Part I
Vectors and Higher-Dimensional Geometry

1 Three-Dimensional Coordinate Systems

Definition 1.1 (Coordinate Axes, Coordinate Planes, Octants) The three fixed lines passing through the fixed point \(O\) (the origin) that are each perpendicular to the other two lines and which are used to represent points in 3-space are referred to as the \(x\), \(y\), and \(z\)-axes. The axes, in turn, determine the three coordinate planes, namely the \(xy\), \(xz\), and \(yz\)-planes. These planes partition 3-space into eight octants.

Definition 1.2 (Right-Hand Rule) The rule that is used to determine the direction of the \(z\)-axis is called the right-hand rule.

Definition 1.3 (Projection) The point that is found when one “drops a perpendicular” from the given point \(P(a, b, c)\) onto a given coordinate plane is referred to as the projection of \(P\) onto said coordinate plane.

Definition 1.4 (Distance Formula in 3-D) The distance \(d(P_1, P_2)\) between the points \(P_1(x_1, y_1, z_1)\) and \(P_2(x_2, y_2, z_2)\) is

\[
d(P_1, P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.
\]

The proof of the distance formula in three dimensions is given in Stewart’s text. This is a special case of the general distance formula for two points \(P_1(x_1, x_2, \ldots, x_n)\) and \(P_2(y_1, y_2, \ldots, y_n)\) in \(n\) dimensions, namely

\[
d(P_1, P_2) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \cdots + (y_n - x_n)^2}
\]  

(See http://mathworld.wolfram.com/EuclideanMetric.html.)

Definition 1.5 (Equation for a Sphere) An equation of a sphere with center \(C(h, k, l)\) and radius \(r\) is

\[
(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2.
\]

Homework: 14, 18, 22, 34, 36

Notes:
2 Vectors

Definition 2.1 (Vector, Vector Components) A **vector** is a quantity that has both magnitude and direction, and can be represented by either an arrow or a directed line segment. Different placements of the vectors are equivalent, for vector length and direction are what is important. A vector \( \vec{a} \) in \( n \) dimensions is written as \( \vec{a} = \langle a_1, a_2, \ldots, a_n \rangle \); \( a_i \) is the \( i \)th component of \( \vec{a} \). We let \( V_n \) represent the set of all \( n \)-tupled vectors with real-valued components.

Definition 2.2 (Representation, Position Vector) A **representation** of the vector \( \vec{a} = \langle a_1, a_2, \ldots, a_n \rangle \) is a directed line segment \( \overrightarrow{PQ} \) from any point \( P(x_1, x_2, \ldots, x_n) \) to the point \( Q(x_1 + a_1, x_2 + a_2, \ldots, x_n + a_n) \). If \( x_i = 0 \) for all \( i \), so that \( P = O \), then the resulting vector is called the **position vector** of the point \( Q(a_1, a_2, \ldots, a_n) \). Hence, given points \( A(a_1, a_2, \ldots, a_n) \) and \( B(b_1, b_2, \ldots, b_n) \), the vector \( \vec{v} \) with representation \( \overrightarrow{AB} \) is \( \vec{v} = \langle b_1 - a_1, b_2 - a_2, \ldots, b_n - a_n \rangle \).

Definition 2.3 (Magnitude, Zero Vector) The **magnitude** (or **norm**) of a vector \( \vec{v} \) is the length of any of its representations, and is denoted by \( |\vec{v}| \) or \( ||\vec{v}|| \). Use of the distance formula shows that, for \( \vec{v} = \langle v_1, v_2, \ldots, v_n \rangle \), we have

\[
||\vec{v}|| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}
\]  

(2)

Thus, the **zero vector**, namely the vector with \( v_1 = v_2 = \cdots = v_n = 0 \), is the only vector of length zero. It is also the only directionless vector.

Definition 2.4 (Vector Addition and Subtraction, Triangle and Parallelogram Laws) If \( \vec{a} = \langle a_1, a_2, \ldots, a_n \rangle \) and \( \vec{b} = \langle b_1, b_2, \ldots, b_n \rangle \), then the **sum vector** \( \vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n \rangle \) while the difference vector \( \vec{a} - \vec{b} = \langle a_1 - b_1, a_2 - b_2, \ldots, a_n - b_n \rangle \). The **Triangle** and **Parallelogram Laws** are particularly useful for illustrating vector addition and vector subtraction in two dimensions, as the corresponding figures in Stewart’s text indicate.

