that there is a natural isomorphism of $\mathbb{F}_q$-vector spaces $\mathsf{C}/(\sigma-1)\mathsf{C} \cong \overline{k}$. Thus $\mathsf{C}/(\sigma-1)\mathsf{C}$ presents the Carlitz module directly.

Proposition 3.4.5. *For Anderson t-motives M and N, the natural map*

$$\mathrm{Hom}_{\overline{k}[t;\sigma]}(\mathsf{M}, \mathsf{N}) \otimes_{\mathbb{F}_q[t]} \mathbb{F}_q(t) \to \mathrm{Hom}_{\mathcal{P}}(M, N)$$

is an isomorphism of $\mathbb{F}_q(t)$-vector spaces.

Proof. Let Θ denote that map in question. It is clearly $\mathbb{F}_q(t)$-linear. To see that it is injective, we first observe that if $\alpha \in \mathrm{Hom}_{\overline{k}[t;\sigma]}(\mathsf{M}, \mathsf{N}) \otimes_{\mathbb{F}_q[t]} \mathbb{F}_q(t)$ then $\alpha = \underline{\phi} \otimes \frac{1}{v}$, for some $\underline{\phi} \in \mathrm{Hom}_{\overline{k}[t;\sigma]}(\mathsf{M}, \mathsf{N})$ and $v \in \mathbb{F}_q[t]$, $v \neq 0$. Then

$$v\Theta(\alpha) = \Theta(v\alpha) = \Theta(\underline{\phi} \otimes 1) =: \phi.$$

But

$$\begin{aligned} &\underline{\phi} \in \mathrm{Hom}_{\overline{k}[t;\sigma]}(\mathsf{M}, \mathsf{N}) \subseteq \mathrm{Hom}_{\overline{k}[t]}(\mathsf{M}, \mathsf{N}), \\ &\phi \in \mathrm{Hom}_{\overline{k}(t)[\sigma,\sigma^{-1}]}(M, N) \subseteq \mathrm{Hom}_{\overline{k}(t)}(M, N), \end{aligned}$$

and so $\phi = 0$ if and only if $\underline{\phi} = 0$. Thus $\Theta(\alpha) = 0$ if and only if $\alpha = 0$.

For surjectivity, suppose that $\phi \in \mathrm{Hom}_{\mathcal{P}}(M, N)$. Fix $\overline{k}[t]$-bases m and n for M and N respectively, and extend these to bases $\mathbf{m}$ and $\mathbf{n}$ of M and N. Then the map $\phi : M \to N$ is represented by a matrix $F \in \mathrm{Mat}_{r\times s}(\overline{k}(t))$ so that

$$F^{(-1)}\Phi_{\mathbf{n}} = \Phi_{\mathbf{m}} F$$

as in §3.2.3. By choice of m and n, $\Phi_{\mathbf{m}}$ and $\Phi_{\mathbf{n}}$ have entries in $\overline{k}[t]$, and it suffices to show that F has entries with denominators in $\mathbb{F}_q[t]$.

For a matrix B with entries in $\overline{k}(t)$, let $\mathrm{den}(B) \in \overline{k}[t]$ be the monic least common multiple of the denominators of the entries of B. Since $\det(\Phi_{\mathbf{n}}) = c(t-\theta)^s$ for some $s \geq 0$ and $c \in \overline{k}^{\times}$, we see that

$$\mathrm{den}(F)(t-\theta)^s \cdot F^{(-1)} = \mathrm{den}(F)(t-\theta)^s \cdot \Phi_{\mathbf{m}}\, F\, \Phi_{\mathbf{n}}^{-1} \in \mathrm{Mat}_{r\times s}(\overline{k}[t]).$$

Therefore, $\mathrm{den}(F^{(-1)})$ divides $\mathrm{den}(F)(t-\theta)^s$. However, $\mathrm{den}(F^{(-1)}) = \mathrm{den}(F)^{(-1)}$ and so $\deg(\mathrm{den}(F^{(-1)})) = \deg(\mathrm{den}(F))$. Thus, it suffices to show that $\mathrm{den}(F^{(-1)})$ is relatively prime to $t-\theta$, since then $\mathrm{den}(F)^{(-1)} = \mathrm{den}(F)$ whence all of the denominators of F are in $\mathbb{F}_q[t]$.

Suppose that $t-\theta$ divides $\mathrm{den}(F^{(-1)})$, and so $t-\theta^q$ divides $\mathrm{den}(F)$. Then $t-\theta^q$ divides $\mathrm{den}(\Phi_{\mathbf{m}}F)$, because otherwise $t-\theta^q$ would divide $\det(\Phi_{\mathbf{m}})$ which is a power of $t-\theta$. Likewise, $t-\theta^q$ divides $\mathrm{den}(\Phi_{\mathbf{m}}F\Phi_{\mathbf{n}}^{-1}) = \mathrm{den}(F^{(-1)})$. By repeating the same argument we see that $\mathrm{den}(F^{(-1)})$ is divisible by each of

$$t-\theta, t-\theta^q, t-\theta^{q^2}, \ldots$$

contradicting that $\mathrm{den}(F^{(-1)}) \in \overline{k}[t]$. □

3.4.6. *Rigid analytic triviality.* Similar to §3.3.1, if M is an Anderson t-motive, then we set

$$\mathsf{M}^{\dagger} := \mathbb{T} \otimes_{\overline{k}[t]} \mathsf{M}.$$

We provide $\mathsf{M}^{\dagger}$ with a $\overline{k}[t;\sigma]$-module structure by setting

$$\sigma(f \otimes m) = f^{(-1)} \otimes \sigma m,$$

and we set

$$\mathsf{M}^{\mathrm{B}} := (\mathsf{M}^{\dagger})^{\sigma} = \{\mu \in \mathsf{M}^{\dagger} \mid \sigma\mu = \mu\}.$$

We say that M is *rigid analytically trivial* if the natural map

$$\mathbb{T} \otimes_{\mathbb{F}_q[t]} \mathsf{M}^{\mathrm{B}} \to \mathsf{M}^{\dagger}$$

is an isomorphism. The following proposition is a companion to Proposition 3.3.9.

Proposition 3.4.7. *Let M be an Anderson t-motive, and let M be its corresponding pre-t-motive. Suppose $\mathsf{m} \in \mathrm{Mat}_{r\times 1}(\mathsf{M})$ is a $\overline{k}[t]$-basis for M, and let $\Phi \in \mathrm{Mat}_{r\times 1}(\overline{k}[t])$ represent multiplication by σ on M with respect to m.*

(a) *M is rigid analytically trivial if and only if it admits a rigid analytic trivialization Ψ with $\Psi \in \mathrm{GL}_r(\mathbb{T})$.*
(b) *If $\Psi \in \mathrm{GL}_r(\mathbb{T})$ is a rigid analytic trivialization of Φ, then the entries of $\Psi^{-1}\mathsf{m}$ form an $\mathbb{F}_q[t]$-basis of M^{B}.*
(c) *M is rigid analytically trivial if and only if M is rigid analytically trivial.*

Proof. The proofs of parts (a) and (b) are in [2, Lem. 4.4.13] and follow the same lines as their counterparts in Proposition 3.3.9. Part (c) is then a consequence of Proposition 3.3.9(c). □

3.4.8. *Definition.* We define the *category $\mathcal{A}^I$ of Anderson t-motives up to isogeny* as follows:

- Objects of $\mathcal{A}^I$: Anderson t-motives;
- Morphisms of $\mathcal{A}^I$: For Anderson t-motives M and N,

$$\mathrm{Hom}_{\mathcal{A}^I}(\mathsf{M},\mathsf{N}) := \mathrm{Hom}_{\overline{k}[t;\sigma]}(\mathsf{M},\mathsf{N}) \otimes_{\mathbb{F}_q[t]} \mathbb{F}_q(t).$$

We also define the full subcategory $\mathcal{AR}^I$ of rigid analytically trivial Anderson t-motives up to isogeny by restriction. We sum up the results of this section in the following theorem.

Theorem 3.4.9. *Let $\mathcal{P}$ be the category of pre-t-motives, and let $\mathcal{R}$ be the category of rigid analytically trivial pre-t-motives.*

(a) *The functor $\mathsf{M} \mapsto M : \mathcal{A}^I \to \mathcal{P}$ is fully faithful.*
(b) *The functor $\mathsf{M} \mapsto M : \mathcal{AR}^I \to \mathcal{R}$ is fully faithful.*

Proof. Part (a) is simply a restatement of Proposition 3.4.5. That the functor in part (b) is well-defined follows from Proposition 3.4.7(c), and its full faithfulness follows from Proposition 3.4.5. □

3.4.10. *The category $\mathcal{T}$.* We define the *category $\mathcal{T}$ of t-motives* to be the strictly full Tannakian subcategory of $\mathcal{R}$ generated by the essential image of the functor

$$\mathsf{M} \mapsto M : \mathcal{AR}^I \to \mathcal{R}.$$

The category of t-motives can further be described as follows:

- Objects of $\mathcal{T}$: rigid analytically trivial pre-t-motives that can be constructed from Anderson t-motives using direct sums, subquotients, tensor products, duals, and internal Hom's.
- Morphisms of $\mathcal{T}$: morphisms of left $\overline{k}(t)[\sigma,\sigma^{-1}]$-modules.

It is worth noting that Proposition 3.4.7(c) says that the category of t-motives is the strictly full Tannakian subcategory of $\mathcal{R}$ generated by the intersection in $\mathcal{P}$ of $\mathcal{R}$ and the image of *all* Anderson t-motives.

3.5. **Galois groups of t-motives.** Having defined a Tannakian category of t-motives, it is now possible to assign to each t-motive a linear algebraic group over $\mathbb{F}_q(t)$, which we call the *Galois group* of the t-motive. For essential facts about Tannakian categories and their associated groups, we refer to [8], [11], [24, App. B].

3.5.1. *Fiber functors.* The functor

$$\begin{array}{rcl} \omega : \mathcal{T} & \to & \mathbf{Vec}(\mathbb{F}_q(t)) \\ M & \mapsto & M^{\mathrm{B}} \end{array}$$

is the fiber functor of $\mathcal{T}$. For any commutative $\mathbb{F}_q(t)$-algebra R, we let $\omega^{(R)} : \mathcal{T} \to \mathbf{Mod}(R)$ be the extension of ω defined by

$$\omega^{(R)}(M) := R \otimes_{\mathbb{F}_q(t)} M^{\mathrm{B}},$$

where $\mathbf{Mod}(R)$ is the category of finitely generated left R-modules. Now fix a t-motive M. We let $\mathcal{T}_M$ be the strictly full Tannakian subcategory of $\mathcal{T}$ generated by M. That is, $\mathcal{T}_M$ consists of all objects of $\mathcal{T}$ isomorphic to subquotients of finite direct sums of $M^{\otimes u} \otimes (M^\vee)^{\otimes v}$ for various u, v. The fiber functor of $\mathcal{T}_M$ is $\omega_M : \mathcal{T}_M \to \mathbf{Vec}(\mathbb{F}_q(t))$, the restriction of ω to $\mathcal{T}_M$, and similarly we restrict $\omega_M^{(R)}$ to $\mathcal{T}_M$ for an $\mathbb{F}_q(t)$-algebra R.

3.5.2. *Galois groups.* As $\mathcal{T}$ is a neutral Tannakian category over $\mathbb{F}_q(t)$, there is an affine group scheme $\Gamma_{\mathcal{T}}$ over $\mathbb{F}_q(t)$ so that $\mathcal{T}$ is equivalent to the category $\mathbf{Rep}(\Gamma_{\mathcal{T}}, \mathbb{F}_q(t))$ of finite dimensional representations of $\Gamma_{\mathcal{T}}$ over $\mathbb{F}_q(t)$:

$$\mathcal{T} \approx \mathbf{Rep}(\Gamma_{\mathcal{T}}, \mathbb{F}_q(t)).$$

The group $\Gamma_{\mathcal{T}}$ is defined to be the group of tensor automorphisms of the fiber functor ω; that is, if R is any $\mathbb{F}_q(t)$-algebra, then

$$\Gamma_{\mathcal{T}}(R) = \mathrm{Aut}_{\mathcal{T}}^{\otimes}\bigl(\omega^{(R)}\bigr).$$

Now for any t-motive M, there is a linear algebraic group $\Gamma_M := \Gamma_{\mathcal{T}_M}$ over $\mathbb{F}_q(t)$ so that $\mathcal{T}_M$ is equivalent to $\mathbf{Rep}(\Gamma_M, \mathbb{F}_q(t))$. As such, for any $\mathbb{F}_q(t)$-algebra R,

$$\Gamma_M(R) = \mathrm{Aut}_{\mathcal{T}_M}^{\otimes}\bigl(\omega_M^{(R)}\bigr).$$

In this way we find that we have a naturally defined faithful representation

$$\Gamma_M \hookrightarrow \mathrm{GL}(M^{\mathrm{B}})$$

over $\mathbb{F}_q(t)$, which provides the basis for constructing the equivalence of categories,

$$\mathcal{T}_M \approx \mathbf{Rep}(\Gamma_M, \mathbb{F}_q(t)).$$

The group Γ_M is called the *Galois group of* M. Furthermore, there is a surjective group homomorphism,

$$\Gamma_{\mathcal{T}} \twoheadrightarrow \Gamma_M.$$

If N is another t-motive in $\mathcal{T}_M$, then there is a natural surjective homomorphism,

$$\Gamma_M \twoheadrightarrow \Gamma_N.$$

In §4, we will show that Γ_M can be calculated using systems of σ-semilinear equations. For now we will calculate the Galois group of the Carlitz motive C.

Lemma 3.5.3. *For m, $n \in \mathbb{Z}$,*

$$\mathrm{Hom}_{\mathcal{T}}(C(m), C(n)) \cong \begin{cases} \mathbb{F}_q(t) & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases}$$

Proof. By tensoring with $C(-m)$, we see that $\mathrm{Hom}_{\mathcal{T}}(C(m), C(n)) \cong \mathrm{Hom}_{\mathcal{T}}(\mathbf{1}, C(n-m))$. Thus it suffices to assume that $m = 0$. If $\phi : \mathbf{1} \to C(n)$ is a morphism in $\mathcal{T}$, then ϕ is represented by some $a \in \overline{k}(t)^{\times}$ such that $a = a^{(-1)}(t-\theta)^n$. By Lemma 3.3.5, this equation has no nonzero solutions $a \in \overline{k}(t)$ unless $n = 0$, in which case a can be anything in $\mathbb{F}_q(t)$. □

Theorem 3.5.4. *For the Carlitz motive C, there is an isomorphism $\Gamma_C \cong \mathbb{G}_{\mathrm{m}}$ over $\mathbb{F}_q(t)$.*

Proof. It is easy enough to check this theorem directly. However, by Lemma 3.5.3, $\mathcal{T}_C$ is equivalent to a $\mathbb{Z}$-graded category of vector spaces over $\overline{k}(t)$ with a fiber functor to $\mathbf{Vec}(\mathbb{F}_q(t))$, and so its Galois group is $\mathbb{G}_{\mathrm{m}}$ over $\mathbb{F}_q(t)$ [11, Ex. II.2.30]. □

4. Galois theory of systems of σ-semilinear equations

In this section we demonstrate how to calculate the Galois group of a t-motive as the Galois group of a certain system of difference equations with respect to the automorphism $\sigma : \overline{k}(t) \to \overline{k}(t)$. These systems of equations and their Galois groups are similar to systems of linear differential equations and their Galois groups, and one should compare our constructions with [23, Ch. 1], [24, Chs. 1–2], which we have used as guides, as well as [4], [6], [10], [17], [19], [22]. For an example of a Galois group of this type in the context of t-motives, see also the proof of [7, Prop. 7.1].

Van der Put and Singer [23] have developed the theory of Picard-Vessiot rings for linear difference equations which is quite useful in our context. However, their treatment generally assumes that the field of constants is algebraically closed. In our case the field of constants is $\mathbb{F}_q(t)$, which presents several difficulties. On the other hand, the Picard-Vessiot rings treated in [23] are not always domains, whereas our central Picard-Vessiot rings *are* domains by construction, which provides several benefits for the characterization of their Galois groups. Because there is inevitable overlap between our constructions and the Galois theory of difference equations with algebraically closed fields of constants and the Galois theory of differential equations, when not crucial to the overall content of the paper we occasionally reference proofs in the literature that suit our purposes with only minor modifications. It is worth noting that some of what is covered here is covered by the theory of Y. André [4], but we present everything from scratch for completeness.

In this section we develop the theory of σ-semilinear equations in some generality. Then in §5 we specialize to the systems of equations related to t-motives.

4.1. Solutions of σ-semilinear equations.

4.1.1. *The automorphism $\overline{\sigma}$.* Throughout this section we fix a positive power σ^n of σ, and we let $\overline{\sigma} := \sigma^n : \mathbb{K}\langle\langle t\rangle\rangle \to \mathbb{K}\langle\langle t\rangle\rangle$.

4.1.2. *Fields of definition.* Let $F \subseteq K \subseteq L$ be extensions of $\mathbb{F}_q(t)$ that are also subfields of $\mathbb{K}\langle\langle t\rangle\rangle$. We say that the triple (F, K, L) is *$\overline{\sigma}$-admissible* if

- $\overline{\sigma}$ restricts to an automorphism of each of F, K, and L;
- $F = F^{\overline{\sigma}} = K^{\overline{\sigma}} = L^{\overline{\sigma}}$;
- K and L are separable extensions of F.

The separability hypothesis is not strictly necessary, but it is crucial for the specific applications we will consider. See Corollary 5.1.3, Proposition 5.1.5, and §5.2.2.

The primary examples of σ-admissible fields that we have in mind are

$$(4.1.2.1)\qquad \begin{aligned}(F,K,L) &= (\mathbb{F}_q(t), \overline{k}(t), \mathbb{L}),\\ (F,K,L) &= (\mathbf{F}, \overline{\overline{k}(t)}, \overline{\mathbb{L}}), \quad \mathbf{F} := \overline{\mathbb{F}_q(t)} \cap \mathbb{F}_q\langle\langle t\rangle\rangle.\end{aligned}$$

For the first example, it should be noted that both $\overline{k}(t)$ and $\mathbb{L}$ are linearly disjoint from $\mathbb{F}_q(t^{1/p})$ and therefore are separable over $\mathbb{F}_q(t)$ [32, §II.15, Thm. 34]. For the second example, $\mathbf{F}$ is the intersection of perfect fields and so is perfect. That $\overline{\overline{k}(t)}^{\sigma} = \mathbf{F}$ was presented in §2.2.5. Moreover, any element of $\overline{\mathbb{L}}^{\sigma}$ is algebraic over $\mathbb{L}^{\sigma} = \mathbb{F}_q(t)$, and so $\overline{\mathbb{L}}^{\sigma} = \mathbf{F}$. Likewise, for general $n \geq 1$, the σ^n-admissible fields of interest to us will be

$$(4.1.2.2)\qquad \begin{aligned}(F,K,L) &= (\mathbb{F}_{q^n}(t), \overline{k}(t), \mathbb{L}),\\ (F,K,L) &= (\mathbf{F}_n, \overline{\overline{k}(t)}, \overline{\mathbb{L}}), \quad \mathbf{F}_n := \overline{\mathbb{F}_q(t)} \cap \mathbb{F}_{q^n}\langle\langle t\rangle\rangle.\end{aligned}$$

Henceforth we shall assume that a $\overline{\sigma}$-admissible triple (F, K, L) has been chosen.

4.1.3. *Convention.* If $\rho : S \to R$ is a homomorphism of modules or rings, and $B \in \mathrm{Mat}_{r\times s}(S)$, we let $\rho(S) \in \mathrm{Mat}_{r\times s}(R)$ be the matrix obtained by applying ρ to the entries of B.

4.1.4. *The ring* $K[\overline{\sigma}, \overline{\sigma}^{-1}]$. We let $K[\overline{\sigma}, \overline{\sigma}^{-1}]$ be the noncommutative ring of Laurent polynomials in $\overline{\sigma}$ with coefficients in K, subject to the relation

$$\overline{\sigma} \cdot f = \overline{\sigma}(f) \cdot \overline{\sigma}, \quad f \in K.$$

Though the symbol "$\overline{\sigma}$" plays the role of automorphism of K and the role ring element, there will be no confusion in what follows since their actions will always be compatible.

4.1.5. *Definition.* The field L is a left $K[\overline{\sigma}, \overline{\sigma}^{-1}]$-module via the automorphism $\overline{\sigma} : L \to L$. This action extends entry-wise to groups of matrices $\mathrm{Mat}_{r\times s}(L)$. Given a matrix $\Phi \in \mathrm{GL}_r(K)$, we consider vectors $\psi \in \mathrm{Mat}_{r\times 1}(L)$ that satisfy

$$\overline{\sigma}(\psi) = \Phi\psi.$$

In this way, we define a *system of* $\overline{\sigma}$*-semilinear equations*, and ψ is a solution. The set of solutions

$$\mathrm{Sol}(\Phi) := \{\psi \in \mathrm{Mat}_{r\times 1}(L) \mid \overline{\sigma}(\psi) = \Phi\psi\}$$

is an F-vector space.

Lemma 4.1.6. *Let* $\Phi \in \mathrm{GL}_r(K)$. *Suppose that* $\psi_1, \dots, \psi_m \in \mathrm{Sol}(\Phi)$ *are linearly independent over* F. *Then they are linearly independent over* L.

Proof. The proof is essentially the same as the one for Lemma 3.3.7, and we omit it. □

Corollary 4.1.7. *Let* $\Phi \in \mathrm{Mat}_r(K)$. *Then* $\mathrm{Sol}(\Phi)$ *is an* F*-vector space of dimension at most* r.