Notes:
Definition 2.5 (Scalar Multiplication, Parallel Vectors) If $c$ is a scalar and $\mathbf{a} = <a_1, a_2, \ldots, a_n>$, then the vector $c\mathbf{a} = <ca_1, ca_2, \ldots, ca_n>$. If two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are such that $\mathbf{b} = c\mathbf{a}$ for some scalar $c$, then $\mathbf{b}$ and $\mathbf{a}$ are parallel vectors.

Remark 2.6 (Properties of Vectors) (1) Vector addition is both commutative and associative. (2) $\mathbf{a} + \mathbf{0} = \mathbf{a}$ for all $\mathbf{a} \in V_n$. (3) $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$ for all $\mathbf{a} \in V_n$. Properties (1) through (3) show that $(V_n, +)$ is an abelian group. (4) Multiplication of a vector by a scalar is (left-)distributive. Likewise, (left) multiplication of a vector by the sum of two scalars is distributive. (5) Multiplication of a vector by a product of scalars is associative. (6) $1\mathbf{a} = \mathbf{a}$ for all $\mathbf{a} \in V_n$.

Definition 2.7 (Standard Basis Vectors) The vectors $\mathbf{\varepsilon}_1 = <1, 0, \ldots, 0>$, $\mathbf{\varepsilon}_2 = <0, 1, \ldots, 0>$, $\ldots, \mathbf{\varepsilon}_n = <0, 0, \ldots, 1>$ are the standard basis vectors in $n$-space. They are referred to as basis vectors because any vector $\mathbf{v} = <v_1, v_2, \ldots, v_n>$ can be represented uniquely in terms of the $\mathbf{\varepsilon}_i$, namely as $\mathbf{v} = \sum_{i=1}^{n} v_i \mathbf{\varepsilon}_i$. When $n = 3$, we adopt the conventions $\mathbf{\varepsilon}_1 = \hat{i}$, $\mathbf{\varepsilon}_2 = \hat{j}$, and $\mathbf{\varepsilon}_3 = \hat{k}$.

Definition 2.8 (Unit Vector) A unit vector is a vector $\mathbf{\hat{a}}$ such that $||\mathbf{\hat{a}}|| = 1$. Every nonzero vector $\mathbf{a}$ has a corresponding parallel unit vector, namely $\mathbf{\hat{a}} = \mathbf{a} / ||\mathbf{a}||$.

Homework: 12, 16, 22, 26, 38

Notes:
3 The Dot (Scalar) Product

This section and the next address the following concern: Is there a way to “multiply” two vectors so that the resulting product is meaningful?

As it turns out, there are two ways to do so. One way comes through the scalar, or dot, product, a method of combining vectors that does not depend upon the dimension in which one operates. The other, the cross product, is restricted to the case in which one works in three dimensions.

Definition 3.1 If \( \vec{a} = \langle a_1, a_2, ..., a_n \rangle \) and \( \vec{b} = \langle b_1, b_2, ..., b_n \rangle \), then the dot (scalar, inner) product of \( \vec{a} \) and \( \vec{b} \) is the number \( \vec{a} \cdot \vec{b} \) given by

\[
\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + \cdots + a_nb_n.
\]

Remark 3.2 (Properties of the Dot Product)

1. \( \vec{a} \cdot \vec{a} = |\vec{a}|^2 \) for any vector \( \vec{a} \in V_n \).
2. The dot product is commutative. \( \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} \) for any set of vectors \( \vec{a}, \vec{b}, \) and \( \vec{c} \) in \( V_n \).
3. \( (d\vec{a}) \cdot \vec{b} = d(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (d\vec{b}) \) for any scalars \( d \) and all vectors \( \vec{a}, \vec{b} \in V_n \).
4. \( \vec{0} \cdot \vec{a} = 0 \) for any vector \( \vec{a} \in V_n \).

An alternate definition of the dot product is given below. One uses the Law of Cosines and the properties of the scalar product given above to prove this.

Theorem 3.3 (Physical Definition of the Dot Product) If \( \theta \) is the angle between vectors \( \vec{a}, \vec{b} \in V_n \), where \( \vec{a} \) and \( \vec{b} \) are position vectors and \( 0 \leq \theta \leq \pi \), then

\[
\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta.
\]

Hence, if \( \vec{a} \) and \( \vec{b} \) are nonzero vectors, then \( \cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} \).

The above theorem gives one a way to determine whether two vectors are perpendicular, or orthogonal, to each other. Namely, \( \vec{a} \) and \( \vec{b} \) are orthogonal if and only if \( \vec{a} \cdot \vec{b} = 0 \) (seen because \( \cos(\pi/2) = 0 \)).

Notes:
Definition 3.4 (Direction Angles, Direction Cosines) The direction angles of a nonzero vector \( \vec{a} \) are the angles \( \alpha, \beta, \) and \( \gamma \) in the interval \([0, \pi]\) that \( \vec{a} \) makes with the positive \( x, y, \) and \( z \)-axes. The cosines of these direction angles are the direction cosines of \( \vec{a} \).

Let \( \vec{a} = \langle a_1, a_2, a_3 \rangle \). Then it is easy to determine that \( \cos \alpha = (\vec{a} \cdot \hat{i}) / (|\vec{a}| |\hat{i}|) = a_1 / |\vec{a}| \), \( \cos \beta = a_2 / |\vec{a}| \), and \( \cos \gamma = a_3 / |\vec{a}| \). Thus, \( \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 \). Moreover, \( \vec{a} = |\vec{a}| \langle \cos \alpha, \cos \beta, \cos \gamma \rangle \), and hence the direction cosines of \( \vec{a} \) are the components of the unit vector in the direction of \( \vec{a} \).

Definition 3.5 (Vector Component, Scalar Component) Set \( \vec{a} = \overrightarrow{PQ} \) and \( \vec{b} = \overrightarrow{PR} \). If \( S \) is the foot of the perpendicular from \( R \) to the line containing \( \overrightarrow{PQ} \), then the vector with representation \( \overrightarrow{PS} \) is called the vector projection of \( \vec{b} \) onto \( \vec{a} \) and is represented by \( \text{proj}_{\vec{a}} \vec{b} \). The scalar projection of \( \vec{b} \) onto \( \vec{a} \), also called the component of \( \vec{b} \) along \( \vec{a} \), is defined to be magnitude of the vector projection. That is, it is the number \( |\vec{b}| \cos \theta \), where \( \theta \) is the angle between \( \alpha \) and \( \beta \). The scalar projection is denoted by \( \text{comp}_{\vec{a}} \vec{b} \).

Note that \( \text{comp}_{\vec{a}} \vec{b} \) is negative if \( \theta \in (\pi/2, \pi] \). Observe that \( \vec{a} \cdot \vec{b} = |\vec{a}| (|\vec{b}| \cos \theta) = |\vec{a}| |\text{comp}_{\vec{a}} \vec{b}| \), and hence \( \text{proj}_{\vec{a}} \vec{b} = \left( \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \right) \vec{a} \).

It is worth noting that we can use the concepts of scalar product and vector projection to define the work done by a constant force in terms of the former. Namely, the work done by a constant force \( \vec{F} \) is the inner product \( \vec{F} \cdot \vec{D} \), where \( \vec{D} \) is the displacement vector.

Homework: 18, 26, 30, 38, 46

Notes:
4 The Cross Product

Definition 4.1 (Cross Product) The cross product of vectors \( \vec{a} = \langle a_1, a_2, a_3 \rangle \) and \( \vec{b} = \langle b_1, b_2, b_3 \rangle \) is the vector \( \vec{a} \times \vec{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle \).

The cross product’s definition is easy to remember as the determinant of the 3 × 3 matrix \( C \), where the first row of \( C \) is \([\hat{i}, \hat{j}, \hat{k}]\), the second row is \([a_1, a_2, a_3]\), and the third row is \([b_1, b_2, b_3]\). Using the determinant representation of the cross product allows us to show that \( \vec{a} \times \vec{a} = \vec{0} \) for all \( \vec{a} \in \mathbb{V}_3 \), and further that \( \vec{a} \times \vec{b} \) is orthogonal to both \( \vec{a} \) and \( \vec{b} \). In fact, the direction of \( \vec{a} \times \vec{b} \) is given by the right-hand rule, as illustrated in the text.

The definitions of cross product and vector magnitude permit us to state the following.