4.1.8. *Fundamental matrix of solutions.* Given $\Phi \in \mathrm{GL}_r(K)$, suppose $\Psi \in \mathrm{GL}_r(L)$ satisfies

$$\overline{\sigma}(\Psi) = \Phi\Psi.$$

Then by Lemma 4.1.6 and Corollary 4.1.7, the columns of Ψ form an F-basis for $\mathrm{Sol}(\Phi)$. The matrix Ψ is called a *fundamental matrix for* Φ.

4.2. **Picard-Vessiot extensions.**

4.2.1. $\overline{\sigma}$*-algebras.* A commutative K-algebra Σ that also has a compatible structure as a left $K[\overline{\sigma}, \overline{\sigma}^{-1}]$-module is called a $\overline{\sigma}$*-algebra*. A *morphism of* $\overline{\sigma}$*-algebras* is simply a K-algebra homomorphism that is also a homomorphism of left $K[\overline{\sigma}, \overline{\sigma}^{-1}]$-modules. In this way $\overline{\sigma}$-algebras form a category.

A $\overline{\sigma}$*-ideal* of Σ is an ideal that is also a $K[\overline{\sigma}, \overline{\sigma}^{-1}]$-submodule. A maximal $\overline{\sigma}$-ideal is an ideal that is maximal among all proper $\overline{\sigma}$-ideals. The $\overline{\sigma}$-algebra Σ is *simple* if its only $\overline{\sigma}$-ideals are (0) and Σ. A $\overline{\sigma}$*-field* is a $\overline{\sigma}$-algebra that is also a field.

For a $\overline{\sigma}$-algebra Σ, we let

$$\Sigma^{\overline{\sigma}} := \{h \in \Sigma \mid \overline{\sigma}(h) = h\}$$

be the F-subalgebra of $\overline{\sigma}$-fixed elements. We also set

$$\mathrm{Aut}^{\overline{\sigma}}(\Sigma) := \{\overline{\sigma}\text{-algebra automorphisms of } \Sigma\}.$$

That is, $\mathrm{Aut}^{\overline{\sigma}}(\Sigma)$ consists of all K-algebra automorphisms of Σ that commute with $\overline{\sigma}$. If $\Sigma' \subseteq \Sigma$ is a $\overline{\sigma}$-subalgebra, then

$$\mathrm{Aut}^{\overline{\sigma}}(\Sigma/\Sigma') := \{\eta \in \mathrm{Aut}^{\overline{\sigma}}(\Sigma) \mid \eta|_{\Sigma'} = \mathrm{id}\}.$$

Lemma 4.2.2. *Let* Σ *be a simple* $\overline{\sigma}$*-algebra.*

(a) Σ *is reduced and* $\Sigma^{\overline{\sigma}}$ *is a field.*
(b) *If* Σ *is finitely generated as a* K*-algebra, then* $\Sigma^{\overline{\sigma}}$ *is an algebraic extension of* F*.*
(c) *If* Σ *is finitely generated as a* K*-algebra,* Σ *is a domain, and* K *is algebraically closed, then* $\Sigma^{\overline{\sigma}} = K^{\overline{\sigma}}$*.*

Proof. Let $\mathfrak{a} \subseteq \Sigma$ be the ideal of nilpotent elements of Σ. It is clearly a $\overline{\sigma}$-ideal. The simplicity of Σ implies that $\mathfrak{a} = 0$, and so Σ is reduced. Now let $h \in \Sigma^{\overline{\sigma}}$, $h \neq 0$. Then $h \cdot \Sigma$ is a $\overline{\sigma}$-ideal different from (0), and so $h \in \Sigma^{\times}$. Easily $\overline{\sigma}(h^{-1}) = h^{-1}$, and so $h \in (\Sigma^{\overline{\sigma}})^{\times}$. Thus $\Sigma^{\overline{\sigma}}$ is a field, which proves (a).

For (b), suppose that Σ is finitely generated as a K-algebra, and let $h \in \Sigma^{\overline{\sigma}} \setminus F$. Since $\Sigma^{\overline{\sigma}}$ is a field, each element of the form $h - a$, with $a \in F$, is invertible in Σ. Set $\nu : K[z] \to \Sigma$, where $K[z]$ is a polynomial ring in one variable, to be the evaluation homomorphism at h. Since Σ is a finitely generated K-algebra, the image of $\nu^\sharp$: $\operatorname{Spec}\Sigma \to \mathbb{A}^1_K$ is constructible by Chevalley's constructibility theorem (see [20, §2.6]). Thus the image of $\nu^\sharp$ is either a finite set or the complement of a finite set. However, the maximal ideal $(z-a) \subseteq K[z]$ is not in the image of $\nu^\sharp$ for any $a \in F$, and so the image of $\nu^\sharp$ must be finite. By passing to $\overline{K}$, we see that $\nu^\sharp \times_K \mathrm{id}_{\overline{K}}$ must be constant on the irreducible components of $\operatorname{Spec}\Sigma \times_K \overline{K}$. Thus ν itself has a nontrivial kernel. Let $F(z) = z^m + f_{m-1}z^{m-1} + \cdots + f_0 \in K[z]$ have minimal positive degree so that $F(h) = 0$. Then

$$F(h) - \overline{\sigma}(F(h)) = (f_{m-1} - \overline{\sigma}(f_{m-1}))h^{m-1} + \cdots + (f_0 - \overline{\sigma}(f_0)) = 0,$$

and the minimality of the degree of F implies that each $f_i \in F$. Thus h is algebraic over F. This proves (b) (cf. [24, Lem. 1.17]).

Now suppose that Σ is a domain and that $K = \overline{K}$, and let $h \in \Sigma^{\overline{\sigma}}$. We make $K[z]$ into a $\overline{\sigma}$-algebra by setting $\overline{\sigma}(z) := z$. Then the map $\mu = \nu|_{F[z]} : F[z] \to \Sigma$ is a map of $\overline{\sigma}$-algebras. The image of μ is the field $F(h) \subseteq \Sigma$. Because Σ is a domain, the compositum $K \cdot F(h)$ is a subfield of Σ. Since K is algebraically closed, we have $K \cdot F(h) = K$, and so $h \in K$. □

4.2.3. *Remark.* A simple $\overline{\sigma}$-algebra need not be a domain and $\Sigma^{\overline{\sigma}}$ need not be contained in $K^{\overline{\sigma}}$ or even $\overline{F}^{\overline{\sigma}}$. For example, take $(F, K, L) = (\mathbb{F}_q(t), \overline{k}(t), \mathbb{L})$ with $\overline{\sigma} = \sigma$. Let $\Sigma = \overline{k}(t) \oplus \overline{k}(t)$, and define a $\overline{k}(t)[\sigma, \sigma^{-1}]$-module structure on Σ by

$$\sigma(f, g) := (g^{(-1)}, f^{(-1)}), \quad f, g \in \overline{k}(t).$$

The proper non-trivial ideals of Σ are $\overline{k}(t) \times 0$ and $0 \times \overline{k}(t)$, and neither is a σ-ideal. Thus Σ is simple but not a domain. We also see that

$$\Sigma^\sigma = \{(f, f^{(1)}) \mid f \in \mathbb{F}_{q^2}(t)\},$$

which is isomorphic to $\mathbb{F}_{q^2}(t) \not\subseteq \overline{\mathbb{F}_q(t)}^{\sigma}$.

4.2.4. *The ring* Σ_0. Fix $r \geq 1$. Let $X := (X_{ij})$ be an $r \times r$ matrix whose entries are independent variables X_{ij}, and set $\Delta := \det(X)$. We now take

$$\Sigma_0 := K[X, \Delta^{-1}] = K[X_{ij}, \Delta^{-1}],$$

which is formed from the commutative polynomial ring $K[X] = K[X_{ij}]$ by inverting Δ. Elements $h \in \Sigma_0$ will be denoted $h(X) := h(X_{ij})$.

Now suppose $\Phi \in \mathrm{GL}_r(K)$. We make Σ_0 into a left $K[\overline{\sigma}, \overline{\sigma}^{-1}]$-module, and thus into a $\overline{\sigma}$-algebra, in the unique way so that

- K acts on Σ_0 by usual left multiplication;
- $\overline{\sigma}$ acts on Σ_0 by setting $\overline{\sigma}X := \Phi X$.

We define a left action of $\mathrm{GL}_r(F)$ on Σ_0 in the following way:
$$\gamma * h := h(X\gamma), \quad \gamma \in \mathrm{GL}_r(F),\ h \in \Sigma_0.$$
This action commutes with the $\overline{\sigma}$-algebra structure on Σ_0, and so we have an injective group homomorphism,
$$\mathrm{GL}_r(F) \hookrightarrow \mathrm{Aut}^{\overline{\sigma}}(\Sigma_0).$$

4.2.5. *Picard-Vessiot extensions.* Let $\Phi \in \mathrm{GL}_r(K)$, and suppose that Ψ is a fundamental matrix for Φ: $\overline{\sigma}(\Psi) = \Phi\Psi$. Then we can define a K-algebra homomorphism
$$\nu_\Psi : \Sigma_0 \to L, \qquad X_{ij} \mapsto \Psi_{ij}$$
and it follows that ν_Ψ is a homomorphism of left $K[\overline{\sigma}, \overline{\sigma}^{-1}]$-modules. Now let
$$\mathfrak{p}_\Psi := \ker(\nu_\Psi).$$
One checks that $\mathfrak{p}_\Psi$ is a prime ideal and also a $\overline{\sigma}$-ideal. Furthermore, $h \in \mathfrak{p}_\Psi$ if and only if $\overline{\sigma}(h) \in \mathfrak{p}_\Psi$. Thus we have an isomorphism of $\overline{\sigma}$-algebras,
$$\Sigma_0/\mathfrak{p}_\Psi \cong K\bigl[\Psi, \Delta(\Psi)^{-1}\bigr] =: \Sigma_\Psi.$$
Since $\overline{\sigma}(\mathfrak{p}_\Psi) = \mathfrak{p}_\Psi$, it follows that $\overline{\sigma} : \Sigma_\Psi \to \Sigma_\Psi$ is bijective. The ring Σ_Ψ is called a *Picard-Vessiot extension of K for Φ* (cf. [24, §1.3–4]).

Lemma 4.2.6. *Let $\Phi \in \mathrm{Mat}_r(K)$, and suppose $\Psi \in \mathrm{GL}_r(L)$ is a fundamental matrix for Φ. Then*
- (a) $(\Sigma_\Psi)^{\overline{\sigma}} = F$.
- (b) *If $\Psi' \in \mathrm{GL}_r(L)$ is also a fundamental matrix for Φ, then $\Sigma_\Psi \cong \Sigma_{\Psi'}$ as $\overline{\sigma}$-algebras.*
- (c) *Let Λ_Ψ be the fraction field of Σ_Ψ. Then $\mathrm{Aut}^{\overline{\sigma}}(\Lambda_\Psi) \cong \mathrm{Aut}^{\overline{\sigma}}(\Sigma_\Psi)$.*

Proof. Since Σ_Ψ is a left $K[\overline{\sigma}, \overline{\sigma}^{-1}]$-submodule of L containing F, we see that $F \subseteq (\Sigma_\Psi)^{\overline{\sigma}} \subseteq L^{\overline{\sigma}} = F$, proving (a).

Since Ψ and Ψ' have entries in L, by part (a) and the definition of fundamental matrix, there is a $\gamma \in \mathrm{GL}_r(F)$ such that $\gamma = (\Psi')^{-1}\Psi$. It follows easily that $\gamma * \mathfrak{p}_\Psi = \mathfrak{p}_{\Psi'}$, which proves (b).

For part (c) (cf. [24, (3) p.19]), we first observe that the $\overline{\sigma}$-algebra structure of Σ_Ψ extends to a $\overline{\sigma}$-algebra structure of Λ_Ψ in a unique way. Also, any $\xi \in \mathrm{Aut}^{\overline{\sigma}}(\Sigma_\Psi)$ extends uniquely to a $\overline{\sigma}$-algebra automorphism of Λ_Ψ. Suppose $\eta \in \mathrm{Aut}^{\overline{\sigma}}(\Lambda_\Psi)$. Then as matrices in $\mathrm{Mat}_r(L)$,
$$\overline{\sigma}(\eta(\Psi)) = \eta(\overline{\sigma}(\Psi)) = \eta(\Phi\Psi) = \Phi\eta(\Psi).$$
Thus $\eta(\Psi) \in \mathrm{Mat}_r(L)$ is a fundamental matrix for Φ. As in the proof of part (b), there is a $\gamma \in \mathrm{GL}_r(F)$ so that $\eta(\Psi) = \Psi\gamma \in \mathrm{Mat}_r(\Sigma_\Psi)$. Therefore, η restricted to Σ_Ψ takes values in Σ_Ψ and is an element of $\mathrm{Aut}^{\overline{\sigma}}(\Sigma_\Psi)$. □

4.3. **Base extensions of Picard-Vessiot rings.**

4.3.1. *Base extension of $\overline{\sigma}$-algebras.* Fix a commutative F-algebra R. If Σ is any $\overline{\sigma}$-algebra, then we set $\Sigma^{(R)} := R \otimes_F \Sigma$. Then $\Sigma^{(R)}$ is made into a $\overline{\sigma}$-algebra by setting
$$\overline{\sigma}(c \otimes h) := c \otimes \overline{\sigma}h, \quad c \in R,\ h \in \Sigma.$$
The map
$$c \mapsto c \otimes 1 : R \to \Sigma^{(R)}$$
is an injective map of F-algebras, and so we can consider R to be an F-subalgebra of $\Sigma^{(R)}$. Also, the map
$$h \mapsto 1 \otimes h : \Sigma \to \Sigma^{(R)}$$

is an injective map of $\overline{\sigma}$-algebras, and so we can consider Σ to be a $\overline{\sigma}$-subalgebra of $\Sigma^{(R)}$. We will often abuse notation and write

$$R \subseteq \Sigma^{(R)}, \quad \Sigma \subseteq \Sigma^{(R)},$$

though for emphasis we may write $c \otimes 1$ and $1 \otimes h$ for the images of $c \in R$ and $h \in \Sigma$ in $\Sigma^{(R)}$.

If $\mathfrak{a}$ is a $\overline{\sigma}$-ideal of Σ, we let $\mathfrak{a}^{(R)}$ be the ideal generated by $\mathfrak{a}$ in $\Sigma^{(R)}$. It is a $\overline{\sigma}$-ideal. In general, since R is an F-vector space,

$$(\Sigma/\mathfrak{a})^{(R)} \cong \Sigma^{(R)}/\mathfrak{a}^{(R)},$$

as K-algebras. We let

$$\mathrm{Aut}_R^{\overline{\sigma}}(\Sigma^{(R)}) := \mathrm{Aut}^{\overline{\sigma}}(\Sigma^{(R)}/K^{(R)}).$$

Thus automorphisms in $\mathrm{Aut}_R^{\overline{\sigma}}(\Sigma^{(R)})$ are $\overline{\sigma}$-algebra automorphisms that are also $K^{(R)}$-algebra homomorphisms. We will say that such automorphisms are *$\overline{\sigma}$-algebra automorphisms over R*.

Suppose Σ and Σ' are $\overline{\sigma}$-algebras and $\phi : S \to R$ is a homomorphism of F-algebras. If $\rho : \Sigma^{(S)} \to \Sigma'^{(S)}$ is a $\overline{\sigma}$-algebra homomorphism, then since $\Sigma^{(R)} \cong R \otimes_S \Sigma^{(S)}$ and $\Sigma'^{(R)} \cong R \otimes_S \Sigma'^{(S)}$, this defines a $\overline{\sigma}$-algebra homomorphism

$$\phi \cdot \rho = 1 \otimes \rho : \Sigma^{(R)} \to \Sigma'^{(R)}.$$

Similar to Lemma 4.1.6, we have the following lemma, whose proof is straightforward.

Lemma 4.3.2. *Let R be an F-algebra. Then* $\bigl(L^{(R)}\bigr)^{\overline{\sigma}} = R$.

4.3.3. *The rings $\Sigma_0^{(R)}$ and $\Sigma_\Psi^{(R)}$.* The ring $\Sigma_0^{(R)}$ has a natural description,

$$\Sigma_0^{(R)} \cong K^{(R)}[X, \Delta^{-1}],$$

and for any $\overline{\sigma}$-ideal $\mathfrak{a}$ of Σ_0,

$$\mathfrak{a}^{(R)} = \mathfrak{a} \cdot \Sigma_0^{(R)} \cong \mathfrak{a} \cdot K^{(R)}[X, \Delta^{-1}].$$

The maps $\overline{\sigma} : \Sigma_0^{(R)} \to \Sigma_0^{(R)}$ and $\overline{\sigma} : \Sigma_\Psi^{(R)} \to \Sigma_\Psi^{(R)}$ are both bijective.

If $\Phi \in \mathrm{GL}_r(K)$ and if $\Psi \in \mathrm{GL}_r(L)$ is a fundamental matrix for Φ, then we set

$$\mathfrak{P}_\Psi^{(R)} := \mathfrak{p}_\Psi \cdot \Sigma_0^{(R)}. \tag{4.3.3.1}$$

Since R is an F-vector space, we have

$$\mathfrak{P}_\Psi^{(R)} = R \otimes_F \mathfrak{p}_\Psi = \ker\bigl(X_{ij} \mapsto \Psi_{ij} : \Sigma_0^{(R)} \to L^{(R)}\bigr).$$

Also $\mathfrak{P}_\Psi^{(R)}$ is a $\overline{\sigma}$-ideal, and for any $h \in \Sigma_0^{(R)}$, $\overline{\sigma}(h) \in \mathfrak{P}_\Psi^{(R)}$ if and only if $h \in \mathfrak{P}_\Psi^{(R)}$. Thus the ring $\Sigma_\Psi^{(R)}$ carries an isomorphism of $\overline{\sigma}$-algebras,

$$\Sigma_\Psi^{(R)} \cong \Sigma_0^{(R)}/\mathfrak{P}_\Psi^{(R)}.$$

If $\Psi' \in \mathrm{GL}_r\bigl(L^{(R)}\bigr)$ also satisfies $\overline{\sigma}(\Psi') = \Phi\Psi'$, then Ψ' is called a *fundamental matrix for Φ over R*.

Lemma 4.3.4. *Let $\Phi \in \mathrm{GL}_r(K)$, and suppose $\Psi \in \mathrm{GL}_r(L)$ is a fundamental matrix for Φ.*

(a) $\bigl(\Sigma_\Psi^{(R)}\bigr)^{\overline{\sigma}} = R$.
(b) *If $\Psi' \in \mathrm{GL}_r\bigl(L^{(R)}\bigr)$ is a fundamental matrix for Φ over R, then $\Psi^{-1}\Psi' \in \mathrm{GL}_r(R)$.*

Proof. Since R is an F-vector space, the natural map of $\overline{\sigma}$-algebras $\Sigma_\Psi^{(R)} \hookrightarrow L^{(R)}$ is injective. Thus we can consider $\Sigma_\Psi^{(R)}$ to be a $\overline{\sigma}$-subalgebra of $L^{(R)}$. Part (a) then follows from Lemma 4.3.2. For part (b), it follows from the definition of fundamental matrix that

$$\overline{\sigma}(\Psi^{-1}\Psi') = \Psi^{-1}\Psi'.$$

By Lemma 4.3.2, $\delta := \Psi^{-1}\Psi'$ has entries in R and is in $\mathrm{GL}_r\bigl(L^{(R)}\bigr)$, and it is easy to check that $\delta \in \mathrm{GL}_r(R)$. □

4.3.5. *Picard-Vessiot extensions over* $K^{(R)}$. For $\Phi \in \mathrm{GL}_r(K)$, suppose that $\Psi \in \mathrm{GL}_r(L)$ is a fundamental matrix for Φ. If $\Psi' \in \mathrm{GL}_r\bigl(L^{(R)}\bigr)$ is a fundamental matrix for Φ over R, we then set

$$\mathfrak{P}_{\Psi'}^{(R)} := \ker\bigl(X_{ij} \mapsto \Psi'_{ij} : \Sigma_0^{(R)} \to L^{(R)}\bigr). \tag{4.3.5.1}$$

It is a straightforward matter to check that

$$\gamma * \mathfrak{P}_\Psi^{(R)} = \mathfrak{P}_{\Psi\gamma^{-1}}^{(R)}$$

for any $\gamma \in \mathrm{GL}_r(R)$. We have the following lemma.

Lemma 4.3.6. *Suppose $\Phi \in \mathrm{GL}_r(K)$, and suppose that $\Psi \in \mathrm{GL}_r(L)$ is a fundamental matrix for Φ. For any F-algebra R, suppose $\gamma * \mathfrak{P}_\Psi^{(R)} \subseteq \mathfrak{P}_\Psi^{(R)}$. Then*

$$\gamma * \mathfrak{P}_\Psi^{(R)} = \mathfrak{P}_\Psi^{(R)}.$$

Proof. Fix $\gamma \in \mathrm{GL}_r(R)$ satisfying $\gamma * \mathfrak{P}_\Psi^{(R)} \subseteq \mathfrak{P}_\Psi^{(R)}$. Then γ induces a $\overline{\sigma}$-algebra automorphism $\gamma : \Sigma_0^{(R)} \to \Sigma_0^{(R)}$. Suppose $\mathfrak{p}_\Psi = (f_1(X), \dots, f_m(X))\cdot\Sigma_0$. Then by hypothesis on γ, there is a matrix $N(X) \in \mathrm{Mat}_m\bigl(\Sigma_0^{(R)}\bigr)$ so that

$$\gamma * \begin{bmatrix} f_1(X) \\ \vdots \\ f_m(X) \end{bmatrix} = N(X) \cdot \begin{bmatrix} f_1(X) \\ \vdots \\ f_m(X) \end{bmatrix}. \tag{4.3.6.1}$$

Fix an F-basis $\{\varepsilon_l\}_{l\in I}$ of $\Sigma_0 = K[X, \Delta^{-1}]$. Then $\{1 \otimes \varepsilon_l\}_{l\in I}$ is an R-basis of $\Sigma_0^{(R)}$. Thus for $1 \le u, v \le m$, we can write the entry $N_{uv}(X)$ of $N(X)$ as

$$N_{uv}(X) = \sum_{l\in I} c(u,v,l) \otimes \varepsilon_l(X), \quad c(u,v,l) \in R,$$

where the $c(u,v,l)$ are unique and zero for all but finitely many l. Set

$$\overline{R} := F[\gamma_{ij}, \det(\gamma)^{-1}, c(u,v,l) : 1 \le u,v \le m;\ l \in I],$$

which is a finitely generated F-subalgebra of R. Then $\Sigma_0^{(\overline{R})} \subseteq \Sigma_0^{(R)}$. Moreover, since $\Sigma_0^{(\overline{R})} = \overline{R} \otimes_F K[X, \Delta^{-1}]$ and since $\overline{R} \otimes_F K$ is finitely generated as a K-algebra, we conclude that $\Sigma_0^{(\overline{R})}$ is noetherian.