Theorem 4.2 (Magnitude of the Cross Product) If \( \theta \) is the angle between \( \vec{a} \) and \( \vec{b} \) (and thus \( \theta \in [0, \pi] \)), then \( |\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta \). Hence, two nonzero vectors \( \vec{a} \) and \( \vec{b} \) are parallel iff \( \vec{a} \times \vec{b} = \vec{0} \).

The above theorem gives us a geometric interpretation of the cross product. Namely, the magnitude of the cross product \( \vec{a} \times \vec{b} \) is equal to the area of the parallelogram determined by \( \vec{a} \) and \( \vec{b} \).

Theorem 4.3 (Additional Properties of Cross Products) If \( \vec{a}, \vec{b}, \) and \( \vec{c} \) are vectors and \( d \) is a scalar, then

1. \( \vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \)
2. \( (d\vec{a}) \times \vec{b} = d(\vec{a} \times \vec{b}) = \vec{a} \times (d\vec{b}) \)
3. \( \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c} \)
4. \( (\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c} \)
5. (Scalar triple product) \( \vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c} \)
6. \( \vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} \)

Notes:
It is clear from the definition of the triple scalar product that we can represent said product using the determinant of a $3 \times 3$ matrix, with the components of $\vec{a}$ constituting the entries of the first row, the components of $\vec{b}$ the second row, and the components of $\vec{c}$ the third. Moreover, the volume of the parallelepiped determined by said vectors is the magnitude of their scalar triple product (see text), and thus we have a geometric interpretation for this product as well.

Cross products appear often in physical contexts. For example, the torque $\tau$ of an object, which measures the tendency of a body to rotate about a defined origin, is given by $\tau = \vec{r} \times \vec{F}$, that is, it is the cross product of the position and force vectors. The direction of $\tau$ indicates the axis of rotation. Moreover, the only component of $\vec{F}$ that can cause a rotation is the one perpendicular to $\vec{r}$, as indicated by the formula for cross product magnitude.

**Homework:** 4, 16, 20, 28, 30, 36

**Notes:**
5 Equations of Lines and Planes

5.1 Lines
A line, whether it lies in two or three dimensions, is determined when we know a point on the line and the direction of said line (call it $L$). To this end, let $\mathbf{r}_0$ be the position vector whose head lies on the known point of $L$, let $\mathbf{r}$ be a position vector whose head also lies on $L$, and let $\mathbf{v}$ be a vector parallel to $L$. Then it is clear, via the notion of vector addition, that for some real number $t$ we have $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, a vector equation of $L$, with $t$ as a parameter. Knowing the components of $\mathbf{r}$, $\mathbf{r}_0$, and $\mathbf{v}$ allows us, in turn, to obtain parametric equations for $L$.

The components of the direction vector $\mathbf{v}$ are, accordingly, referred to as direction numbers of $L$. Provided none of said components are zero, you can solve each of the equations for $t$, and equate the results (see text) to obtain symmetric equations for $L$. Note that if one of the components is zero, you can still eliminate $t$ (again, refer to the text). Refer to the book for examples that illustrate the above concepts as well as the notion of skew lines (lines that do not intersect and are not parallel, and therefore do not lie in the same plane).

5.2 Planes
Unlike lines, vectors parallel to a plane are insufficient to tell one the plane’s orientation. However, a vector perpendicular to the plane is sufficient to determine the plane’s direction, so long as a point on the plane is also specified. Let $P_0(x_0, y_0, z_0)$ be the specified point, let $P(x, y, z)$ be an arbitrary point on the plane, and let $\mathbf{r}_0$ and $\mathbf{r}$ represent the corresponding position vectors, respectively. Further, let $\mathbf{n} = \langle a, b, c \rangle$ represent a vector that is normal, or perpendicular, to the plane’s surface. Then it is clear that

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0,$$

or equivalently, $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$. Either of these equations is a vector equation of the plane. This equation gives rise to the scalar equation $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$, which in turn yields the linear equation $ax + by + cz + d = 0$, where $d = -(ax_0 + by_0 + cz_0)$. As the text notes, it can be shown that if $a$, $b$, and $c$ are not all zero, then the linear equation just given represents a plane with normal vector $\mathbf{n}$.