By design, $\gamma : \Sigma_0^{(\overline{R})} \to \Sigma_0^{(\overline{R})}$ is a $\overline{\sigma}$-algebra automorphism that commutes with $K^{(\overline{R})}$, and each $N_{uv}(X) \in \Sigma_0^{(\overline{R})}$. Let

$$\mathfrak{a} := \mathfrak{p}_\Psi \cdot \Sigma_0^{(\overline{R})} = (f_1(X), \dots, f_m(X)) \cdot \Sigma_0^{(\overline{R})}.$$

By (4.3.6.1), it follows that $\gamma * f_j(X) \in \mathfrak{a}$ for each j, and so $\gamma * \mathfrak{a} \subseteq \mathfrak{a}$. Since $\gamma : \Sigma_0^{(\overline{R})} \to \Sigma_0^{(\overline{R})}$ is an automorphism over $K^{(\overline{R})}$, we can form an ascending chain of ideals

$$\mathfrak{a} \subseteq \gamma^{-1} * \mathfrak{a} \subseteq \gamma^{-2} * \mathfrak{a} \subseteq \cdots \subseteq \Sigma_0^{(\overline{R})}.$$

Because $\Sigma_0^{(R)}$ is noetherian, we conclude that this chain stabilizes, and so

$$\gamma * \mathfrak{a} = \mathfrak{a}.$$

Finally, since

$$\mathfrak{P}_\Psi^{(R)} = (f_1(X), \cdots, f_m(X)) \cdot \Sigma_0^{(R)} = \mathfrak{a} \cdot \Sigma_0^{(R)},$$

it follows that $\gamma * \mathfrak{P}_\Psi^{(R)} = \mathfrak{P}_\Psi^{(R)}$. □

4.3.7. *Ideal correspondence.* The following lemma provides a correspondence between the extension and contraction of ideals in certain $\overline{\sigma}$-algebras. The proof is similar to the one for [24, Lem.1.23] with minor modifications, so we omit it.

Lemma 4.3.8. *Let Λ be a $\overline{\sigma}$-field, and let $F = \Lambda^{\overline{\sigma}}$. We fashion a $\overline{\sigma}$-algebra structure on $\Lambda[Y, \Delta(Y)^{-1}]$ by setting $\overline{\sigma}(Y_{ij}) := (Y_{ij})$. The correspondences*

$$\begin{array}{ccc} \{\text{ideals of } F[Y,\Delta(Y)^{-1}]\} & \longleftrightarrow & \{\overline{\sigma}\text{-ideals of } \Lambda[Y,\Delta(Y)^{-1}]\} \\ \mathfrak{a} & \rightarrow & \mathfrak{a}\cdot\Lambda[Y,\Delta(Y)^{-1}] \\ \mathfrak{b}\cap F[Y,\Delta(Y)^{-1}] & \leftarrow & \mathfrak{b} \end{array}$$

are bijections.

Lemma 4.3.9. *Let $\Phi \in \mathrm{GL}_r(K)$, and suppose $\Psi \in \mathrm{GL}_r(L)$ is a fundamental matrix for Φ. For any ideal $\mathfrak{a} \subseteq \Sigma_0$ and any F-algebra R,*

$$\mathfrak{a} = \left(\mathfrak{a}\cdot\Sigma_0^{(R)}\right) \cap \Sigma_0.$$

In particular, $\mathfrak{p}_\Psi = \mathfrak{P}_\Psi^{(R)} \cap \Sigma_0$.

Proof. Since R is an F-vector space, the natural map $\Sigma_0 \to \Sigma_0^{(R)}$ is injective, and it makes $\Sigma_0^{(R)}$ into a free Σ_0-module. Therefore, $\Sigma_0^{(R)}$ is faithfully flat as a Σ_0-algebra. The lemma follows from standard facts on faithfully flat algebras. □

Theorem 4.3.10. *Let $\Phi \in \mathrm{GL}_r(K)$, and suppose $\Psi \in \mathrm{GL}_r(L)$ is a fundamental matrix for Φ. Then the ring Σ_Ψ is a simple $\overline{\sigma}$-algebra.*

Proof. (cf. [24, Prop. 1.22]) Let $\mathfrak{q}$ be a maximal $\overline{\sigma}$-ideal of Σ_0 that contains $\mathfrak{p}_\Psi$. As before, let Λ_Ψ be the fraction field of Σ_Ψ, and consider

$$\Sigma_0 \subseteq \Lambda_\Psi \otimes_K \Sigma_0 \cong \Lambda_\Psi[X, \Delta^{-1}].$$

We can define a matrix $Y = (Y_{ij})$ with entries in $\Lambda_\Psi[X,\Delta^{-1}]$ by

$$(Y_{ij}) := \Psi^{-1}\cdot(X_{ij}).$$

Now $\Lambda_\Psi[X,\Delta^{-1}] = \Lambda_\Psi[Y,\Delta(Y)^{-1}]$, and also $\overline{\sigma}(Y_{ij}) = (Y_{ij})$. Therefore we obtain an isomorphism of $\overline{\sigma}$-algebras

$$\Lambda_\Psi \otimes_K \Sigma_0 \cong \Lambda_\Psi \otimes_F F[Y,\Delta(Y)^{-1}]. \tag{4.3.10.1}$$

Now let $\mathfrak{Q} := \mathfrak{q}\cdot(\Lambda_\Psi\otimes_K\Sigma_0)$. Since

$$(\Lambda_\Psi\otimes_K\Sigma_0)/\mathfrak{Q} \cong \Lambda_\Psi\otimes_K(\Sigma_0/\mathfrak{q})$$

is not trivial, we see that $\mathfrak{Q}$ is a proper $\overline{\sigma}$-ideal. Let $\Sigma_1 := F[Y,\Delta(Y)^{-1}]$, and then set $\overline{\mathfrak{Q}}$ to be the ideal of Σ_1,

$$\overline{\mathfrak{Q}} := \mathfrak{Q}\cap\Sigma_1.$$

Since $(\Lambda_\Psi)^{\overline{\sigma}} = F$, we can apply Lemma 4.3.8, and we find that $\overline{\mathfrak{Q}}$ is a proper ideal and that $\mathfrak{Q}$ is generated by $\overline{\mathfrak{Q}}$. Let $\overline{\mathfrak{M}}$ be a maximal ideal of Σ_1 containing $\overline{\mathfrak{Q}}$. Then $\Sigma_1/\overline{\mathfrak{M}}$

is a field that is a finite extension of F. Also if we let $\mathfrak{M} := \overline{\mathfrak{M}} \cdot (\Lambda_\Psi \otimes_F \Sigma_1)$, then $\overline{\mathfrak{M}} = \mathfrak{M} \cap \Sigma_1$ by Lemma 4.3.8, and so

$$(\Lambda_\Psi \otimes_F \Sigma_1)/\mathfrak{M} \cong \Lambda_\Psi \otimes_F (\Sigma_1/\overline{\mathfrak{M}}).$$

Using (4.3.10.1), we have a natural surjection

$$\Pi : \Lambda_\Psi \otimes_K \Sigma_0 \twoheadrightarrow \Lambda_\Psi \otimes_F (\Sigma_1/\overline{\mathfrak{M}}),$$

and the kernel of Π contains $\mathfrak{Q}$. If we restrict Π to $\Sigma_0 \subseteq \Lambda_\Psi \otimes_K \Sigma_0$, we obtain

$$\pi := \Pi|_{\Sigma_0} : \Sigma_0 \to \Lambda_\Psi \otimes_F (\Sigma_1/\overline{\mathfrak{M}}),$$

and then $\mathfrak{q} \subseteq \ker \pi$. Since $\mathfrak{q}$ is a maximal $\overline{\sigma}$-ideal, we in fact have $\mathfrak{q} = \ker \pi$.

Now we take $R := \Sigma_1/\overline{\mathfrak{M}}$, and note that $\Lambda_\Psi \otimes_F (\Sigma_1/\overline{\mathfrak{M}}) \cong \Lambda_\Psi^{(R)} \subseteq L^{(R)}$. Let $\Psi' \in \mathrm{GL}(\Lambda_\Psi^{(R)})$ be defined by $(\Psi'_{ij}) := \pi(X_{ij})$. Then Ψ' is a fundamental matrix for Φ over $L^{(R)}$, and so by Lemma 4.3.4(b), there is a $\gamma \in \mathrm{GL}_r(R)$ so that $\Psi' = \Psi\gamma^{-1}$. Let π' be the natural map,

$$\pi' := 1 \otimes \pi : \Sigma_0^{(R)} \to \Lambda_\Psi^{(R)}.$$

Then as in §4.3.5, we see that

$$\mathfrak{q} \cdot \Sigma_0^{(R)} \subseteq \ker \pi' = \mathfrak{P}_{\Psi'}^{(R)} = \gamma * \mathfrak{P}_\Psi^{(R)}.$$

Since $\mathfrak{p}_\Psi \subseteq \mathfrak{q} = \ker \pi$, it follows that

$$\mathfrak{P}_\Psi^{(R)} \subseteq \mathfrak{q} \cdot \Sigma_0^{(R)} \subseteq \gamma * \mathfrak{P}_\Psi^{(R)}.$$

By Lemma 4.3.6, we conclude that

$$\mathfrak{P}_\Psi^{(R)} = \mathfrak{q} \cdot \Sigma_0^{(R)} = \gamma * \mathfrak{P}_\Psi^{(R)}.$$

After intersecting with Σ_0, Lemma 4.3.9 implies that

$$\mathfrak{p}_\Psi = \mathfrak{q},$$

and since $\mathfrak{q}$ is a maximal $\overline{\sigma}$-ideal, we conclude that Σ_Ψ is a simple $\overline{\sigma}$-algebra. □

4.4. **The Galois group Γ_Ψ.**

4.4.1. *Definition.* Let $\Phi \in \mathrm{GL}_r(K)$, and suppose that $\Psi \in \mathrm{GL}_r(L)$ is a fundamental matrix for Φ. For any F-algebra R, the left action of $\mathrm{GL}_r(F)$ on Σ_0 extends to a left action of $\mathrm{GL}_r(R)$ on $\Sigma_0^{(R)}$. Now set

$$\Gamma_\Psi(R) := \bigl\{\gamma \in \mathrm{GL}_r(R) \bigm| \gamma * \mathfrak{P}_\Psi^{(R)} \subseteq \mathfrak{P}_\Psi^{(R)}\bigr\}.$$

As a result of Lemma 4.3.6, we see that $\Gamma_\Psi(R)$ is a subgroup of $\mathrm{GL}_r(R)$. We shall see in Theorem 4.4.6 that $\Gamma_\Psi(R)$ is the set of R-valued points of a linear algebraic group Γ_Ψ over F.

Proposition 4.4.2. *Suppose $\Phi \in \mathrm{GL}_r(K)$, and suppose that $\Psi \in \mathrm{GL}_r(L)$ is a fundamental matrix for Φ. For for any F-algebra R, there is an isomorphism*

$$\kappa_R : \Gamma_\Psi(R) \xrightarrow{\sim} \mathrm{Aut}_R^{\overline{\sigma}}\bigl(\Sigma_\Psi^{(R)}\bigr)$$

that is functorial in R.

Proof. By Lemma 4.3.6, each $\gamma \in \Gamma_\Psi(R)$ induces a $\overline{\sigma}$-algebra automorphism of $\Sigma_\Psi^{(R)} \cong \Sigma_0^{(R)}/\mathfrak{P}_\Psi^{(R)}$ that fixes $K^{(R)}$. This defines a map

$$\kappa_R : \Gamma_\Psi(R) \to \mathrm{Aut}_R^{\overline{\sigma}}\bigl(\Sigma_\Psi^{(R)}\bigr),$$

which is a group homomorphism. It is straightforward to show that κ_R is injective. Now suppose $\xi \in \mathrm{Aut}_R^{\overline{\sigma}}\bigl(\Sigma_\Psi^{(R)}\bigr)$. Then for any $h(\Psi) \in \Sigma_\Psi^{(R)} = K^{(R)}[\Psi, \Delta(\Psi)^{-1}]$, given by $h \in K^{(R)}[X, \Delta^{-1}]$, we have that

$$\xi(h(\Psi)) = h(\xi(\Psi)),$$

where $\xi(\Psi) \in \mathrm{Mat}_r\bigl(\Sigma_\Psi^{(R)}\bigr) \cap \mathrm{GL}_r\bigl(L^{(R)}\bigr)$. It follows that,

$$h(\Psi) = 0 \Leftrightarrow h(\xi(\Psi)) = 0, \quad h(X) \in K^{(R)}[X, \Delta^{-1}].$$

Thus,

$$\mathfrak{P}_\Psi^{(R)} = \{h(X) \in \Sigma_0^{(R)} \mid h(\Psi) = 0\} = \{h(X) \in \Sigma_0^{(R)} \mid h(\xi(\Psi)) = 0\}.$$

We note that

$$\overline{\sigma}(\xi(\Psi)) = \xi(\overline{\sigma}(\Psi)) = \xi(\Phi\Psi) = \Phi\xi(\Psi),$$

and so $\xi(\Psi)$ is also a fundamental matrix for Φ over R. By Lemma 4.3.4(b), we obtain $\delta := \Psi^{-1}\xi(\Psi) \in \mathrm{GL}_r(R)$, which induces a $\overline{\sigma}$-algebra automorphism $\delta : \Sigma_0^{(R)} \to \Sigma_0^{(R)}$. We have a commutative diagram of $\overline{\sigma}$-algebra homomorphisms,

$$\begin{array}{ccc} \Sigma_0^{(R)} & \twoheadrightarrow & \Sigma_\Psi^{(R)} \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\xi} \\ \Sigma_0^{(R)} & \twoheadrightarrow & \Sigma_\Psi^{(R)}, \end{array}$$

and it follows that $\delta * \mathfrak{P}_\Psi^{(R)} \subseteq \mathfrak{P}_\Psi^{(R)}$ and that $\xi = \kappa_R(\delta)$.

We now prove functoriality in R. Fix an F-algebra homomorphism $\phi : S \to R$. There are natural isomorphisms

$$\Sigma_0^{(R)} \cong R \otimes_S \Sigma_0^{(S)}, \quad \Sigma_\Psi^{(R)} \cong R \otimes_S \Sigma_\Psi^{(S)},$$

which will facilitate our considerations. The map $\phi : \Gamma_\Psi(S) \to \Gamma_\Psi(R)$ is induced by the map $\phi : \mathrm{GL}_r(S) \to \mathrm{GL}_r(R)$. This is well-defined: if $\delta \in \Gamma_\Psi(S)$, then $\delta * \mathfrak{P}_\Psi^{(S)} \subseteq \mathfrak{P}_\Psi^{(S)}$; however, $\mathfrak{P}_\Psi^{(R)} \cong R \otimes_S \mathfrak{P}_\Psi^{(S)}$, which implies

$$\phi(\delta) * \mathfrak{P}_\Psi^{(R)} \cong R \otimes_S \delta * \mathfrak{P}_\Psi^{(S)} \subseteq R \otimes_S \mathfrak{P}_\Psi^{(S)} \cong \mathfrak{P}_\Psi^{(R)},$$

and thus $\phi(\delta) \in \Gamma_\Psi(R)$.

The map $\phi : \mathrm{Aut}_S^{\overline{\sigma}}\bigl(\Sigma_\Psi^{(S)}\bigr) \to \mathrm{Aut}_R^{\overline{\sigma}}\bigl(\Sigma_\Psi^{(R)}\bigr)$ is induced by base extension. If $\xi : \Sigma_\Psi^{(S)} \to \Sigma_\Psi^{(S)}$ is a $\overline{\sigma}$-algebra automorphism, then

$$\phi \cdot \xi = 1 \otimes \xi : R \otimes_S \Sigma_\Psi^{(S)} \to R \otimes_S \Sigma_\Psi^{(S)}$$

induces a $\overline{\sigma}$-algebra homomorphism $\phi \cdot \xi : \Sigma_\Psi^{(R)} \to \Sigma_\Psi^{(R)}$. We will see in the end that $\phi \cdot \xi$ is an automorphism. We claim that

$$\phi \cdot \xi = \kappa_R \circ \phi \circ \kappa_S^{-1}(\xi),$$

which will show that $\phi \cdot \xi \in \mathrm{Aut}_R^{\overline{\sigma}}\bigl(\Sigma_\Psi^{(R)}\bigr)$ and also complete the proof. Let $\delta := \kappa_S^{-1}(\xi) \in \Gamma_\Psi(S)$. To show that $\phi \cdot \xi = \kappa_R(\phi(\delta))$ on $\Sigma_\Psi^{(R)}$, it suffices to show that they both induce the same map on $R \otimes_S \Sigma_\Psi^{(S)}$. Suppose $\delta = (d_{ij}) \in \mathrm{GL}_r(S)$. Then the equality $\phi(\delta) * X = X\phi(\delta)$ in $\mathrm{Mat}_r\bigl(\Sigma_0^{(R)}\bigr)$ translates into an equality in $R \otimes_S \Sigma_0^{(S)}$,

$$\phi(\delta) * (1 \otimes X_{ij}) = \sum_s (\phi(d_{sj}) \otimes X_{is}) = 1 \otimes \left(\sum_s d_{sj} X_{is} \right).$$

The last term is the extension of the ij-th entry of $X\delta$ to $R \otimes_S \Sigma_0^{(S)}$. This induces a commutative diagram

$$\begin{array}{ccc} \Sigma_0^{(S)} & \xrightarrow{\iota} & \Sigma_0^{(R)} \\ \big\downarrow \delta & & \big\downarrow \phi(\delta) \\ \Sigma_0^{(S)} & \xrightarrow{\iota} & \Sigma_0^{(R)} \end{array} \quad , \tag{4.4.2.1}$$

where the top and bottom maps are the ones induced by $h \mapsto 1 \otimes h : \Sigma_0^{(S)} \to R \otimes_S \Sigma_0^{(S)}$. Thus for all $h(X) \in K^{(S)}[X, \Delta^{-1}]$, we have an equality in $R \otimes_S \Sigma_\Psi^{(S)}$,

$$\phi(\delta) * (1 \otimes h(\Psi)) = 1 \otimes h(\Psi\delta).$$

Now, also in $R \otimes_S \Sigma_\Psi^{(S)}$,

$$(\phi \cdot \xi)(1 \otimes h(\Psi)) = 1 \otimes \xi(h(\Psi)) = 1 \otimes h(\Psi\delta).$$

Thus $\phi \cdot \xi$ and $\kappa_R(\phi(\delta))$ induce the same map on $R \otimes_S \Sigma_\Psi^{(S)}$. □

4.4.3. *Representability of* Γ_Ψ. Proposition 4.4.2 defines a functor $\Gamma_\Psi : \mathbf{Alg}(F) \to \mathbf{Set}$ from the category of F-algebras to the category of sets. Using a modification of the argument in [24, Thm. 1.27(1)], it is possible to show that this functor is representable. We omit the details.

Proposition 4.4.4. *The functor* $\Gamma_\Psi : \mathbf{Alg}(F) \to \mathbf{Set}$ *is representable.*

4.4.5. *The Galois group of* Ψ. Since the induced functor $\Gamma_\Psi : \mathbf{Alg}(F) \to \mathbf{Grp}$ to the category of groups is representable as a functor to the category of sets by an F-algebra U. The Yoneda lemma can be used to show that U is a commutative Hopf algebra over F [30, §1.2–1.4]. Thus as a functor on F-algebras, Γ_Ψ is isomorphic to an affine group scheme over F, and we will say that Γ_Ψ *is* an affine group scheme over F. Furthermore,

$$F[Y, \Delta(Y)^{-1}] \twoheadrightarrow U$$

is a surjective homomorphism of Hopf algebras over F, and so Γ_Ψ is a closed linear algebraic subgroup of GL_r over F, which is unique up to unique isomorphism. We formulate this in the following theorem.

Theorem 4.4.6. *Let* $\Phi \in \mathrm{GL}_r(K)$*, and suppose* $\Psi \in \mathrm{GL}_r(L)$ *is a fundamental matrix for* Φ*. Then the functor*

$$R \mapsto \Gamma_\Psi(R) : \mathbf{Alg}(F) \to \mathbf{Grp}$$

is the functor of points of a closed linear algebraic subgroup of GL_r *over* F*.*

5. σ-semilinear equations and t-motives

We now consider the case $(F, K, L) = (\mathbb{F}_q(t), \overline{k}(t), \mathbb{L})$ with $\overline{\sigma} = \sigma$. We fix $\Phi \in \mathrm{GL}_r(\overline{k}(t))$ and suppose that $\Psi \in \mathrm{GL}_r(\mathbb{L})$ is a fundamental matrix for Φ. As in §4.2, we define $\mathfrak{p}_\Psi$ to be the kernel of the $\overline{k}(t)$-algebra map $\nu : \overline{k}(t)[X, \Delta^{-1}] \to \mathbb{L}$ such that $\nu(X_{ij}) = \Psi_{ij}$; we set $\Sigma_\Psi := \mathrm{im}(\nu) \subseteq \mathbb{L}$; and we take Λ_Ψ to be the fraction field of Σ_Ψ. From this data we construct the group Γ_Ψ over $\mathbb{F}_q(t)$ as in §4.4.

5.1. **The Lie algebra of Γ_Ψ.**

5.1.1. *Derivations.* If $S \subseteq R$ are commutative rings, we set $\mathrm{Der}(R/S)$ to be the R-module of derivations $D : R \to R$ that are trivial on S. If Σ is a σ-algebra, we let

$$\mathrm{Der}(\Sigma) := \mathrm{Der}(\Sigma/\overline{k}(t)).$$

Since $\sigma : \Sigma \to \Sigma$ is bijective, $\mathrm{Der}(\Sigma/\overline{k}(t))$ carries the structure of a left $\overline{k}(t)[\sigma, \sigma^{-1}]$-module via

$$\sigma \cdot D := \sigma \circ D \circ \sigma^{-1}, \quad D \in \mathrm{Der}(\Sigma/\overline{k}(t)).$$

We set

$$\mathrm{Der}^\sigma(\Sigma) := \{D \in \mathrm{Der}(\Sigma/\overline{k}(t)) \mid \sigma \circ D \circ \sigma^{-1} = D\}.$$

It follows that $\mathrm{Der}^\sigma(\Sigma)$ is an $\mathbb{F}_q(t)$-vector space. If Σ is a domain and Λ is its fraction field, then $\mathrm{Der}(\Lambda) \cong \mathrm{Der}(\Sigma) \otimes_\Sigma \Lambda$, and we can canonically identify $\mathrm{Der}(\Sigma) \subseteq \mathrm{Der}(\Lambda)$.

Lemma 5.1.2. *Let $\Phi \in \mathrm{GL}_r(\overline{k}(t))$, and suppose that $\Psi \in \mathrm{GL}_r(\mathbb{L})$ is a fundamental matrix for Φ.*

(a) *Suppose that $D \in \mathrm{Der}^\sigma(\Lambda_\Psi)$, and consider the matrix $D(\Psi) \in \mathrm{Mat}_r(\Lambda_\Psi)$. Then $D(\Psi) = \Psi\delta$ for some $\delta \in \mathrm{Mat}_r(\mathbb{F}_q(t))$.*
(b) $\mathrm{Der}^\sigma(\Lambda_\Psi) = \mathrm{Der}^\sigma(\Sigma_\Psi)$.
(c) *If $D_1, \ldots, D_m \in \mathrm{Der}^\sigma(\Lambda_\Psi)$ are linearly independent over $\mathbb{F}_q(t)$, then they are linearly independent over Λ_Ψ in $\mathrm{Der}(\Lambda_\Psi)$.*

Proof. Suppose $D \in \mathrm{Der}^\sigma(\Lambda_\Psi)$ is given, and set $\Upsilon := D(\Psi)$. Since $D \in \mathrm{Der}^\sigma(\Lambda_\Psi)$, it follows that

$$\Upsilon = \sigma(D(\sigma^{-1}(\Psi))) = \sigma\bigl(D\bigl((\Phi^{-1})^{(1)} \cdot \Psi\bigr)\bigr) = \sigma\bigl((\Phi^{-1})^{(1)} \cdot D(\Psi)\bigr) = \Phi^{-1}\Upsilon^{(-1)}.$$

Thus $\Upsilon^{(-1)} = \Phi\Upsilon$. This implies that $(\Psi^{-1}\Upsilon)^{(-1)} = \Psi^{-1}\Upsilon$, and thus $\Upsilon = \Psi\delta$, where $\delta \in \mathrm{Mat}_r(\mathbb{F}_q(t))$. This proves part (a). For (b), we note that $\Upsilon \in \mathrm{Mat}_r(\Sigma_\Psi)$, and so $D \in \mathrm{Der}^\sigma(\Sigma_\Psi)$. Part (c) is essentially the same as the proof of Lemma 3.3.7, and we omit the details. ☐

Corollary 5.1.3. *Let $\Phi \in \mathrm{GL}_r(\overline{k}(t))$, and suppose that $\Psi \in \mathrm{GL}_r(\mathbb{L})$ is a fundamental matrix for Φ. Then*

$$\dim_{\mathbb{F}_q(t)} \mathrm{Der}^\sigma(\Lambda_\Psi) \leq \mathrm{tr.\,deg}_{\overline{k}(t)}\, \Lambda_\Psi.$$

Proof. Since Λ_Ψ is finitely generated over $\overline{k}(t)$ and separable over $\overline{k}(t)$,

$$\dim_{\Lambda_\Psi} \mathrm{Der}(\Lambda_\Psi) = \mathrm{tr.\,deg}_{\overline{k}(t)}\, \Lambda_\Psi,$$

by [32, §II.17, Thm. 41]. The result follows from Lemma 5.1.2(c). ☐

5.1.4. *The Lie algebra of Γ_Ψ.* Let $\Phi \in \mathrm{GL}_r(\overline{k}(t))$, and suppose that $\Psi \in \mathrm{GL}_r(\mathbb{L})$ is a fundamental matrix for Φ. Let $\mathrm{Lie}\,\Gamma_\Psi$ be the Lie algebra of Γ_Ψ over $\mathbb{F}_q(t)$. Let $\mathbb{F}_q(t)[\varepsilon] = \mathbb{F}_q(t)[\varepsilon]/(\varepsilon^2)$, and let $\nu : \mathbb{F}_q(t)[\varepsilon] \to \mathbb{F}_q(t)$ be defined by $\nu(\varepsilon) = 0$. Then there is a canonical isomorphism of $\mathbb{F}_q(t)$-vector spaces

$$\mathrm{Lie}\,\Gamma_\Psi \cong \ker\bigl(\Gamma_\Psi(\mathbb{F}_q(t)[\varepsilon]) \xrightarrow{\nu} \Gamma_\Psi(\mathbb{F}_q(t))\bigr).$$

We have a canonical isomorphism $\lambda : \mathrm{Mat}_r(\mathbb{F}_q(t)) \to \mathrm{Lie}\,\mathrm{GL}_r$. This translates into the identification

$$\lambda : \mathrm{Lie}\,\Gamma_\Psi \xleftarrow{\sim} \bigl\{\delta \in \mathrm{Mat}_r(\mathbb{F}_q(t)) \bigm| (I + \varepsilon\delta) * \mathfrak{P}_\Psi^{(\varepsilon)} \subseteq \mathfrak{P}_\Psi^{(\varepsilon)}\bigr\},$$

where $\mathfrak{P}_\Psi^{(\varepsilon)} := \mathfrak{P}_\Psi^{(\mathbb{F}_q(t)[\varepsilon])}$. This furthermore translates into the characterization that for $\delta \in \mathrm{Mat}_r(\mathbb{F}_q(t))$, $\lambda(\delta) \in \mathrm{Lie}\,\Gamma_\Psi$ if and only if

$$\sum_{i,j=1}^{r} (X \cdot \delta)_{ij} \frac{\partial h(X)}{\partial X_{ij}} \in \mathfrak{p}_\Psi, \quad \forall\, h \in \mathfrak{p}_\Psi, \tag{5.1.4.1}$$

where $X = (X_{ij}) \in \mathrm{GL}_r(\Sigma_0)$.

Proposition 5.1.5. *Let $\Phi \in \mathrm{GL}_r(\overline{k}(t))$, and suppose that $\Psi \in \mathrm{GL}_r(\mathbb{L})$ is a fundamental matrix for Φ. Then*

$$\mathrm{Lie}\,\Gamma_\Psi \cong \mathrm{Der}^\sigma(\Lambda_\Psi).$$

Thus $\dim_{\mathbb{F}_q(t)} \mathrm{Lie}\,\Gamma_\Psi \leq \mathrm{tr.\,deg}_{\overline{k}(t)}\, \Lambda_\Psi$.

Proof. Suppose $D \in \mathrm{Der}^\sigma(\Lambda_\Psi)$. Since $D \in \mathrm{Der}(\Sigma_\Psi)$ by Lemma 5.1.2(b), we can set $\Upsilon := D(\Psi) \in \mathrm{Mat}_r(\Sigma_\Psi)$. For D to be in $\mathrm{Der}(\Sigma_\Psi)$ is equivalent to,

$$0 = D(h(\Psi)) = \sum_{i,j=1}^{r} \Upsilon_{ij} \frac{\partial h}{\partial X_{ij}}(\Psi), \quad \forall\, h(X) \in \mathfrak{p}_\Psi. \tag{5.1.5.1}$$

By Lemma 5.1.2(a), $\Upsilon = \Psi\delta \in \mathrm{Mat}_r(\Sigma_\Psi)$ for some $\delta \in \mathrm{Mat}_r(\mathbb{F}_q(t))$. We claim that $(I + \varepsilon\delta) * \mathfrak{P}_\Psi^{(\varepsilon)} \subseteq \mathfrak{P}_\Psi^{(\varepsilon)}$. Suppose $h(X) \in \mathfrak{p}_\Psi$. Then

$$\sum_{i,j=1}^{r} (\Psi \cdot \delta)_{ij} \frac{\partial h}{\partial X_{ij}}(\Psi) = \sum_{i,j=1}^{r} \Upsilon_{ij} \frac{\partial h}{\partial X_{ij}}(\Psi) = D(h(\Psi)) = 0.$$

By the definition of $\mathfrak{p}_\Psi$ and (5.1.4.1), it follows that $(I + \varepsilon\delta) * \mathfrak{P}_\Psi^{(\varepsilon)} \subseteq \mathfrak{P}_\Psi^{(\varepsilon)}$, and so $\lambda(\delta) \in \mathrm{Lie}\,\Gamma_\Psi$. Thus we have defined a map

$$\begin{array}{rcll} \mu : \mathrm{Der}^\sigma(\Sigma_\Psi) & \to & \mathrm{Lie}\,\Gamma_\Psi, & (D(\Psi) = \Psi \cdot \delta). \\ D & \mapsto & \lambda(\delta) & \end{array}$$

We will show this is an isomorphism. Suppose $\lambda(\delta) \in \mathrm{Lie}\,\Gamma_\Psi$ for $\delta \in \mathrm{Mat}_r(\mathbb{F}_q(t))$. Then by (5.1.4.1) and the definition of $\mathfrak{p}_\Psi$,

$$\sum_{i,j=1}^{r} (\Psi \cdot \delta)_{ij} \frac{\partial h}{\partial X_{ij}}(\Psi) = 0, \quad \forall\, h(X) \in \mathfrak{p}_\Psi.$$

If we let $\Upsilon = \Psi\delta \in \mathrm{Mat}_r(\Sigma_\Psi)$, then setting $D \cdot (\Psi) := \Upsilon$ determines a derivation in $\mathrm{Der}(\Sigma_\Psi)$ by (5.1.5.1). Since $\Upsilon^{(-1)} = \Phi\Upsilon$, it follows that $\sigma \cdot D = D$. The final conclusion of the proposition then follows from Corollary 5.1.3. ☐

5.2. A $\widetilde{\Gamma}_\Psi$-torsor.

Lemma 5.2.1. *Let $\Phi \in \mathrm{GL}_r(\overline{k}(t))$ and suppose that $\Psi \in \mathrm{GL}_r(\mathbb{L})$ is a fundamental matrix for Φ. Then $\Lambda_\Psi \cap \overline{k(t)} = \overline{k}(t)$.*

Proof. Let $f \in \Lambda_\Psi \cap \overline{k(t)}$, and consider the field

$$L := \overline{k}\big(t;\ f^{(i)} : i \in \mathbb{Z}\big)$$

obtained by adjoining all of the twists of f to $\overline{k}(t)$. Each $f^{(i)}$ is algebraic over $\overline{k}(t)$, and so $L/\overline{k}(t)$ is algebraic. Since Λ_Ψ is finitely generated as a field over $\overline{k}(t)$, so is L. Thus $[L : \overline{k}(t)] < \infty$. Furthermore, L is invariant under σ and σ^{-1}.

The field L is the function field of a smooth projective curve X over $\overline{k}$, and the inclusion $\overline{k}(t) \subseteq L$ provides a surjective morphism $X \to \mathbb{P}^1_{\overline{k}}$ over $\overline{k}$. Now $\sigma : L \to L$ induces an automorphism $\tau : X \to X$ as a scheme over $\mathbb{F}_q$. Because σ leaves the integral closure of $\overline{k}[t]$ in L invariant, the points $\infty_1, \dots, \infty_d$ in X above the point ∞ in $\mathbb{P}^1_{\overline{k}}$ are permuted by σ. Thus we can construct an effective divisor I of X such that $\tau(I) = I$ and $\mathrm{Supp}(I) = \{\infty_1, \dots, \infty_d\}$. Now for $N \geq 1$ sufficiently large, the field L is generated over $\overline{k}(t)$ by the functions in the finite dimensional $\overline{k}$-vector space

$$S := \Gamma(X, N \cdot I) \subseteq L.$$

By our assumptions on I, it is invariant under σ and σ^{-1}. If the entries of $\mathbf{f} := [f_1, \dots, f_m]^{\mathrm{tr}}$ form a $\overline{k}$-basis for S, then there is a matrix $A \in \mathrm{GL}_m(\overline{k})$ so that $\sigma(\mathbf{f}) = A\mathbf{f}$. If $\mathbf{g} \in \mathrm{Mat}_{m\times 1}(S)$ and $\mathbf{g} = B\mathbf{f}$ for some $B \in \mathrm{GL}_m(\overline{k})$, then

$$\sigma(\mathbf{g}) = B^{(-1)}AB^{-1}\mathbf{g}.$$

By the theory of Lang isogenies [18], we can pick a $B \in \mathrm{GL}_m(\overline{k})$ so that

$$B^{-1}B^{(1)} = A^{(1)},$$

and if we let $\mathbf{g} := B\mathbf{f}$, then

$$\sigma(\mathbf{g}) = \mathbf{g}.$$

Thus S contains a $\overline{k}$-basis $\mathbf{g}$ that is fixed by σ, and $L = \overline{k}(t, \mathbf{g})$. Let g be an entry of $\mathbf{g}$. Then $g \in \overline{k(t)}^{\sigma} \cap \mathbb{L}^{\sigma}$, but

$$\overline{k(t)}^{\sigma} \cap \mathbb{L}^{\sigma} = (\overline{\mathbb{F}_q(t)} \cap \mathbb{F}_q\langle\langle t\rangle\rangle) \cap \mathbb{F}_q(t) = \mathbb{F}_q(t).$$

Thus $[L : \overline{k}(t)] = 1$. □

5.2.2. *Absolute primality.* Lemma 5.2.1 shows that $\overline{k}(t)$ is algebraically closed in Λ_Ψ. Because Λ_Ψ is also separable over $\overline{k}(t)$, it follows that the ideal $\mathfrak{p}_\Psi \subseteq \overline{k}(t)[X, \Delta^{-1}]$ is absolutely prime [33, §VII.11, Thm. 39]. In particular, the extension of $\mathfrak{p}_\Psi$ to $\overline{k(t)}[X, \Delta^{-1}]$ is also prime.

Lemma 5.2.3. *Let $\mu : \overline{k(t)}[X, \Delta^{-1}] \to \overline{\mathbb{L}}$ be the $\overline{k(t)}$-algebra homomorphism defined by $\mu(X_{ij}) = \Psi_{ij}$ and let $\mathfrak{q}_\Psi$ be its kernel. Then $\mathfrak{q}_\Psi = \mathfrak{p}_\Psi \cdot \overline{k(t)}[X, \Delta^{-1}]$*

Proof. Let $\mathfrak{Q} := \mathfrak{p}_\Psi \cdot \overline{k(t)}[X, \Delta^{-1}]$. By the remarks in the preceding paragraph $\mathfrak{Q}$ is a prime ideal, and it is contained in $\mathfrak{q}_\Psi$. Taking dimensions of prime ideals, $\dim \mathfrak{Q} = \dim \mathfrak{p}_\Psi$. That $\mathrm{im}(\mu) = \Sigma_\Psi \cdot \overline{k(t)} \subseteq \overline{\mathbb{L}}$ implies $\dim \mathfrak{p}_\Psi = \dim \mathfrak{q}_\Psi$. Thus $\mathfrak{Q} = \mathfrak{q}_\Psi$. □

5.2.4. *The groups $\widetilde{\Gamma}^n_\Psi$.* For $n \geq 1$, the matrix Ψ is a fundamental matrix for

$$\widetilde{\Phi}_n := \Phi^{(-n+1)} \cdots \Phi^{(-1)}\Phi.$$

with respect to the automorphism $\sigma^n : \mathbb{L} \to \mathbb{L}$. Following the notation in (4.1.2.1) and (4.1.2.2), we consider the σ^n-admissible fields $(\mathbf{F}_n, \overline{k(t)}, \overline{\mathbb{L}})$. We let $\mathfrak{q}_\Psi$ be the kernel of the map $\mu : \overline{k(t)}[X, \Delta^{-1}] \to \overline{\mathbb{L}}$; we set $\widetilde{\Sigma}_\Psi := \mathrm{im}(\mu) \subseteq \overline{\mathbb{L}}$; and we let $\widetilde{\Lambda}_\Psi$ be the fraction field of $\widetilde{\Sigma}_\Psi$. The ideal $\mathfrak{q}_\Psi$ is the extension of $\mathfrak{p}_\Psi$ to $\overline{k(t)}[X, \Delta^{-1}]$ by Lemma 5.2.3. The $\overline{k(t)}$-algebra $\overline{k(t)}[X, \Delta^{-1}]$ has a σ^n-algebra structure induced by $\widetilde{\Phi}_n$ for each n, and we note that $\mathfrak{q}_\Psi$ is a maximal σ^n-ideal for every $n \geq 1$ by Theorem 4.3.10. Finally we let $\widetilde{\Gamma}^n_\Psi$ be the group over $\mathbf{F}_n$ associated to $\widetilde{\Phi}_n$ and Ψ, which is a subgroup of GL_r over $\mathbf{F}_n$.

5.2.5. *The groups Γ_Ψ and $\widetilde{\Gamma}^n_\Psi$ over extension fields.* Let $E/\mathbb{F}_q(t)$ be a finite extension with $E \subseteq \mathbb{K}\langle\langle t\rangle\rangle$. Then we can form the group $\Gamma_\Psi(E) \subseteq \mathrm{GL}_r(E)$. However, since $\bigcup_{n=1}^{\infty} \mathbf{F}_n = \overline{\mathbb{F}_q(t)}$, it follows that $E \subseteq \mathbf{F}_n$ for some $n \geq 1$. Thus we also have the group $\widetilde{\Gamma}^n_\Psi(E) := \widetilde{\Gamma}^n_\Psi(\mathbf{F}_n) \cap \mathrm{GL}_r(E)$ for n sufficiently large. We will appeal to the following lemma, whose proof is another variation on the one for Lemma 3.3.7.

Lemma 5.2.6. *Suppose E and H are σ-invariant subfields of $\mathbb{K}\langle\langle t\rangle\rangle$, and suppose that $E^\sigma = E$ and $H^\sigma \subseteq E$. Then E and H are linearly disjoint over H^σ.*

Proposition 5.2.7. *Let $\Phi \in \mathrm{GL}_r(\bar{k}(t))$, and suppose that $\Psi \in \mathrm{GL}_r(\mathbb{L})$ is a fundamental matrix for Φ. If $E/\mathbb{F}_q(t)$ is a finite extension of fields and $E \subseteq \mathbf{F}_n$ for $n \geq 1$, then*

$$\Gamma_\Psi(E) = \widetilde{\Gamma}_\Psi^n(E).$$

Proof. Recall that $\mathbf{F} = \mathbf{F}_1$. We observe that $\mathbb{F}_q(t) = \bar{k}(t)^\sigma = \mathbb{L}^\sigma$; that $\mathbb{F}_q(t) \subseteq E \cap \mathbf{F}$; and that $(E \cap \mathbf{F})^\sigma = E \cap \mathbf{F}$. By Lemma 5.2.6, $E \cap \mathbf{F}$ is linearly disjoint over $\mathbb{F}_q(t)$ from $\bar{k}(t)$ and from $\mathbb{L}$. Likewise, $E \cap \mathbf{F} = E^\sigma$, $E \cap \mathbf{F} \subseteq \mathbf{F}$, and $\mathbf{F}^\sigma = \mathbf{F}$. Thus Lemma 5.2.6 shows that E and $\mathbf{F}$ are linearly disjoint over $E \cap \mathbf{F}$. Together these imply,

$$\begin{aligned} E \otimes_{\mathbb{F}_q(t)} \bar{k}(t) &\cong E \otimes_{E\cap\mathbf{F}} (E \cap \mathbf{F} \otimes_{\mathbb{F}_q(t)} \bar{k}(t)) \\ &\hookrightarrow E \otimes_{E\cap\mathbf{F}} \overline{k(t)} \\ &\cong E \otimes_{E\cap\mathbf{F}} (\mathbf{F} \otimes_{\mathbf{F}} \overline{k(t)}) \\ &\hookrightarrow E\mathbf{F} \otimes_{\mathbf{F}} \overline{k(t)}. \end{aligned}$$

The same is true if $\bar{k}(t)$ is replaced by $\mathbb{L}$ and $\overline{k(t)}$ is replaced by $\overline{\mathbb{L}}$. Thus we can form the commutative diagram with exact rows,

$$\begin{array}{ccccccc} 0 \longrightarrow & E \otimes_{\mathbb{F}_q(t)} \mathfrak{p}_\Psi & \longrightarrow & (E \otimes_{\mathbb{F}_q(t)} \bar{k}(t))[X, \Delta^{-1}] & \longrightarrow & E \otimes_{\mathbb{F}_q(t)} \mathbb{L} \\ & \downarrow \phi & & \downarrow \phi & & \downarrow \psi \\ 0 \longrightarrow & E\mathbf{F} \otimes_{\mathbf{F}} \mathfrak{q}_\Psi & \longrightarrow & (E\mathbf{F} \otimes_{\mathbf{F}} \overline{k(t)})[X, \Delta^{-1}] & \longrightarrow & E\mathbf{F} \otimes_{\mathbf{F}} \overline{\mathbb{L}} \\ & \downarrow & & \downarrow & & \downarrow \rho \\ 0 \longrightarrow & \mathfrak{q}_\Psi & \longrightarrow & \overline{k(t)}[X, \Delta^{-1}] & \longrightarrow & \overline{\mathbb{L}}. \end{array}$$

All of the maps commute with the action of $\mathrm{GL}_r(E)$. The maps ϕ and ψ are injective by our considerations above, and the bottom row of vertical maps are surjective.