Notes:
The book gives several helpful examples, so far as finding equations of planes, et cetera, are concerned. For example, if one is given three distinct points \( P, Q, \) and \( R \) on a given plane, one can obtain the vector equation of the plane by forming the vectors \( \overrightarrow{PQ} \) and \( \overrightarrow{PR} \), then taking the cross product of said vectors to obtain the normal vector in order to put oneself in position to write the vector equation of the plane.

Two planes are parallel if their normal vectors are parallel. The angle between two vectors is defined as the acute angle between their normal vectors. Note that non-parallel planes intersect in a straight line. In order to find the equation of the line common to said planes, one first finds a point on the line (usually done by setting one of the variables equal to zero, then solving the resulting pair of equations in the two remaining unknowns). Then, one notes that, since the line lies on both planes, it is perpendicular to both normal vectors, hence one obtains a vector parallel to the line by taking the cross product of said normal vectors.

The book gives the derivation of the formula for the distance \( D \) between a given point \( P_1(x_1, y_1, z_1) \) and a given plane \( ax + by + cz + d = 0 \). Specifically, the formula is

\[
D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}.
\]

Take careful note of the examples which illustrate the use of said formula.

**Homework:** 10, 16, 24, 30, 44, 64

**Notes:**
6 Cylinders and Quadric Surfaces

Planes and spheres have now been considered. We move to a study of cylinders and quadric surfaces.

When sketching the graph of a three-dimensional object (that is, a surface), it is helpful to determine the traces, or cross-sections, of said surface.

6.1 Cylinders

Definition 6.1 (Cylinder) A cylinder is a surface consisting of all lines (called rulings) parallel to a given line and passing through a given plane curve.

Examples of such objects are parabolic and circular cylinders, each addressed in the text. As the text notes, whenever one of the three variables is missing from the equation of a surface, the resulting shape is cylindrical.

6.2 Quadric Surfaces

Definition 6.2 (Quadric Surface) A quadric surface is the graph of a second-degree equation in the variables $x$, $y$, and $z$. The corresponding (unaltered) general equation takes the form

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0,$$

where $A$, $B$, ..., $J$ are constants. Translations and rotations permit us to simplify the above equation to one of the two standard forms $ax^2 + by^2 + cz^2 + j = 0$ or $sx^2 + ty^2 + uz = 0$, where again, $a$, $b$, $c$, $j$, $s$, $t$, and $u$ are all real constants.

There are six types of quadric surfaces, which can each be drawn via the use of trace curves, and whose properties are outlined in the text: ellipsoids, cones, elliptic paraboloids, hyperboloids of one sheet, hyperbolic paraboloids, and hyperboloids of two sheets. You need to familiarize yourself with the shapes and properties of each quadric surface!

**Homework:** 6, 12, 16, 36, 44

**Notes:**
7 Cylindrical and Spherical Coordinates

In the cylindrical coordinate system (used when symmetry about an axis, chosen to be the $z$-axis, is involved), a point $P$ in three dimensions is represented by $(r, \theta, z)$ where $r$ and $\theta$ are the polar coordinates of $P$‘s projection onto the $xy$-plane and $z$ is the directed distance from the $xy$-plane to $P$. Thus, $x = r \cos \theta$, $y = r \sin \theta$, and $z = z$, and hence $r^2 = x^2 + y^2$, $\tan \theta = y/x$, and $z = z$. Note that the rectangular equation $x^2 + y^2 = c^2$, $c > 0$, has as its corresponding cylindrical equation $r = c$, which shows why we refer to this system as “cylindrical.”

In the spherical coordinate system (used when symmetry about a point, chosen to be the origin $O$, is involved), a point $P$ in three dimensions is represented by $(\rho, \theta, \phi)$, where $0 \leq \rho$ is the distance from the origin to $P$, $\theta$ retains its meaning from the cylindrical coordinate system, and $0 \leq \phi \leq \pi$ is the angle between the positive $z$-axis and the line segment $OP$. The relationship between spherical and rectangular coordinates can be found by first noting that $z = \rho \cos \phi$ and $r = \rho \sin \phi$, then using the facts that $x = r \cos \theta$ and $y = r \sin \theta$ to obtain $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$, and $z = \rho \cos \phi$. Hence, $\rho^2 = x^2 + y^2 + z^2$ (which shows why we refer to these coordinates as “spherical”).