Now suppose that $\gamma \in \mathrm{GL}_r(E)$. The ideal $E \otimes_{\mathbb{F}_q(t)} \mathfrak{p}_\Psi$ is generated by $1 \otimes \mathfrak{p}_\Psi$, and the ideal $E\mathbf{F} \otimes_{\mathbf{F}} \mathfrak{q}_\Psi$ is generated by $1 \otimes \mathfrak{q}_\Psi$. Since $\mathfrak{q}_\Psi$ is the extension of $\mathfrak{p}_\Psi$ to $\overline{k(t)}[X, \Delta^{-1}]$, if $\gamma * (E \otimes_{\mathbb{F}_q(t)} \mathfrak{p}_\Psi) \subseteq E \otimes_{\mathbb{F}_q(t)} \mathfrak{p}_\Psi$, then $\gamma * (E\mathbf{F} \otimes_{\mathbf{F}} \mathfrak{q}_\Psi) \subseteq E\mathbf{F} \otimes_{\mathbf{F}} \mathfrak{q}_\Psi$. Likewise, if $\gamma * (E\mathbf{F} \otimes_{\mathbf{F}} \mathfrak{q}_\Psi) \subseteq E\mathbf{F} \otimes_{\mathbf{F}} \mathfrak{q}_\Psi$, then a diagram chase combined with the facts that ϕ and ψ are injective imply that $\gamma * (E \otimes_{\mathbb{F}_q(t)} \mathfrak{p}_\Psi) \subseteq E \otimes_{\mathbb{F}_q(t)} \mathfrak{p}_\Psi$. Thus,

$$\gamma * (E \otimes_{\mathbb{F}_q(t)} \mathfrak{p}_\Psi) \subseteq E \otimes_{\mathbb{F}_q(t)} \mathfrak{p}_\Psi \Leftrightarrow \gamma * (E\mathbf{F} \otimes_{\mathbf{F}} \mathfrak{q}_\Psi) \subseteq E\mathbf{F} \otimes_{\mathbf{F}} \mathfrak{q}_\Psi.$$

On the other hand it is straightforward to see that $\gamma * (E\mathbf{F} \otimes_{\mathbf{F}} \mathfrak{q}_\Psi) \subseteq E\mathbf{F} \otimes_{\mathbf{F}} \mathfrak{q}_\Psi \Rightarrow \gamma * \mathfrak{q}_\Psi \subseteq \mathfrak{q}_\Psi$ by Lemma 4.3.9. To verify the converse, it suffices to check that $\gamma * (1 \otimes f) \in E\mathbf{F} \otimes_{\mathbf{F}} \mathfrak{q}_\Psi$ whenever $f \in \mathfrak{q}_\Psi$. For such an $f(X) \in \mathfrak{q}_\Psi$, we have $\gamma * f(X) = f(X\gamma)$, and so we ask whether $1 \otimes f(X\gamma) \in E\mathbf{F} \otimes_{\mathbf{F}} \mathfrak{q}_\Psi$. The image of $1 \otimes f(X\gamma)$ in $E\mathbf{F} \otimes_{\mathbf{F}} \overline{\mathbb{L}}$ is $1 \otimes f(\Psi\gamma)$. Then $\rho(1 \otimes f(\Psi\gamma)) = 0$ because $f \in \mathfrak{q}_\Psi$. However, $1 \otimes \overline{\mathbb{L}}$ injects into $\overline{\mathbb{L}}$ under ρ, so therefore $1 \otimes f(\Psi\gamma) = 0$, and so $1 \otimes f(X\gamma) \in E\mathbf{F} \otimes_{\mathbf{F}} \mathfrak{q}_\Psi$. Thus $\gamma \in \Gamma_\Psi(E)$ if and only if $\gamma \in \widetilde{\Gamma}_\Psi^n(E)$. □

5.2.8. *Local notation.* Fix an integer $n \geq 1$, and consider the group $\widetilde{\Gamma}_\Psi^n$ as defined above. We set $\Sigma_0 := \overline{k(t)}[X, \Delta^{-1}]$, and consider the ideal $\mathfrak{q} := \mathfrak{q}_\Psi \subseteq \Sigma_0$ defined in §5.2.4, which is a maximal σ^n-ideal of Σ_0. Letting $\widetilde{\Sigma}_\Psi := \mu(\Sigma_0)$, consider the σ^n-algebra,

$$\widetilde{\Sigma}_\Psi[X, \Delta^{-1}] = \widetilde{\Sigma}_\Psi \otimes_{\overline{k(t)}} \Sigma_0.$$

By taking (Ψ_{ij}) to be the image of (X_{ij}) in $\widetilde{\Sigma}_\Psi$, we define the matrix $Y = (Y_{ij})$ by $(Y_{ij}) := (\Psi_{ij})^{-1}(X_{ij})$. The action of σ^n on Y, and thus on $\overline{\Sigma}_n := \mathbf{F}_n[Y, \Delta(Y)^{-1}]$ is the identity.

Now there are natural isomorphisms,

$$\widetilde{\Sigma}_\Psi \otimes_{\overline{k(t)}} \Sigma_0 \cong \widetilde{\Sigma}_\Psi \otimes_{\mathbf{F}_n} \overline{\Sigma}_n \cong \widetilde{\Sigma}_\Psi[Y, \Delta(Y)^{-1}]. \tag{5.2.8.1}$$

Thus we can let $\mathfrak{Q} := \mathfrak{q} \cdot (\widetilde{\Sigma}_\Psi \otimes_{\overline{k(t)}} \Sigma_0)$ and $\mathfrak{b}_n := \mathfrak{Q} \cap \overline{\Sigma}_n$.

Lemma 5.2.9. *With the notation as above,* $\mathfrak{b}_n \cdot (\widetilde{\Sigma}_\Psi[Y, \Delta(Y)^{-1}]) = \mathfrak{Q}$.

Proof. The proof is identical to [23, Lem. 1.11], so we omit it. However, it is worth noting that the simplicity of $\widetilde{\Sigma}_\Psi$ as a σ^n-algebra is essential. □

Proposition 5.2.10. *For each $n \geq 1$, there is a natural isomorphism*

$$\widetilde{\Sigma}_\Psi \otimes_{\overline{k(t)}} \widetilde{\Sigma}_\Psi \cong \widetilde{\Sigma}_\Psi \otimes_{\mathbf{F}_n} (\overline{\Sigma}_n/\mathfrak{b}_n).$$

Let $V_n := \operatorname{Spec} \overline{\Sigma}_n/\mathfrak{b}_n$. Then

$$V_n(\mathbf{F}_n) = \widetilde{\Gamma}^n_\Psi(\mathbf{F}_n)$$

as subsets of $\mathrm{GL}_r(\mathbf{F}_n)$.

Proof. (cf. [23, Lem. 1.12]) The isomorphism follows from Lemma 5.2.9 by quotienting the appropriate terms in (5.2.8.1) by $\mathfrak{Q}$ and $\mathfrak{b}_n$. Since $\overline{k(t)}$ is algebraically closed, $\widetilde{\Sigma}_\Psi$ is a separable $\overline{k(t)}$-algebra. Thus, $\widetilde{\Sigma}_\Psi \otimes_{\overline{k(t)}} \widetilde{\Sigma}_\Psi$ is reduced by [32, §III.15, Thm. 39] (in fact it is a domain by [32, §III.15, Cor. 1]). Therefore $\overline{\Sigma}_n/\mathfrak{b}_n$ is reduced. For $\gamma \in \mathrm{GL}_r(\mathbf{F}_n)$,

$$\begin{aligned} \gamma \in \widetilde{\Gamma}^n_\Psi(\mathbf{F}_n) &\Leftrightarrow \gamma * \mathfrak{q} = \mathfrak{q} \\ &\Leftrightarrow \gamma * \mathfrak{Q} = \mathfrak{Q} \\ &\Leftrightarrow \gamma * \mathfrak{b}_n = \mathfrak{b}_n. \end{aligned}$$

These three equivalences follow from Lemma 4.3.6, Lemma 5.2.9, and the definition of $\widetilde{\Gamma}^n_\Psi(\mathbf{F}_n)$. Now suppose that $\gamma \in \widetilde{\Gamma}^n_\Psi(\mathbf{F}_n)$. Then by the first equivalence above, the map

$$f \otimes g \mapsto f \cdot (\gamma * g) : \widetilde{\Sigma}_\Psi \otimes_{\overline{k(t)}} \widetilde{\Sigma}_\Psi \to \widetilde{\Sigma}_\Psi$$

is a $\overline{k(t)}$-algebra homomorphism that commutes with σ^n. The image of $\overline{\Sigma}_n/\mathfrak{b}_n \subseteq \widetilde{\Sigma}_\Psi \otimes_{\overline{k(t)}} \widetilde{\Sigma}_\Psi$ under this map must lie in $\widetilde{\Sigma}_\Psi^{\sigma^n}$, since σ^n acts trivially on $\overline{\Sigma}_n$. Since

$$\widetilde{\Sigma}_\Psi^{\sigma^n} \subseteq \overline{\mathbb{L}}^{\sigma^n} = \mathbf{F}_n,$$

we have defined an $\mathbf{F}_n$-linear homomorphism $\overline{\Sigma}_n/\mathfrak{b}_n \to \mathbf{F}_n$, thus providing a map $\widetilde{\Gamma}^n_\Psi(\mathbf{F}_n) \to V_n(\mathbf{F}_n)$.

On the other hand, if $\gamma \in V_n(\mathbf{F}_n)$, then we can define a $\overline{k(t)}$-algebra homomorphism that commutes with σ^n,

$$f \otimes c(Y) \mapsto f \cdot c(\gamma) : \widetilde{\Sigma}_\Psi \otimes_{\overline{k(t)}} \overline{\Sigma}_n/\mathfrak{b}_n \to \widetilde{\Sigma}_\Psi.$$

Via the isomorphism $\widetilde{\Sigma}_\Psi \otimes_{\overline{k(t)}} \widetilde{\Sigma}_\Psi \cong \widetilde{\Sigma}_\Psi \otimes_{\mathbf{F}_n} (\overline{\Sigma}_n/\mathfrak{b}_n)$, we can restrict this map to $1 \otimes \widetilde{\Sigma}_\Psi \subseteq \widetilde{\Sigma}_\Psi \otimes_{\overline{k(t)}} \widetilde{\Sigma}_\Psi$, and so we obtain a $\overline{k(t)}$-algebra homomorphism $\widetilde{\Sigma}_\Psi \to \widetilde{\Sigma}_\Psi$ that commutes with σ^n. One checks that this map is given by $f(\Psi) \mapsto f(\Psi\gamma)$, and so we have $\gamma * \mathfrak{q} = \mathfrak{q}$ and $\gamma \in \Gamma^n_\Psi(\mathbf{F}_n)$. Finally, it follows easily that these two maps $\widetilde{\Gamma}^n_\Psi(\mathbf{F}_n) \to V_n(\mathbf{F}_n)$ and $V_n(\mathbf{F}_n) \to \widetilde{\Gamma}^n_\Psi(\mathbf{F}_n)$ are inverses of each other. □

5.2.11. *The variety V over $\overline{\mathbb{F}_q(t)}$.* Let $\overline{\Sigma} = \overline{\mathbb{F}_q(t)}[Y, \Delta(Y)^{-1}]$. By Lemma 5.2.9, it follows that the ideals generated by each $\mathfrak{b}_n$ in $\overline{\Sigma}$ are the same ideal $\overline{\mathfrak{b}} \subseteq \overline{\Sigma}$. That is, there are $g_1, \dots, g_s \in \mathbf{F}_1[Y, \Delta(Y)^{-1}]$ so that for every $n \geq 1$,

$$\mathfrak{b}_n = (g_1, \dots, g_s) \cdot \overline{\Sigma}_n.$$

Let $V = \operatorname{Spec} \overline{\Sigma}/\overline{\mathfrak{b}}$. Then for each $n \geq 1$,

$$V = V_n \times_{\mathbf{F}_n} \overline{\mathbb{F}_q(t)},$$

and V is a reduced variety over $\overline{\mathbb{F}_q(t)}$. Also, by Propositions 5.2.7 and 5.2.10, it follows that for all $n \geq 1$,

$$V(\overline{\mathbb{F}_q(t)}) = V_n(\overline{\mathbb{F}_q(t)}) = \Gamma_\Psi(\overline{\mathbb{F}_q(t)}).$$

Because V is reduced, the Zariski closure of $\Gamma_\Psi(\overline{\mathbb{F}_q(t)})$ in GL_r over $\overline{\mathbb{F}_q(t)}$ is precisely V. Moreover,

$$\dim V = \dim \Gamma_\Psi.$$

We arrive at the following theorem.

Theorem 5.2.12. *Let $\Phi \in \mathrm{GL}_r(\overline{k}(t))$ and suppose that $\Psi \in \mathrm{GL}_r(\mathbb{L})$ is a fundamental matrix for Φ. The following hold.*

(a) $\dim \Gamma_\Psi = \operatorname{tr.deg}_{\overline{k}(t)} \Lambda_\Psi$.
(b) Γ_Ψ *is smooth over* $\mathbb{F}_q(t)$.

Proof. The transcendence degree of Λ_Ψ over $\overline{k}(t)$ is the same as the transcendence degree of the fraction field $\widetilde{\Lambda}_\Psi$ of $\widetilde{\Sigma}_\Psi$ over $\overline{k(t)}$. From Proposition 5.2.10 it follows that the dimension of V is equal to the latter quantity, and so by the discussion above,

$$\operatorname{tr.deg}_{\overline{k}(t)} \Lambda_\Psi = \dim V = \dim \Gamma_\Psi.$$

By Proposition 5.1.5, it follows that

$$\dim_{\mathbb{F}_q(t)} \operatorname{Lie} \Gamma_\Psi \leq \dim \Gamma_\Psi.$$

However, the Lie algebra of an algebraic group always has dimension bounded below by the dimension of the group, and so

$$\dim_{\mathbb{F}_q(t)} \operatorname{Lie} \Gamma_\Psi = \dim \Gamma_\Psi.$$

By [30, Cor., p. 94], Γ_Ψ is smooth. □

5.2.13. *The group $\widetilde{\Gamma}_\Psi$ and the space $\widetilde{Z}_\Psi$.* We now set $\widetilde{\Gamma}_\Psi := \Gamma_\Psi \times_{\mathbb{F}_q(t)} \overline{\mathbb{F}_q(t)}$. By the discussion in §5.2.11, the coordinate ring of $\widetilde{\Gamma}_\Psi$ is $\overline{\Sigma}/\overline{\mathfrak{b}}$. Moreover, from Proposition 5.2.10 we see that

$$\widetilde{\Sigma}_\Psi \otimes_{\overline{k(t)}} \widetilde{\Sigma}_\Psi \cong \widetilde{\Sigma}_\Psi \otimes_{\overline{\mathbb{F}_q(t)}} \overline{\Sigma}/\overline{\mathfrak{b}}.$$

Let $\widetilde{Z}_\Psi := \operatorname{Spec} \widetilde{\Sigma}_\Psi$.

Theorem 5.2.14. *The space $\widetilde{Z}_\Psi$ is a $\widetilde{\Gamma}_\Psi$-torsor over some finite extension of $\overline{k}(t)$.*

Proof. Let $\widetilde{\Gamma}'_\Psi := \widetilde{\Gamma}_\Psi \times_{\overline{\mathbb{F}_q(t)}} \overline{k(t)}$. Since $\widetilde{\Sigma}_\Psi \otimes_{\overline{\mathbb{F}_q(t)}} \overline{\Sigma}/\overline{\mathfrak{b}} \cong \widetilde{\Sigma}_\Psi \otimes_{\overline{k(t)}} (\overline{k(t)} \otimes_{\overline{\mathbb{F}_q(t)}} \overline{\Sigma}/\overline{\mathfrak{b}})$, the natural injection

$$\widetilde{\Sigma}_\Psi \hookrightarrow \widetilde{\Sigma}_\Psi \otimes_{\overline{\mathbb{F}_q(t)}} \overline{\Sigma}/\overline{\mathfrak{b}}$$

induces a surjective morphism

$$(z, \gamma) \mapsto z\gamma : \widetilde{Z}_\Psi \times_{\overline{k(t)}} \widetilde{\Gamma}'_\Psi \to \widetilde{Z}_\Psi$$

of varieties over $\overline{k(t)}$. In this way $\widetilde{\Gamma}'_\Psi$ acts on the right on $\widetilde{Z}_\Psi$. Moreover, the isomorphism

$$\widetilde{\Sigma}_\Psi \otimes_{\overline{k(t)}} \widetilde{\Sigma}_\Psi \cong \widetilde{\Sigma}_\Psi \otimes_{\overline{k(t)}} (\overline{k(t)} \otimes_{\overline{\mathbb{F}_q(t)}} \overline{\Sigma}/\overline{\mathfrak{b}}),$$

induces the isomorphism

$$(z,\gamma) \mapsto (z\gamma, z) : \widetilde{Z}_\Psi \times_{\overline{k(t)}} \widetilde{\Gamma}'_\Psi \to \widetilde{Z}_\Psi \times_{\overline{k(t)}} \widetilde{Z}_\Psi.$$

Thus $\widetilde{Z}_\Psi$ is a principal homogeneous space for $\widetilde{\Gamma}_\Psi$ over $\overline{k(t)}$. This isomorphism is defined over some finite extension of $\overline{k}(t)$. ☐

5.3. **The Galois action on $\widetilde{\Lambda}_\Psi$.**

5.3.1. *The action of* $\Gamma_\Psi(\overline{\mathbb{F}_q(t)})$ *on* $\widetilde{\Lambda}_\Psi$. Since $\bigcup_{n=1}^\infty \mathbf{F}_n = \overline{\mathbb{F}_q(t)}$, every finite extension $E/\mathbb{F}_q(t)$ is contained in $\mathbf{F}_n$ for some $n \geq 1$. For such an $E \subseteq \mathbf{F}_n$, we have an isomorphism

$$\widetilde{\Gamma}^n_\Psi(\mathbf{F}_n) \cong \operatorname{Aut}^{\sigma^n}(\widetilde{\Lambda}_\Psi/\overline{k(t)}),$$

by Lemma 4.2.6 and Proposition 4.4.2. The actions of each $\widetilde{\Gamma}^n_\Psi(E)$ are the same once n is large enough so that $E \subseteq \mathbf{F}_n$, and so by Proposition 5.2.7, we have an action of $\Gamma_\Psi(\overline{\mathbb{F}_q(t)})$ induced on $\widetilde{\Sigma}_\Psi$ and $\widetilde{\Lambda}_\Psi$.

Theorem 5.3.2. *Let $\Phi \in \mathrm{GL}_r(\overline{k}(t))$, and suppose that $\Psi \in \mathrm{GL}_r(\mathbb{L})$ is a fundamental matrix for Φ.*

(a) *The subfield of $\widetilde{\Lambda}_\Psi$ fixed by $\Gamma_\Psi(\overline{\mathbb{F}_q(t)})$ is $\overline{k(t)}$.*
(b) *The elements of Λ_Ψ fixed by $\Gamma_\Psi(\overline{\mathbb{F}_q(t)})$ are precisely $\overline{k}(t)$.*

Proof. See [23, Lem. 1.28]. Suppose $f \in \widetilde{\Lambda}_\Psi$ is fixed by $\Gamma_\Psi(\overline{\mathbb{F}_q(t)}) = \widetilde{\Gamma}_\Psi(\overline{\mathbb{F}_q(t)})$. We consider $f \in \widetilde{\Lambda}_\Psi$ to be a function $f : \widetilde{Z}_\Psi \to \mathbb{P}^1$ over $\overline{k(t)}$. For $i = 1, 2$, we consider the two maps,

$$g_i : \widetilde{Z}_\Psi \times_{\overline{\mathbb{F}_q(t)}} \widetilde{\Gamma}_\Psi \to \widetilde{Z}_\Psi \times_{\overline{k(t)}} \widetilde{Z}_\Psi \xrightarrow{\pi_i} \widetilde{Z}_\Psi \xrightarrow{f} \mathbb{P}^1,$$

where π_i is the i-th projection. Because f is $\widetilde{\Gamma}_\Psi$-invariant, we must have $g_1 = g_2$. Therefore, $f \circ \pi_1 = f \circ \pi_2$, which implies that f is constant. This proves part (a). Part (b) follows from part (a) and Lemma 5.2.1. ☐

5.4. **Γ_Ψ and t-motives.** Given a t-motive M, we defined the Galois group Γ_M of M in §3.5.2. Associated to M we can also choose a matrix $\Phi \in \mathrm{GL}_r(\overline{k}(t))$ that represents multiplication by σ on M. Let $\Psi \in \mathrm{GL}_r(\mathbb{L})$ be a rigid analytic trivialization of Φ. We will show that Γ_M is isomorphic to Γ_Ψ over $\mathbb{F}_q(t)$.

5.4.1. *t-motives and σ-semilinear equations.* Let M be a t-motive. We fix the following notation throughout this section. Let $\mathbf{m}$ be a basis for M, and let $\Phi \in \mathrm{GL}_r(\overline{k}(t))$ represent multiplication by σ on M. We pick a rigid analytic trivialization $\Psi \in \mathrm{GL}_r(\mathbb{L})$ for M, which is at the same time a fundamental matrix for Φ.

Let $M^u_v := M^{\otimes u} \otimes (M^\vee)^{\otimes v}$, for u, $v \geq 0$, with corresponding natural basis $\mathbf{m}^u_v$. By Proposition 3.3.9(b), $\Psi^{-1}\mathbf{m}$ is an $\mathbb{F}_q(t)$-basis for M^{B}. Moreover, there is a natural $\Theta^u_v \in \mathrm{GL}_w(\mathbb{F}_q[X, \Delta^{-1}])$, with $w = r^{u+v}$, so that

$$\Psi^{\otimes u} \otimes (\Psi^\vee)^{\otimes v} = \Theta^u_v(\Psi).$$

Since $\Theta^u_v(AB) = \Theta^u_v(A)\Theta^u_v(B)$ for $r \times r$ matrices A and B, it follows that

$$\Theta^u_v(\Psi)^{-1} \cdot \mathbf{m}^u_v = \Theta^u_v(\Psi^{-1}) \cdot \mathbf{m}^u_v$$

is an $\mathbb{F}_q(t)$-basis for $(M^u_v)^{\mathrm{B}}$.

Because $\mathcal{T}_M$ is Tannakian, if N is any t-motive in $\mathcal{T}_M$, then N is the subquotient of a direct sum of various M_v^u, and vice versa. If $\mathbf{n}$ is a basis for N and $\Psi_N \in \mathrm{GL}_s(\mathbb{L})$ is a rigid analytic trivialization for N with respect to $\mathbf{n}$, then it follows from the above paragraphs and Propositions 3.3.9(b) and 3.3.11(b) that the entries of Ψ_N are in Σ_Ψ. Thus in fact $\Psi_N \in \mathrm{GL}_s(\Sigma_\Psi)$.

Lemma 5.4.2. *For any t-motive N in $\mathcal{T}_M$ and $\mathbb{F}_q(t)$-algebra R, the natural map,*

$$\Sigma_\Psi^{(R)} \otimes_{\mathbb{F}_q(t)} N^{\mathrm{B}} \to \Sigma_\Psi^{(R)} \otimes_{\overline{k}(t)} N$$

is bijective.

Proof. Let κ be the map defined in the statement of the lemma. Thus as above we can pick a basis $\mathbf{n}$ for N and a rigid analytic trivialization $\Psi_N \in \mathrm{GL}_s(\Sigma_\Psi)$ with respect to $\mathbf{n}$. Now $1 \otimes (\Psi_N^{-1}\mathbf{n})$ is a $\Sigma_\Psi^{(R)}$-basis of $\Sigma_\Psi^{(R)} \otimes_{\mathbb{F}_q(t)} N^{\mathrm{B}}$. If $\mathbf{f} \in \mathrm{Mat}_{1\times s}(\Sigma_\Psi^{(R)})$, then

$$\kappa\bigl((\mathbf{f} \otimes 1)\cdot(1 \otimes (\Psi_N^{-1}\mathbf{n}))\bigr) = (\mathbf{f}\Psi_N^{-1} \otimes 1)\cdot(1 \otimes \mathbf{n}).$$

The entries of $(\Psi_N^{-1} \otimes 1)\cdot(1\otimes \mathbf{n})$ are in the image of κ, and

$$\Psi_N \cdot (\Psi_N^{-1} \otimes 1)\cdot(1 \otimes \mathbf{n}) = 1 \otimes \mathbf{n}.$$

Thus κ is surjective. Since $\Psi_N \in \mathrm{GL}_s(\Sigma_\Psi)$, the map κ is bijective. □

Theorem 5.4.3. *Let M be a t-motive, and let N be a t-motive in $\mathcal{T}_M$. If we consider N^{B} to be an algebraic group over $\mathbb{F}_q(t)$, then there is a natural representation*

$$\xi_N : \Gamma_\Psi \to \mathrm{GL}(N^{\mathrm{B}})$$

over $\mathbb{F}_q(t)$ that is functorial in N.

Proof. To define the representation, it suffices by the Yoneda lemma [30, §1.2–1.4] to define a representation

$$\xi_N^{(R)} : \Gamma_\Psi(R) \to \mathrm{GL}(R \otimes_{\mathbb{F}_q(t)} N^{\mathrm{B}})$$

for every $\mathbb{F}_q(t)$-algebra R and show that it is functorial in R. Now N is isomorphic to a subquotient of a direct sum of M_v^u for various u, v. Because for any t-motives $N_1 \subseteq N_2$, we have $(N_1/N_2)^{\mathrm{B}} \cong N_1^{\mathrm{B}}/N_2^{\mathrm{B}}$ by Proposition 3.3.11(b), it suffices to assume that N is a sub-t-motive of such a direct sum. Henceforth, fix $Q := \bigoplus_{i=1}^n M_{v_i}^{u_i}$ and $\mathbf{q} := \bigoplus_{i=1}^n \mathbf{m}_{v_i}^{u_i}$, and assume that $N \subseteq Q$.

Fix a basis $\mathbf{n}$ for N, a matrix $\Phi_N \in \mathrm{GL}_s(\overline{k}(t))$ such that $\sigma\mathbf{n} = \Phi_N\mathbf{n}$, and a rigid analytic trivialization $\Psi_N \in \mathrm{GL}_s(\Sigma_\Psi)$ for Φ_N. Let R be an $\mathbb{F}_q(t)$-algebra, and let $\gamma \in \Gamma_\Psi(R)$. Define

$$\Xi^{(R)}(\gamma) := \gamma \otimes 1 : \Sigma_\Psi^{(R)} \otimes_{\overline{k}(t)} N \to \Sigma_\Psi^{(R)} \otimes_{\overline{k}(t)} N, \tag{5.4.3.1}$$

which is an isomorphism of $\overline{k}(t)$-vector spaces. Now by Lemma 5.4.2, $R \otimes_{\mathbb{F}_q(t)} N^{\mathrm{B}}$ spans $\Sigma_\Psi^{(R)} \otimes_{\overline{k}(t)} N$ as a $\Sigma_\Psi^{(R)}$-module. Let $\xi^{(R)}(\gamma)$ be the restriction of $\Xi^{(R)}(\gamma)$ to $R \otimes_{\mathbb{F}_q(t)} N^{\mathrm{B}}$.

We claim that the image of $\xi^{(R)}(\gamma)$ is $R \otimes_{\mathbb{F}_q(t)} N^{\mathrm{B}}$. Extend $\mathbf{n}$ to a basis for Q, say $\mathbf{q}' = [\mathbf{n}, \mathbf{p}]^{\mathrm{tr}}$. Then there is a matrix $B \in \mathrm{GL}_w(\overline{k}(t))$ so that $\mathbf{q}' = B\mathbf{q}$ with $\sigma\mathbf{q}' = \Phi'\mathbf{q}'$. It follows that $B^{(-1)}\Theta(\Phi) = \Phi' B$ (where $\Theta := \bigoplus_{i=1}^n \Theta_{v_i}^{u_i} \in \mathrm{GL}_w(\mathbb{F}_q[X, \Delta^{-1}])$) and that

$$\Phi' = \begin{bmatrix} \Phi_N & 0 \\ * & * \end{bmatrix}.$$

We check that $B\Theta(\Psi)$ is a rigid analytic trivialization for Φ', and so $\Theta(\Psi)^{-1}B^{-1}\mathbf{q}'$ is an $\mathbb{F}_q(t)$-basis for Q^{B} by Proposition 3.3.9(b). By the same token, $\Psi_N^{-1}\mathbf{n}$ is an $\mathbb{F}_q(t)$-basis for $N^{\mathrm{B}} \subseteq Q^{\mathrm{B}}$, and so we can extend $\Psi_N^{-1}\mathbf{n}$ to an $\mathbb{F}_q(t)$-basis $\boldsymbol{\mu} := [\Psi_N^{-1}\mathbf{n}, \mathbf{h}]$ of Q^{B}. Then for some $\delta \in \mathrm{GL}_w(\mathbb{F}_q(t))$, we have $\boldsymbol{\mu} = \delta\Theta(\Psi)^{-1}B^{-1}\mathbf{q}'$, and it follows that

$$\delta\Theta(\Psi)^{-1}B^{-1} = \begin{bmatrix} \Psi_N^{-1} & 0 \\ * & * \end{bmatrix}. \tag{5.4.3.2}$$

Consider the matrix $\delta\Theta(X)^{-1}B^{-1} \in \mathrm{GL}_w(\bar{k}(t)[X, \Delta^{-1}])$. The polynomials in the upper right $s \times (w-s)$ block of $\delta\Theta(X)^{-1}B^{-1}$ all vanish when evaluated at the entries of Ψ by (5.4.3.2). By the definition of $\Gamma_\Psi(R)$, and in particular (4.3.5.1), it follows that for any $\gamma \in \Gamma_\Psi(R)$,

$$\delta\Theta(\Psi\gamma)^{-1}B^{-1} = \begin{bmatrix} * & 0 \\ * & * \end{bmatrix}, \tag{5.4.3.3}$$

where the 0 is the upper right $s \times (w-s)$ block of $\delta\Theta(\Psi\gamma)^{-1}B^{-1}$. Now let $J := [I_s, 0] \in \mathrm{Mat}_{s\times w}(\mathbb{L})$, where I_s is the $s \times s$ identity matrix. It follows that

$$\Psi_N^{-1} = J\delta\Theta(\Psi)^{-1}B^{-1}J^{\mathrm{tr}}, \quad \Phi_N = J\Phi' J^{\mathrm{tr}}.$$

Let $\gamma \in \Gamma_\Psi(R)$. We see that

$$\begin{aligned} \xi^{(R)}(\gamma)(\Psi_N^{-1}\mathbf{n}) &= \xi^{(R)}(\gamma)(J\delta\Theta(\Psi)^{-1}B^{-1}J^{\mathrm{tr}}\mathbf{n}) \\ &= J\delta\Theta(\Psi\gamma)^{-1}B^{-1}J^{\mathrm{tr}}\mathbf{n} \\ &= J\delta\Theta(\gamma^{-1})\Theta(\Psi)^{-1}B^{-1}J^{\mathrm{tr}}\mathbf{n} \\ &= J\delta\Theta(\gamma^{-1})\delta^{-1}\delta\Theta(\Psi)^{-1}B^{-1}J^{\mathrm{tr}}. \end{aligned}$$

It follows from (5.4.3.2) and (5.4.3.3) that $\delta\Theta(\gamma^{-1})\delta^{-1}$ is block lower triangular, and so

$$\begin{aligned} \xi^{(R)}(\gamma)(\Psi_N^{-1}\mathbf{n}) &= J\delta\Theta(\gamma^{-1})\delta^{-1}J^{\mathrm{tr}} \cdot J\delta\Theta(\Psi)^{-1}B^{-1}J^{\mathrm{tr}} \\ &= J\delta\Theta(\gamma^{-1})\delta^{-1}J^{\mathrm{tr}} \cdot \Psi_N^{-1}\mathbf{n}. \end{aligned}$$

Let $\rho(\gamma) := J\delta\Theta(\gamma^{-1})\delta^{-1}J^{\mathrm{tr}} \in \mathrm{GL}_s(R)$. Then we find that $\xi^{(R)}$ is given by

$$\begin{aligned} \xi^{(R)}(\gamma) : R \otimes_{\mathbb{F}_q(t)} N^{\mathrm{B}} &\rightarrow R \otimes_{\mathbb{F}_q(t)} N^{\mathrm{B}}, \\ \mathbf{f} \cdot \Psi_N^{-1}\mathbf{n} &\mapsto \mathbf{f} \cdot \rho(\gamma) \cdot \Psi_N^{-1}\mathbf{n} \end{aligned}$$

for $\mathbf{f} \in \mathrm{Mat}_{1\times s}(R)$. Thus $\xi^{(R)}(\gamma)$ is an R-linear automorphism, and it follows easily that we have a group homomorphism

$$\xi^{(R)} : \Gamma_\Psi(R) \rightarrow \mathrm{GL}(R \otimes_{\mathbb{F}_q(t)} N^{\mathrm{B}})$$

for each $\mathbb{F}_q(t)$-algebra R. Finally, it is straightforward to check that this construction is functorial in R, and so we have defined a group homorphism $\xi_N := \xi : \Gamma_\Psi \rightarrow \mathrm{GL}(N^{\mathrm{B}})$ over $\mathbb{F}_q(t)$.

To check for functoriality in N, it suffices to show that, for $N \subseteq \bigoplus_{i=1}^n M_{v_i}^{u_i}$ and $P \subseteq \bigoplus_{j=1}^m M_{x_j}^{w_j}$, if $\phi : N \rightarrow P$ is a morphism of t-motives, then there is an $\mathbb{F}_q(t)$-linear map $N^{\mathrm{B}} \rightarrow P^{\mathrm{B}}$ that commutes with ξ_N and ξ_P. Of course the natural map to take is $\phi^{\mathrm{B}} : N^{\mathrm{B}} \rightarrow P^{\mathrm{B}}$. Thus we need to show that, for every $\mathbb{F}_q(t)$-algebra R, the diagram

$$\begin{array}{ccc} R \otimes_{\mathbb{F}_q(t)} N^{\mathrm{B}} & \xrightarrow{\xi_N^{(R)}} & R \otimes_{\mathbb{F}_q(t)} N^{\mathrm{B}} \\ \downarrow{\scriptstyle 1\otimes\phi^{\mathrm{B}}} & & \downarrow{\scriptstyle 1\otimes\phi^{\mathrm{B}}} \\ R \otimes_{\mathbb{F}_q(t)} P^{\mathrm{B}} & \xrightarrow{\xi_P^{(R)}} & R \otimes_{\mathbb{F}_q(t)} P^{\mathrm{B}} \end{array} \tag{5.4.3.4}$$

commutes. However, (5.4.3.4) is simply obtained by restriction to $R \otimes_{\mathbb{F}_q(t)} N^{\mathrm{B}}$ and $R \otimes_{\mathbb{F}_q(t)} P^{\mathrm{B}}$ from

$$\begin{array}{ccc} \Sigma_\Psi^{(R)} \otimes_{\overline{k}(t)} N & \xrightarrow{\Xi_N^{(R)}} & \Sigma_\Psi^{(R)} \otimes_{\overline{k}(t)} N \\ \downarrow{\scriptstyle 1\otimes\phi} & & \downarrow{\scriptstyle 1\otimes\phi} \\ \Sigma_\Psi^{(R)} \otimes_{\overline{k}(t)} P & \xrightarrow{\Xi_P^{(R)}} & \Sigma_\Psi^{(R)} \otimes_{\overline{k}(t)} P, \end{array}$$

which clearly commutes. □

Corollary 5.4.4. *Let M be a t-motive. The representation $\xi_M : \Gamma_\Psi \to \mathrm{GL}(M^{\mathrm{B}})$ is faithful.*

Proof. Let $\mathbf{m}$ be a $\overline{k}(t)$-basis for M so that $\sigma\mathbf{m} = \Phi\mathbf{m}$. Let R be an $\mathbb{F}_q(t)$-algebra. For any $\mathbf{f} \in \mathrm{Mat}_{1\times r}(R)$ and $\gamma \in \Gamma_\Psi(R)$,

$$\xi^{(R)}(\gamma)(\mathbf{f} \cdot \Psi^{-1}\mathbf{m}) = \mathbf{f} \cdot \gamma^{-1} \cdot \Psi^{-1}\mathbf{m}.$$

Thus $\xi^{(R)} : \Gamma_\Psi(R) \to \mathrm{GL}(R \otimes_{\mathbb{F}_q(t)} M^{\mathrm{B}})$ is injective. □

5.4.5. *The functor ξ_M.* For a t-motive M, if $\Phi \in \mathrm{GL}_r(\overline{k}(t))$ represents multiplication by σ on M and if $\Psi \in \mathrm{GL}_r(\mathbb{L})$ is a rigid analytic trivialization of Φ, then Theorem 5.4.3 defines a functor

$$\xi_M : \mathcal{T}_M \to \mathbf{Rep}(\Gamma_\Psi, \mathbb{F}_q(t)).$$

It is straightforward to check that ξ_M is a tensor functor. Let

$$\eta_M : \mathbf{Rep}(\Gamma_M, \mathbb{F}_q(t)) \xrightarrow{\sim} \mathcal{T}_M$$

be the equivalence of categories defined in §3.5.2. Letting

$$F_\Psi : \mathbf{Rep}(\Gamma_\Psi, \mathbb{F}_q(t)) \to \mathbf{Vec}(\mathbb{F}_q(t))$$

be the forgetful functor, we see immediately that

$$\omega_M = F_\Psi \circ \xi_M.$$

Thus by [11, Cor. II.2.9], there is a unique homomorphism $\pi_M : \Gamma_\Psi \to \Gamma_M$ over $\mathbb{F}_q(t)$ so that the natural functor $\tau_M : \mathbf{Rep}(\Gamma_M, \mathbb{F}_q(t)) \to \mathbf{Rep}(\Gamma_\Psi, \mathbb{F}_q(t))$ induced by π_M satisfies

$$\xi_M \circ \eta_M = \tau_M.$$

Proposition 5.4.6. *Let M be a t-motive. Suppose that $\Phi \in \mathrm{GL}_r(\overline{k}(t))$ represents multiplication by σ on M and that $\Psi \in \mathrm{GL}_r(\mathbb{L})$ is a rigid analytic trivialization for Φ. Then the functor*

$$\xi_M : \mathcal{T}_M \to \mathbf{Rep}(\Gamma_\Psi, \mathbb{F}_q(t))$$

is fully faithful.

Proof. For any t-motives N and P in $\mathcal{T}_M$, there is a natural isomorphism of $\mathbb{F}_q(t)$-vector spaces,

$$\mathrm{Hom}_{\mathcal{T}_M}(P, N) \cong \mathrm{Hom}_{\mathcal{T}_M}(\mathbf{1}, \mathrm{Hom}(P, N)).$$

Thus it suffices to prove full faithfulness when $P = \mathbf{1}$. Now

$$\mathrm{Hom}_{\mathcal{T}_M}(\mathbf{1}, N) \cong N \cap N^{\mathrm{B}} = \{n \in N \mid \sigma n = n\},$$

and this provides an injection

$$\mathrm{Hom}_{\mathcal{T}_M}(\mathbf{1}, N) \hookrightarrow \mathrm{Hom}_{\Gamma_\Psi}(\mathbf{1}^{\mathrm{B}}, N^{\mathrm{B}}).$$

Conversely suppose that $\phi : \mathbf{1}^{\mathrm{B}} \to N^{\mathrm{B}}$ is a Γ_Ψ-morphism. Pick a $\overline{k}(t)$-basis $\mathbf{n}$ for N. Then $\phi(1) = \mathbf{h}(\Psi) \cdot \mathbf{n}$ for some $\mathbf{h}(\Psi) \in \mathrm{Mat}_{1\times s}(\Sigma_\Psi)$ by Lemma 5.4.2. Let $E/\mathbb{F}_q(t)$ be a finite extension of fields. Using the argument in the proof of Theorem 5.4.3, we see that for $\gamma \in \Gamma_\Psi(E)$, the action of $\xi^{(E)}(\gamma) := \xi_M^{(E)}(N)(\gamma)$ on $E \otimes_{\mathbb{F}_q(t)} N^{\mathrm{B}}$ is simply the restriction of the map $\Xi^{(E)}(\gamma) : \Sigma_\Psi^{(E)} \otimes_{\overline{k}(t)} N \to \Sigma_\Psi^{(E)} \otimes_{\overline{k}(t)} N$ from (5.4.3.1) to $E \otimes_{\mathbb{F}_q(t)} N^{\mathrm{B}}$. Since ϕ is a Γ_Ψ-morphism, it follows that $\xi^{(E)}(\gamma)(\phi(1)) = \phi(1)$ for all $\gamma \in \Gamma_\Psi(E)$. Thus,

$$\mathbf{h}(\Psi) \cdot \mathbf{n} = \phi(1) = \xi^{(E)}(\gamma)(\phi(1)) = \mathbf{h}(\Psi\gamma) \cdot \mathbf{n}, \quad \gamma \in \Gamma_\Psi(E).$$

Because $\mathbf{n}$ is a $\Sigma_\Psi^{(E)}$-basis of $\Sigma_\Psi^{(E)} \otimes_{\overline{k}(t)} N$, the entries of $\mathbf{h}(\Psi)$ must each be fixed by every $\gamma \in \Gamma_\Psi(E)$. By varying over all $E/\mathbb{F}_q(t)$ finite, Theorem 5.3.2 implies that $\mathbf{h}(\Psi) \in \mathrm{Mat}_{1\times s}(\overline{k}(t))$. Thus $\phi(1) \in N \cap N^{\mathrm{B}}$. □

Lemma 5.4.7. *Let $\Phi \in \mathrm{GL}_r(\overline{k}(t))$, and suppose that $\Psi \in \mathrm{GL}_r(\mathbb{L})$ is a fundamental matrix for Φ. Suppose that $W \subseteq \Lambda_\Psi^{\oplus s}$ is a vector subspace over Λ_Ψ such that for every finite extension of fields $E/\mathbb{F}_q(t)$,*

$$\Gamma_\Psi(E) \cdot (E \otimes_{\mathbb{F}_q(t)} W) \subseteq E \otimes_{\mathbb{F}_q(t)} W.$$

Then W has a system of defining equations over $\overline{k}(t)$.

Proof. Suppose that W has dimension $s - m$, and let $A(\Psi) \in \mathrm{Mat}_{m\times s}(\Lambda_\Psi)$ be a coefficient matrix for a system of defining equations for W. By changing the order of the variables if necessary, we can use Gaussian elimination on $A(\Psi)$ to obtain

$$G(\Psi) = [I_m, C(\Psi)],$$

where $C(\Psi) \in \mathrm{Mat}_{m\times(s-m)}(\Lambda_\Psi)$. Both $A(\Psi)$ and $G(\Psi)$ provide coefficient matrices for equations for W, and so it suffices to show that $C(\Psi)$ has entries in $\overline{k}(t)$.

Let $E/\mathbb{F}_q(t)$ be a finite extension of fields. Since $E \otimes_{\mathbb{F}_q(t)} W$ is invariant under $\Gamma_\Psi(E)$, it follows that, for every $\gamma \in \Gamma_\Psi(E)$, the matrix $G(\Psi\gamma^{-1})$ is also the coefficient matrix of a defining set of equations for $E \otimes_{\mathbb{F}_q(t)} W$. Now the columns of the matrix $[-C(\Psi),\ I_{s-m}]^{\mathrm{tr}} \in \mathrm{Mat}_{m\times s}(\Lambda_\Psi)$ form a basis for W. Thus,

$$\begin{bmatrix} I_m & C(\Psi\gamma^{-1}) \end{bmatrix} \cdot \begin{bmatrix} -C(\Psi) \\ I_{s-m} \end{bmatrix} = 0, \quad \forall\, \gamma \in \Gamma_\Psi(E),$$

and so

$$C(\Psi\gamma) = C(\Psi), \quad \forall\, \gamma \in \Gamma_\Psi(E).$$

After varying over all $E/\mathbb{F}_q(t)$ finite, it follows from Theorem 5.3.2 that $C(\Psi) \in \mathrm{Mat}_{m\times(s-m)}(\overline{k}(t))$. □

Proposition 5.4.8. *Let M be a t-motive. Suppose that $\Phi \in \mathrm{GL}_r(\overline{k}(t))$ represents multiplication by σ on M and that $\Psi \in \mathrm{GL}_r(\mathbb{L})$ is a rigid analytic trivialization for Φ. For every t-motive N in $\mathcal{T}_M$ and every Γ_Ψ-subrepresentation V of N^{B}, there is a sub-t-motive $P \subseteq N$ so that $\xi_M(P) = V$.*

Proof. Pick a $\overline{k}(t)$-basis $\mathbf{n} \in \mathrm{Mat}_{s\times 1}$ for N with $\sigma\mathbf{n} = \Phi_N\mathbf{n}$, and let $\Psi_N \in \mathrm{GL}_s(\mathbb{L})$ be a rigid analytic trivialization for Φ_N. Let $\mathbf{v} \in \mathrm{Mat}_{v\times 1}(N^{\mathrm{B}})$ be an $\mathbb{F}_q(t)$-basis for V, and extend $\mathbf{v}$ to a basis $\mathbf{u}$ of N^{B}, $\mathbf{u} = [\mathbf{v}, \mathbf{w}]^{\mathrm{tr}}$. By Lemma 5.4.2, there is a $H(\Psi) \in \mathrm{GL}_s(\Sigma_\Psi)$ so that

$$\mathbf{u} = H(\Psi) \cdot \mathbf{n}.$$

We note that $H(\Psi) = \delta^{-1}\Psi_N^{-1}$ for some $\delta \in \mathrm{GL}_s(\mathbb{F}_q(t))$ by Proposition 3.3.9(b).

Let $E/\mathbb{F}_q(t)$ be a finite extension of fields, and let $\gamma \in \Gamma_\Psi(E)$. The action of γ on $E \otimes_{\mathbb{F}_q(t)} N^{\mathrm{B}}$ is given by the restriction of $\Xi^{(E)}$ in (5.4.3.1) to $E \otimes_{\mathbb{F}_q(t)} N^{\mathrm{B}}$. Thus,

$$\xi^{(E)}(\gamma)(\mathbf{u}) = H(\Psi\gamma)\mathbf{n} = H(\Psi\gamma)H(\Psi)^{-1}\mathbf{u}.$$

Since V is invariant under Γ_Ψ, it follows that the upper right $v \times (s-v)$ block of $H(\Psi\gamma)H(\Psi)^{-1}$ is 0 for every $\gamma \in \Gamma_\Psi(E)$.

Let $D(\Psi) \in \mathrm{Mat}_{s\times(s-v)}(\Lambda_\Psi)$ be the $s-v$ right-most columns of $H(\Psi)^{-1}$, and consider the subspace $W \subseteq \mathrm{Mat}_{1\times s}(\Lambda_\Psi)$,

$$W = \{\mathbf{x} \in \mathrm{Mat}_{1\times s}(\Lambda_\Psi) \mid \mathbf{x}\cdot D(\Psi) = 0\}.$$

By our considerations on $H(\Psi)$ at the end of the preceding paragraph, we see from Lemma 5.4.7 that W has a set of defining equations over $\overline{k}(t)$. Thus there is a $C \in \mathrm{Mat}_{v\times s}(\overline{k}(t))$ of maximal rank so that

$$C \cdot D(\Psi) = 0.$$

Extend C to a matrix $B \in \mathrm{GL}_s(\overline{k}(t))$ such that C forms the top rows of B. Now let $\mathbf{n}' = B\cdot\mathbf{n} = [\mathbf{p},\mathbf{q}]^{\mathrm{tr}}$, with $\sigma\mathbf{n}' = \Phi'\mathbf{n}'$, and let P be the $\overline{k}(t)$-span of $\mathbf{p} = C\cdot\mathbf{n}$. Notice that

$$\begin{aligned}\sigma\begin{bmatrix}\mathbf{p}\\ \mathbf{q}\end{bmatrix} = \sigma(B\cdot\mathbf{n}) &= \sigma(BH(\Psi)^{-1}H(\Psi)\mathbf{n})\\ &= \sigma(B\cdot H(\Psi)^{-1}\cdot\mathbf{u})\\ &= \bigl(B\cdot H(\Psi)^{-1}\bigr)^{(-1)}\cdot\mathbf{u}\\ &= \bigl(B\cdot H(\Psi)^{-1}\bigr)^{(-1)}H(\Psi)\cdot B^{-1}\cdot\begin{bmatrix}\mathbf{p}\\ \mathbf{q}\end{bmatrix}.\end{aligned}$$

By construction, the upper right-hand $v\times(s-v)$ block of $B\cdot H(\Psi)^{-1}$ is 0. Thus,

$$\sigma\begin{bmatrix}\mathbf{p}\\ \mathbf{q}\end{bmatrix} = \begin{bmatrix}\Phi_P & 0\\ * & *\end{bmatrix}\cdot\begin{bmatrix}\mathbf{p}\\ \mathbf{q}\end{bmatrix} = \Phi'\cdot\begin{bmatrix}\mathbf{p}\\ \mathbf{q}\end{bmatrix}.$$

Since $\Phi' \in \mathrm{GL}_s(\overline{k}(t))$, it follows that $\Phi_P \in \mathrm{GL}_v(\overline{k}(t))$. Thus P is a sub-t-motive of N. Furthermore, as $H(\Psi)^{-1} = \Psi_N\delta$, $\delta \in \mathrm{GL}_s(\mathbb{F}_q(t))$, it follows that $B\cdot H(\Psi)^{-1}$ is a rigid analytic trivialization of Φ'. If we set

$$B\cdot H(\Psi)^{-1} =: \begin{bmatrix}\Psi_P & 0\\ * & *\end{bmatrix},$$

then Ψ_P is a rigid analytic trivialization for Φ_P. Moreover,

$$\begin{bmatrix}\mathbf{v}\\ \mathbf{w}\end{bmatrix} = H(\Psi)\cdot B^{-1}\cdot\begin{bmatrix}\mathbf{p}\\ \mathbf{q}\end{bmatrix} = \begin{bmatrix}\Psi_P^{-1} & 0\\ * & *\end{bmatrix}\cdot\begin{bmatrix}\mathbf{p}\\ \mathbf{q}\end{bmatrix},$$

and so $P^{\mathrm{B}} = V$ by Proposition 3.3.9(b). □

Proposition 5.4.9. *Let M be a t-motive. Suppose that $\Phi \in \mathrm{GL}_r(\overline{k}(t))$ represents multiplication by σ on M and that $\Psi \in \mathrm{GL}_r(\mathbb{L})$ is a rigid analytic trivialization for Φ. To every representation W in $\mathbf{Rep}(\Gamma_\Psi, \mathbb{F}_q(t))$ there is a t-motive N in $\mathcal{T}_M$ so that W is isomorphic to a subquotient of $\xi_M(N)$.*

Proof. The representation M^{B} is faithful by Corollary 5.4.4. Thus any object in $\mathbf{Rep}(\Gamma_\Psi, \mathbb{F}_q(t))$ is isomorphic to a subquotient of a direct sum of representations of the form $(M^{\mathrm{B}})^u_v := (M^{\mathrm{B}})^{\otimes u} \otimes ((M^{\mathrm{B}})^\vee)^{\otimes v}$. Since $\xi_M(M^u_v) = (M^u_v)^{\mathrm{B}} \cong (M^{\mathrm{B}})^u_v$, the proposition follows. □

Theorem 5.4.10. *Let M be a t-motive. Suppose that $\Phi \in \mathrm{GL}_r(\overline{k}(t))$ represents multiplication by σ on M and that $\Psi \in \mathrm{GL}_r(\mathbb{L})$ is a rigid analytic trivialization for Φ. Then the functor*

$$\xi_M : \mathcal{T}_M \to \mathbf{Rep}(\Gamma_\Psi, \mathbb{F}_q(t))$$

is an equivalence of Tannakian categories. Equivalently, the homomorphism $\pi_M : \Gamma_\Psi \to \Gamma_M$ is an isomorphism over $\mathbb{F}_q(t)$.

Proof. By Propositions 5.4.6 and 5.4.8, the map π_M is faithfully flat [11, Prop. II.2.21(a)]. By Proposition 5.4.9, π_M is a closed immersion [11, Prop. II.2.21(b)]. Thus π_M is an isomorphism of affine group schemes over $\mathbb{F}_q(t)$. □

6. Galois groups and transcendence

In this section we first recall the linear independence criterion introduced in [2] by Anderson, Brownawell, and the author, and one of its applications to t-motives. We then link this together with our study of the Galois groups of certain t-motives, whose matrices representing multiplication by σ have entries in $\overline{k}[t]$ and whose fundamental matrices have entries in $\mathbb{E}$. These t-motives include as a subset rigid analytically trivial Anderson t-motives. In what follows our primary goal will be to consider the fundamental matrix Ψ associated to such a t-motive M and to equate the transcendence degree over $\overline{k}$ of $\Psi(\theta)$ and the dimension of the Galois group of M.

6.1. **Linear independence criterion.**

Theorem 6.1.1 ([2, Thm. 3.1.1])**.** *Let $\Phi \in \mathrm{Mat}_r(\overline{k}[t])$ be given such that $\det \Phi = c(t-\theta)^s$, $c \in \overline{k}^\times$, and suppose that $\psi \in \mathrm{Mat}_{r\times 1}(\mathbb{E})$ satisfies*

$$\psi^{(-1)} = \Phi\psi.$$

For every $\rho \in \mathrm{Mat}_{1\times r}(\overline{k})$ such that

$$\rho\psi(\theta) = 0,$$

there is a $P \in \mathrm{Mat}_{1\times r}(\overline{k}[t])$ so that

$$P(\theta) = \rho, \quad P\psi = 0.$$

6.1.2. *Connection with solutions of σ-semilinear equations.* At first glance at the above theorem, the solutions ψ of the σ-semilinear equation associated to Φ are quite special in that their entries are assumed to be in $\mathbb{E}$. However, the following proposition demonstrates that this situation is not unusual.

Proposition 6.1.3 ([2, Prop. 3.1.3])**.** *Suppose we are given $\Phi \in \mathrm{Mat}_r(\overline{k}[t])$ and $\psi \in \mathrm{Mat}_{r\times 1}(\mathbb{T})$ so that*

$$\det \Phi(0) \neq 0, \quad \psi^{(-1)} = \Phi\psi.$$

Then we necessarily have $\psi \in \mathrm{Mat}_{r\times 1}(\mathbb{E})$.

6.1.4. *Connection with left $\overline{k}[t;\sigma]$-modules.* The following is a variation on [2, Prop. 4.4.3] with slightly milder hypotheses. We do not assume that the representing matrix Φ is one directly associated to an Anderson t-motive. However, we do obtain the same equality of dimensions (with the same proof).

Proposition 6.1.5 ([2, Prop. 4.4.3])**.** *Let $\Phi \in \mathrm{Mat}_r(\overline{k}[t])$ and $\psi \in \mathrm{Mat}_{r\times 1}(\mathbb{E})$ be given as in Theorem 6.1.1. Let*

$$N := \overline{k}[t]\text{-span in } \mathbb{E} \text{ of the entries of } \psi,$$

$$V := \overline{k}\text{-span in } \overline{k_\infty} \text{ of the entries of } \psi(\theta).$$

Then $\operatorname{rk}_{\overline{k}[t]} N = \dim_{\overline{k}} V$.

Proof. Let $N_1 := \{P \in \operatorname{Mat}_{1\times r}(\overline{k}[t]) \mid P\psi = 0\}$. We then obtain an exact sequence of $\overline{k}[t]$-modules,

$$0 \to N_1 \to \operatorname{Mat}_{1\times r}(\overline{k}[t]) \to N \to 0,$$

where the second map is given by $P \mapsto P\psi$. It is easy to check that this is an exact sequence of left $\overline{k}[t;\sigma]$-modules. Every $\overline{k}[t]$-basis for N_1 can be extended to a basis of $\operatorname{Mat}_{1\times r}(\overline{k}[t])$, and so the number of $\overline{k}$-linearly independent relations of $\overline{k}$-linear dependence among the entries of $\psi(\theta)$ is at least as great as $\operatorname{rk}_{\overline{k}[t]} N_1$. Thus $\operatorname{rk}_{\overline{k}[t]} N \geq \dim_{\overline{k}} V$. Moreover, Theorem 6.1.1 implies that every $\overline{k}$-linear relation among the entries of $\psi(\theta)$ lifts to a $\overline{k}[t]$-linear relation among the entries of ψ. Thus $\operatorname{rk}_{\overline{k}[t]} N \leq \dim_{\overline{k}} V$. □

6.2. **Dimensions and transcendence degrees.**

6.2.1. *Rigid analytic trivializations over* $\mathbb{E}$. Let M be a t-motive. Suppose that $\Phi \in \operatorname{GL}_r(\overline{k}(t)) \cap \operatorname{Mat}_r(\overline{k}[t])$ represents multiplication by σ on M and that $\det \Phi = c(t-\theta)^s$, $c \in \overline{k}^{\times}$. An important observation is that by Propositions 3.3.9(c) and 6.1.3, there is a rigid analytic trivialization Ψ for Φ such that $\Psi \in \operatorname{GL}_r(\mathbb{T}) \cap \operatorname{Mat}_r(\mathbb{E})$.

Theorem 6.2.2. *Let M be a t-motive, and let Γ_M be its Galois group. Suppose that $\Phi \in \operatorname{GL}_r(\overline{k}(t)) \cap \operatorname{Mat}_r(\overline{k}[t])$ represents multiplication by σ on M and that* $\det \Phi = c(t-\theta)^s$, $c \in \overline{k}^{\times}$. *Let Ψ be a rigid analytic trivialization of Φ in* $\operatorname{GL}_r(\mathbb{T}) \cap \operatorname{Mat}_r(\mathbb{E})$. *Finally, let L be the subfield of $\overline{k_\infty}$ generated over $\overline{k}$ by the entries of $\Psi(\theta)$. Then*

$$\operatorname{tr.deg}_{\overline{k}} L = \dim \Gamma_M.$$

Proof. By Theorem 5.4.10, the groups Γ_M and Γ_Ψ are isomorphic. Moreover, by Theorem 5.2.12, their dimension is the same as $\operatorname{tr.deg}_{\overline{k}(t)} \Lambda_\Psi$, where Λ_Ψ is the subfield of $\mathbb{L}$ generated over $\overline{k}(t)$ by the entries of Ψ. Now let $Q = \overline{k}[\Psi(\theta)] \subseteq L$, and let $S = \overline{k}(t)[\Psi] \subseteq \Lambda_\Psi$. Then as rings,

$$Q \cong \overline{k}[X_{ij}]/\mathfrak{a}, \quad S \cong \overline{k}(t)[X_{ij}]/\mathfrak{b},$$

for ideals $\mathfrak{a}$ and $\mathfrak{b}$. For $d \geq 1$, let $\overline{k}[X_{ij}]_d$ and $\mathfrak{a}_d$ denote the elements of $\overline{k}[X_{ij}]$ and $\mathfrak{a}$ of total degree $\leq d$, and let $Q_d \subseteq Q$ correspond to their quotient. Similarly define $\overline{k}(t)[X_{ij}]_d$, $\mathfrak{b}_d$, and S_d.

Fix $d \geq 1$. Now for any $n \geq 1$, the entries of $\Psi^{\otimes n}$ comprise all monomials of total degree n in the Ψ_{ij}. If ψ is a column of $\Psi^{\otimes n}$, then $\psi^{(-1)} = \Phi^{\otimes n}\psi$. Thus let $\overline{\psi} \in \operatorname{Mat}_{N\times 1}(\mathbb{E})$ be the column vector whose entries are the concatenation of 1 and each of the columns of $\Psi^{\otimes n}$ for $n \leq d$. (Here $N = (r^{2d+2}-1)/(r^2-1)$.) Then if $\overline{\Phi} \in \operatorname{Mat}_N(\overline{k}[t]) \cap \operatorname{GL}_N(\overline{k}(t))$ is the block diagonal matrix

$$\overline{\Phi} := [1] \oplus \Phi^{\oplus r} \oplus \left(\Phi^{\otimes 2}\right)^{\oplus r^2} \oplus \cdots \left(\Phi^{\otimes d}\right)^{\oplus r^d},$$

it follows that

$$\overline{\psi}^{(-1)} = \overline{\Phi}\,\overline{\psi}.$$

Now it is easy to see that

$$Q_d = \overline{k}\text{-span of the columns of } \overline{\psi}(\theta),$$
$$S_d = \overline{k}(t)\text{-span of the columns of } \overline{\psi}.$$

Since $\overline{\Phi}$ and $\overline{\psi}$ satisfy the hypotheses for Proposition 6.1.5, we see that for all $d \geq 1$,

$$\dim_{\overline{k}} Q_d = \dim_{\overline{k}(t)} S_d.$$

Thus the homogenizations of Q and S have the same Hilbert series (see [33, Ch. VII, §12]), and so tr. $\deg_{\overline{k}} L =$ tr. $\deg_{\overline{k}(t)} \Lambda_\Psi$. $\square$

7. Application to Carlitz logarithms

7.1. Carlitz logarithms and t-motives.

7.1.1. *The power series L_α.* For $\alpha \in \overline{k}^\times$ with $|\alpha|_\infty < |\theta|_\infty^{q/(q-1)}$, define the power series

$$L_\alpha(t) := \alpha + \sum_{i=1}^{\infty} \frac{\alpha^{q^i}}{(t-\theta^q)(t-\theta^{q^2})\cdots(t-\theta^{q^i})}.$$

It is easy to see that

$$\left\| \frac{\alpha^{q^i}}{(t-\theta^q)(t-\theta^{q^2})\cdots(t-\theta^{q^i})} \right\| = \frac{|\alpha|_\infty^{q^i}}{|\theta|_\infty^{(q^{i+1}-q)/(q-1)}}, \quad i \geq 1,$$

and so $L_\alpha \in \mathbb{T}$. Moreover, $L_\alpha(z)$ converges for all $z \in \mathbb{K}$ with $|z|_\infty < |\theta|_\infty^q$. By §1.2.2, we see that

$$L_\alpha(\theta) = \log_C(\alpha).$$

Furthermore, as a power series in $\mathbb{T}$, L_α also satisfies the functional equation

$$L_\alpha^{(-1)} = \alpha^{(-1)} + \frac{L_\alpha}{t-\theta}. \tag{7.1.1.1}$$

7.1.2. *t-motives for Carlitz logarithms.* Fix $\alpha_1, \dots, \alpha_r \in \overline{k}^\times$ with $|\alpha_i|_\infty < |\theta|_\infty^{q/(q-1)}$ for $i = 1, \dots, r$. Set

$$\Phi := \Phi(\alpha_1, \dots, \alpha_r) := \begin{bmatrix} t-\theta & 0 & \cdots & 0 \\ \alpha_1^{(-1)}(t-\theta) & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_r^{(-1)}(t-\theta) & 0 & \cdots & 1 \end{bmatrix} \in \mathrm{Mat}_{r+1}(\overline{k}[t]).$$

Note that Φ defines a pre-t-motive $X := X(\alpha_1, \dots, \alpha_r)$ that is an extension of $\mathbf{1}^r$ by the Carlitz motive C:

$$0 \to C \to X \to \mathbf{1}^r \to 0.$$

In spite of the restrictions on $\alpha_1, \dots, \alpha_r$, we will be able to use the objects $X(\alpha_1, \dots, \alpha_r)$ to accommodate *all* Carlitz logarithms using Lemma 7.4.1.

Proposition 7.1.3. *Let $\alpha_1, \dots, \alpha_r \in \overline{k}^\times$ with $|\alpha_i|_\infty < |\theta|_\infty^{q/(q-1)}$ for $i = 1, \dots, r$. The pre-t-motive $X = X(\alpha_1, \dots, \alpha_r)$ is a t-motive.*

Proof. We prove first that X is rigid analytically trivial and then that X is an object in $\mathcal{T}$. Define

$$\Psi := \Psi(\alpha_1, \dots, \alpha_r) := \begin{bmatrix} \Omega & 0 & \cdots & 0 \\ \Omega L_{\alpha_1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Omega L_{\alpha_r} & 0 & \cdots & 1 \end{bmatrix} \in \mathrm{GL}_{r+1}(\mathbb{T})$$

It is a simple matter to check that Ψ is a rigid analytic trivialization for Φ using (7.1.1.1). We note by Proposition 6.1.3 that the entries of Ψ are in $\mathbb{E}$.

Consider the pre-t-motive $C \otimes X$. We claim that $C \otimes X$ is in the essential image of the functor $\mathsf{M} \mapsto M : \mathcal{AR}^I \to \mathcal{R}$ of Theorem 3.4.9. By the definition of the category $\mathcal{T}$ in §3.4.10, it will follow that X is a t-motive.

Let $\mathsf{M} := \overline{k}[t]^{r+1}$ with standard $\overline{k}[t]$-basis $m_0, \dots, m_r$. Letting $\mathsf{m} := [m_1, \dots, m_r]^{\mathrm{tr}}$, we give M the structure of a left $\overline{k}[t;\sigma]$-module by setting

$$\sigma \mathsf{m} := (t-\theta)\Phi \mathsf{m}.$$

Now M sits in an exact sequence of left $\overline{k}[t;\sigma]$-modules,

$$0 \to \mathsf{C}^{\otimes 2} \to \mathsf{M} \to \mathsf{C} \to 0,$$

where C is the Carlitz motive in the category of Anderson t-motives of §3.4.3. Since C and $\mathsf{C}^{\otimes}$ are finitely generated as left $\overline{k}[\sigma]$-modules, so is M, and it follows from [2, Prop. 4.3.2] that M is free and finitely generated as a left $\overline{k}[\sigma]$-module. For any $u_0, \dots, u_r \in \overline{k}[t]$,

$$\sigma[u_0, \dots, u_r]\mathsf{m} = \Bigl[\Bigl(u_0^{(-1)} + \textstyle\sum_{i=1}^r \alpha_i^{(-1)} u_i^{(-1)} \Bigr)(t-\theta)^2, \\ u_1^{(-1)}(t-\theta), \dots, u_r^{(-1)}(t-\theta) \Bigr] \mathsf{m},$$

from which it follows that

$$\sigma \mathsf{M} = \langle (t-\theta)^2 m_0, (t-\theta)m_1, \dots, (t-\theta)m_r \rangle_{\overline{k}[t]}.$$

Thus $(t-\theta)^n \mathsf{M} \subseteq \sigma \mathsf{M}$ for all $n \geq 2$, and M is an Anderson t-motive by §3.4.1. □

7.2. **The Galois group Γ_X.** We continue with the notations of the previous section, including choices of $\alpha_1, \dots, \alpha_r \in \overline{k}^{\times}$ with $|\alpha_i|_\infty < |\theta|_\infty^{q/(q-1)}$ for $i = 1, \dots, r$.

7.2.1. *The group G.* Let G be the algebraic subgroup of GL_{r+1} over $\mathbb{F}_q(t)$ such that for all $\mathbb{F}_q(t)$-algebras R,

$$G(R) = \left\{ \begin{bmatrix} * & 0 \\ * & I_r \end{bmatrix} \in \mathrm{GL}_{r+1}(R) \right\}.$$

7.2.2. *Preliminary calculations.* Let $\mathcal{T}_X$ be the sub-Tannakian category of $\mathcal{T}$ generated by $X = X(\alpha_1, \dots, \alpha_r)$, and let $\Gamma_X \subseteq \mathrm{GL}_{r+1}$ be its Galois group over $\mathbb{F}_q(t)$. We can identify Γ_X with Γ_Ψ, where $\Psi = \Psi(\alpha_1, \dots, \alpha_r)$, by Theorem 5.4.10. Consider $\mathfrak{p}_\Psi := \ker \nu_\Psi = \ker(X_{ij} \mapsto \Psi_{ij} : \overline{k}(t)[X, \Delta^{-1}] \to \mathbb{L})$. It is clear that

$$X_{ij} - \delta_{ij} \in \mathfrak{p}_\Psi, \quad \forall\, i, \forall\, j \geq 2,$$

where δ_{ij} the usual Kronecker delta. If $\gamma \in \Gamma_X(\mathbb{F}_q(t))$, then we must have

$$(X\gamma)_{ij} - \delta_{ij} \in \mathfrak{p}_\Psi, \quad \forall\, i, \forall\, j \geq 2.$$

It is straightforward to check that this implies that $\gamma \in G(\mathbb{F}_q(t))$. Moreover, by the definition in §4.4 of $\Gamma_X(R)$ for any $\mathbb{F}_q(t)$-algebra R, one checks that

$$\Gamma_X \subseteq G.$$

It will be convenient henceforth to label the non-trivial coordinates of $G \subseteq \mathrm{GL}_{r+1}$ as $X_0, \dots, X_r$.

Because the Carlitz motive C is contained in X, it is an object in $\mathcal{T}_X$, and hence there is a surjection

$$\pi : \Gamma_X \twoheadrightarrow \mathbb{G}_{\mathrm{m}}$$

over $\mathbb{F}_q(t)$ by Theorem 3.5.4. Since now under $\nu_\Psi : \overline{k}(t)[X_0, X_0^{-1}, X_1, \dots, X_r] \to \mathbb{L}$ we have $\nu_\Psi(X_0) = \Omega$, the action of any $\gamma \in \Gamma_X(\overline{\mathbb{F}_q(t)})$ on Ω agrees with the action of the X_0-coordinate of γ on Ω. That is, the surjection π coincides with the natural projection on the X_0-coordinate of G. Let V be the kernel of π so that we have an exact sequence of algebraic groups over $\mathbb{F}_q(t)$,

$$1 \to V \to \Gamma_X \to \mathbb{G}_{\mathrm{m}} \to 1.$$

The group V is a subgroup of the group of unipotent matrices of G, which itself is naturally isomorphic to $\mathbb{G}_{\mathrm{a}}^r$. Thus we can think of $V \subseteq \mathbb{G}_{\mathrm{a}}^r$ with coordinates $X_1, \dots, X_r$.

Proposition 7.2.3. *With notation as above, the group V is a linear subspace of $\mathbb{G}_{\mathrm{a}}^r$ over $\mathbb{F}_q(t)$.*

Proof. Since Γ_X is a smooth over $\mathbb{F}_q(t)$ by Theorem 5.2.12, the fact that $\pi : \Gamma_X \to \mathbb{G}_{\mathrm{m}}$ coincides with projection from G implies that π is a smooth morphism. Thus V is also smooth, and so it is determined by the Zariski closure of $V(\overline{\mathbb{F}_q(t)})$ in $\mathbb{G}_{\mathrm{a}}^r$. Because π is surjective, for any nonzero $\alpha \in \overline{\mathbb{F}_q(t)}$, we can choose $\gamma \in \Gamma_X(\overline{\mathbb{F}_q(t)})$ so that $\pi(\gamma) = \alpha$. Suppose that

$$\mu = \begin{bmatrix} 1 & 0 \\ v & I_r \end{bmatrix} \in V(\overline{\mathbb{F}_q(t)}).$$

Then direct calculation gives

$$\gamma^{-1}\mu\gamma = \begin{bmatrix} 1 & 0 \\ \alpha v & I_r \end{bmatrix} \in V(\overline{\mathbb{F}_q(t)}),$$

and thus $V(\overline{\mathbb{F}_q(t)})$ is a linear subspace of $\mathbb{G}_{\mathrm{a}}^r(\overline{\mathbb{F}_q(t)})$. Since V is smooth, its defining equations over $\overline{\mathbb{F}_q(t)}$ are linear forms in $X_1, \dots, X_r$. These forms can be defined over $\mathbb{F}_q(t)$ since V is simply a linear subspace. □

7.2.4. *Defining polynomials for Γ_X.* Because the map $\Gamma_X \to \mathbb{G}_{\mathrm{m}}$ is a smooth morphism over $\mathbb{F}_q(t)$, Hilbert's Theorem 90 provides an exact sequence

$$1 \to V(\mathbb{F}_q(t)) \to \Gamma_X(\mathbb{F}_q(t)) \to \mathbb{G}_{\mathrm{m}}(\mathbb{F}_q(t)) \to 1$$

by [30, §18.5]. Let $b_0 \in \mathbb{F}_q(t)^\times \setminus \mathbb{F}_q^\times$, and fix a matrix

$$\gamma = \begin{bmatrix} b_0 & 0 & \dots & 0 \\ b_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_r & 0 & \cdots & 1 \end{bmatrix} \in \Gamma_X(\mathbb{F}_q(t)) \tag{7.2.4.1}$$

One checks that the Zariski closure in Γ_X of the cyclic group generated by γ is the line in G connecting γ to the identity matrix. Translating this line by any element of V shows that Γ_X contains the linear space spanned by V and γ. Since Γ_X is irreducible and of dimension 1 greater than the dimension of V, we see that Γ_X is this linear subspace. Moreover, this implies the following proposition.

Proposition 7.2.5. *Suppose $F_1, \dots, F_s \in \mathbb{F}_q(t)[X_1, \dots, X_r]$ are linear forms defining V, and suppose that $\gamma \in \Gamma_X(\mathbb{F}_q(t))$ is defined as in* (7.2.4.1)*. Then the linear polynomials in $\mathbb{F}_q(t)[X_0, \dots, X_r]$,*

$$G_i := (b_0 - 1)F_i - F_i(b_1, \dots, b_r)(X_0 - 1), \quad i = 1, \dots, s,$$

are defining polynomials for Γ_X.

7.3. **Linear relations among Carlitz logarithms.**

7.3.1. *Defining polynomials for Z_Ψ.* Let $Z_\Psi := \operatorname{Spec} \Sigma_\Psi$, where $\Psi = \Psi(\alpha_1, \dots, \alpha_m)$. From Theorem 5.2.14 we see that Z_Ψ and Γ_X are isomorphic over $\overline{k(t)}$. Since Γ_X is a linear space, Z_Ψ is also a linear space and isomorphic to Γ_X over $\overline{k}(t)$. Thus we can pick

$$\zeta = \begin{bmatrix} f_0 & 0 & \dots & 0 \\ f_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f_r & 0 & \cdots & 1 \end{bmatrix} \in Z_\Psi(\overline{k}(t)),$$

and then

$$Z_\Psi(\overline{k}(t)) = \zeta \cdot \Gamma_X(\overline{k}(t)).$$

It is a simple matter to check that the linear polynomials in $\overline{k}(t)[X_0, \dots, X_r]$,

$$H_i := G_i - X_0 G(f_0, \dots, f_r)/f_0, \quad i = 1, \dots, s,$$

are defining polynomials for Z_Ψ. That is, its defining ideal $\mathfrak{p}_\Psi$ is $(H_1, \dots, H_s)$.

The following theorems show how the above constructions can be used to characterize all k-linear relations among $\widetilde{\pi}$, $\log_C(\alpha_1), \dots, \log_C(\alpha_r)$.

Theorem 7.3.2. *Let* $\alpha_1, \dots, \alpha_r \in \overline{k}^\times$ *with* $|\alpha_i|_\infty < |\theta|_\infty^{q/(q-1)}$ *for* $i = 1, \dots, r$. *Let* $X = X(\alpha_1, \dots, \alpha_r)$ *be the associated* t*-motive.*

(a) *Let*

$$F = c_1 X_1 + \cdots + c_r X_r, \quad c_1, \dots, c_r \in \mathbb{F}_q(t),$$

be a defining linear form for V *so that*

$$G = (b_0 - 1)F - F(b_1, \dots, b_r)(X_0 - 1), \quad b_0, \dots, b_r \in \mathbb{F}_q(t), b_0 \notin \mathbb{F}_q,$$

is a defining polynomial for Γ_X. *Then*

$$(b_0(\theta) - 1) \sum_{i=1}^r c_i(\theta) \log_C(\alpha_i) - \sum_{i=1}^r c_i(\theta) b_i(\theta) \widetilde{\pi} = 0.$$

(b) *Every* k*-linear relation among* $\widetilde{\pi}$, $\log_C(\alpha_1), \dots, \log_C(\alpha_r)$ *is a* k*-linear combination of the relations from part (a).*

(c) *Let* N *be the* k*-linear span of* $\widetilde{\pi}$, $\log_C(\alpha_1), \dots, \log_C(\alpha_r)$. *Then*

$$\dim \Gamma_X = \dim_k N.$$

Proof. Choose $f \in \overline{k}(t)$ as in §7.3.1 so that $H := G - fX_0$ is a defining polynomial for Z_Ψ. Then as in §4.2.5, we must have

$$H(\Omega, \Omega L_{\alpha_1}, \dots, \Omega L_{\alpha_r}) = G(\Omega, \Omega L_{\alpha_1}, \dots, \Omega L_{\alpha_r}) - f\Omega = 0. \tag{7.3.2.1}$$

We note that

$$\begin{aligned} \sigma G(\Omega, \Omega L_{\alpha_1}, \dots, \Omega L_{\alpha_r}) &= G\big((t-\theta)\Omega - \Omega, \alpha_1^{(-1)}(t-\theta)\Omega, \dots \alpha_r^{(-1)}(t-\theta)\Omega\big) \\ &\quad + G(\Omega, \Omega L_{\alpha_1}, \dots, \Omega L_{\alpha_r}\big) - F(b_1, \dots, b_r) \\ &= \Omega G\big(t - \theta - 1, \alpha_1^{(-1)}(t-\theta), \dots, \alpha_r^{(-1)}(t-\theta)\big) \\ &\quad + f\Omega - F(b_1, \dots, b_r)\Omega \\ &= f^{(-1)} \Omega^{(-1)}. \end{aligned}$$

The first two equalities result from explicit computations, whereas the third is a consequence of (7.3.2.1). Thus

$$(t-\theta) f^{(-1)} - f = G\big(t - \theta - 1, \alpha_1^{(-1)}(t-\theta), \dots, \alpha_r^{(-1)}(t-\theta)\big) - F(b_1, \dots, b_r).$$

The right-hand side is a polynomial in $\overline{k}[t]$, so it follows that f is regular at $t = \theta$. Indeed if not, then $f^{(-1)}$ must have a pole at $t = \theta^{(-1)}$, whence f must also have a pole at $t = \theta^{(-1)}$. Continuing in this way we see that if f has a pole at $t = \theta$, then it must have a pole at each $t = \theta^{(-i)}$, $i \geq 1$, which is not possible. By a similar argument we deduce that $f^{(-1)}$ is also regular at $t = \theta$. Thus we see that

$$f(\theta) = -G(-1, 0, \ldots, 0)|_{t=\theta} + \sum_{i=1}^{r} c_i(\theta) b_i(\theta) = -\sum_{i=1}^{r} c_i(\theta) b_i(\theta).$$

Equation (7.3.2.1) transforms into

$$(b_0 - 1) \sum_{i=1}^{r} c_i \Omega L_{\alpha_i} - \sum_{i=1}^{r} c_i b_i (\Omega - 1) - f\Omega = 0.$$

Dividing through by Ω and evaluating at $t = \theta$, we obtain part (a). Part (b) is a consequence of (a) and (c), since Γ_X is a linear space in G over $\mathbb{F}_q(t)$. For part (c), part (a) implies that $\dim_k N \leq \dim \Gamma_X$, since the defining polynomials for Γ_X generate a set of k-linear relations on $\widetilde{\pi}$, $\log_C(\alpha_1), \ldots, \log_C(\alpha_r)$ of dimension $r + 1 - \dim \Gamma_X$. However,

$$\dim_k N \geq \text{tr. deg}_{\overline{k}}\ \overline{k}(\widetilde{\pi}, \log_C(\alpha_1), \ldots, \log_C(\alpha_r)) = \dim \Gamma_X.$$

by Theorem 6.2.2. ☐

7.3.3. *Example.* Let $\zeta_\theta = \sqrt[q-1]{-\theta}$, let X be the t-motive $X(\zeta_\theta)$ of dimension 2 over $\overline{k}(t)$, and let $\Psi = \Psi(\zeta_\theta)$. Since ζ_θ satisfies

$$\mathfrak{C}_t(\zeta_\theta) = \theta\zeta_\theta + \zeta_\theta^q = 0,$$

we see that ζ_θ is a t-torsion point on the Carlitz module. Moreover, $\exp_C(\theta \log_C(\zeta_\theta)) = 0$, and one calculates that

$$\log_C(\zeta_\theta) = \frac{\widetilde{\pi}}{\theta}.$$

Thus Γ_X is 1-dimensional by Theorem 7.3.2(c). If we consider the function in $\mathbb{T}$

$$\Upsilon := tL_{\zeta_\theta} - \zeta_\theta(t - \theta),$$

then $\Upsilon^{(-1)} = \Upsilon/(t-\theta)$. Thus $\Upsilon = f/\Omega$ for some $f \in \mathbb{F}_q[t]$ by Lemma 3.3.5. Evaluation at $t = \theta$ shows that $f = -1$ identically. Therefore, Z_Ψ is defined by

$$Z_\Psi : \zeta_\theta(t - \theta)X_0 - tX_1 - 1 = 0.$$

It follows that the defining equation for Γ_X is

$$\Gamma_X : tX_1 - X_0 + 1 = 0.$$

In the notation of Theorem 7.3.2, we have

$$F := X_1, \quad b_0 := t + 1, \quad b_1 := 1$$
$$G := tX_1 - X_0 + 1, \quad H := G - fX_0, \quad f := \zeta_\theta(t - \theta) - 1.$$

7.4. **Algebraic independence of Carlitz logarithms.** Before proving the main result on Carlitz logarithms, we prove a reduction lemma.

Lemma 7.4.1. *Let $\lambda \in \mathbb{K}^\times$. If $\exp_C(\lambda) \in \overline{k}^\times$, then there is an $\alpha \in \overline{k}^\times$ with $|\alpha|_\infty < |\theta|_\infty^{q/(q-1)}$, an $f \in \mathbb{F}_q[\theta]$, and an $n \geq 1$, so that*

$$\lambda = \theta^n \log_C(\alpha) + f\widetilde{\pi}.$$

Proof. Let $\beta = \exp_C(\lambda)$, and assume that $|\beta|_\infty \geq |\theta|_\infty^{q/(q-1)}$. We solve the equation
$$\mathfrak{C}_t(x) = \theta x + x^q = \beta;$$
that is, we find the t-division points of β on the Carlitz module. The Newton polygon for this equation, along with our assumptions on β, imply that any solution $\alpha \in \overline{k}^\times$ of this equation must satisfy
$$|\alpha|_\infty = |\beta|_\infty^{1/q}.$$
Moreover, if for some $\eta \in \mathbb{K}$ we have $\exp_C(\eta) = \alpha$, then
$$\exp_C(\theta\eta) = \beta = \exp_C(\lambda).$$
If $|\beta|_\infty < |\theta|_\infty^{q^2/(q-1)}$, then α is sufficiently small and we can pick $\eta = \log_C(\alpha)$. The result then follows with $n = 1$. Otherwise, we continue to take t-division values, and for some $n \geq 1$, we have
$$\mathfrak{C}_{t^n}(\alpha) = \beta, \quad |\alpha|_\infty < |\theta|_\infty^{q/(q-1)},$$
for which $\exp_C(\theta^n \log_C(\alpha)) = \beta$. □

Theorem 7.4.2. *Let $\lambda_1, \ldots, \lambda_r \in \mathbb{K}$ satisfy $\exp_C(\lambda_i) \in \overline{k}$ for $i = 1, \ldots, r$. If $\lambda_1, \ldots, \lambda_r$ are linearly independent over k, then they are algebraically independent over $\overline{k}$.*

Proof. Assume that $\lambda_1, \ldots, \lambda_r$ are linearly independent over k. By Lemma 7.4.1, for each λ_i we can pick $\alpha_i \in \overline{k}^\times$ with $|\alpha_i|_\infty < |\theta|_\infty^{q/(q-1)}$ so that the k-linear span of $\lambda_1, \ldots, \lambda_r$ is contained in the k-linear span of $\widetilde{\pi}$, $\log_C(\alpha_1), \ldots, \log_C(\alpha_r)$. Let $X = X(\alpha_1, \ldots, \alpha_r)$ be the t-motive associated to these logarithms as in the previous sections, and let Γ_X be its Galois group. Let
$$L = \overline{k}(\widetilde{\pi}, \log_C(\alpha_1), \ldots, \log_C(\alpha_r)),$$
and let
$$N = k\text{-linear span of } \widetilde{\pi},\ \log_C(\alpha_1), \ldots, \log_C(\alpha_r).$$
Because $\lambda_1, \ldots, \lambda_r$ are linearly independent over k, we see that $r \leq \dim_k N \leq r + 1$. Theorems 6.2.2 and 7.3.2 imply that
$$\operatorname{tr.deg}_{\overline{k}} L = \dim \Gamma_X = \dim_k N.$$
If $\widetilde{\pi}$, $\log_C(\alpha_1), \ldots, \log_C(\alpha_r)$ are linearly independent over k, then they are algebraically independent over $\overline{k}$, whence the same follows for $\lambda_1, \ldots, \lambda_r$ since $L = \overline{k}(\widetilde{\pi}, \lambda_1, \ldots, \lambda_r)$. If there is a linear dependence among $\widetilde{\pi}$, $\log_C(\alpha_1), \ldots, \log_C(\alpha_r)$ over k, then N is equal to the k-span of $\lambda_1, \ldots, \lambda_r$ and
$$L = \overline{k}(\lambda_1, \ldots, \lambda_r).$$
Thus in that case $\lambda_1, \ldots, \lambda_r$ are algebraically independent over $\overline{k}$. □

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Department of Mathematics, Texas A&M University, College Station, TX 77843
E-mail address: `map@math.tamu.edu`