**Homework:** 6, 12, 18, 24, 28, 44, 52

**Notes:**
Part II

Vector Functions

8 Vector Functions and Space Curves

Definition 8.1 (Vector(-Valued) Function, Component Functions) A vector-valued function $\vec{r}(t)$ is a function whose domain is a subset of the real numbers, and whose range is a set of vectors in $\mathbb{V}_n$. Such a function takes the form $\vec{r}(t) = \langle r_1(t), r_2(t), ..., r_n(t) \rangle$ where $r_1$, ..., $r_n$, the component functions of $\vec{r}$, are real-valued functions.

Definition 8.2 (Limit of a Vector Function) If $\vec{r}(t) = \langle r_1(t), r_2(t), ..., r_n(t) \rangle$, then 
\[
\lim_{t \to a} \vec{r}(t) = \langle \lim_{t \to a} r_1(t), \lim_{t \to a} r_2(t), ..., \lim_{t \to a} r_n(t) \rangle,
\]
provided the limits of the component functions exist.

As an example, consider Problem 4 on page 855. We have 
\[
\lim_{t \to 0} \langle (e^t - 1)/t, (\sqrt{1+t} - 1)/t, 3/(1+t) \rangle = \langle 1, 0, 5/3 \rangle.
\]

Definition 8.3 (Continuous Vector Function) A vector function $\vec{r}(t)$ is continuous at $t = a$ if 
\[
\lim_{t \to a} \vec{r}(t) = \vec{r}(a).
\]

NB (page 860): “[A]ny continuous vector function $\vec{r}$ defines a space curve $C$ that is traced out by the tip of the moving vector $\vec{r}(t)$.”

Carefully study the text’s examples. Note that computer packages (including MAPLE, available on a limited basis in Neckers A-258) can be used to draw space curves. Specifically, invoke the plots package once you open MAPLE (type in \texttt{with(plots):}), then use the spacecurve command. (Note: Use of the colon after a command suppresses output in MAPLE, while the semicolon does not. No MAPLE statement is complete without use of exactly one of the above punctuation symbols.)

As an example, the following command set, when invoked, draws the toroidal spiral on page 853, with $t \in [0, 4\pi]$.

\[
\text{with(plots):} \\
\text{spacecurve(}[(4+\sin(20*t))\cos(t),(4+\sin(20*t))\sin(t),\cos(20*t)],t=0..4\Pi,\text{axes=FRAME});
\]

Homework: 6, 14, 18, 20, 22, 24, 30, 34, 40

Notes:
9 Derivatives and Integrals of Vector Functions

Definition 9.1 (Derivative of a Vector Function) For a vector function \( \vec{r}(t) \) we have

\[
\frac{d\vec{r}}{dt} = \vec{r}'(t) = \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}.
\]

Definition 9.2 (Tangent Vector, Tangent Line) The vector \( \vec{r}'(t) \) is called the tangent vector to the curve defined by \( \vec{r} \) at the point \( P(x(t), y(t), z(t)) \), provided \( \vec{r}'(t) \) exists and does not equal the zero vector. The tangent line to the curve at \( P \) is the line through \( P \) parallel to \( \vec{r}'(t) \).

Theorem 9.3 (Calculation of the Derivative) If \( \vec{r}(t) = <f(t), g(t), h(t)> \) where \( f, g \) and \( h \) are differentiable functions, then \( \vec{r}'(t) = <f'(t), g'(t), h'(t)> \).

Higher derivatives are calculated in a similar fashion (see page 858 of the text).

For a given interval \( I \), let \( o(I) \) represent the largest open subinterval contained in \( I \). For example, if \( I = [2, 3) \) then \( o(I) = (2, 3) \).

Definition 9.4 (Smooth Curve) A curve given by a vector function \( \vec{r}(t) \), \( t \in I \) for interval \( I \), is smooth on \( I \) if \( \vec{r}' \) is continuous on \( I \) and \( \vec{r}'(t) \neq \vec{0} \) for all \( t \in o(I) \).

See Example 4 in the text for an illustration of a curve that is not smooth throughout its domain.

Refer to the text as well (page 859) for a list of differentiation rules for vector-valued functions. The important matter to note is that the sum, difference, and constant multiple rules carry over in a "natural" way, as do the product (for the scalar and cross products) and chain rules.

So far as integrals are concerned, we have the following. For \( \vec{r}(t) = <f(t), g(t), h(t)> \) where \( f, g \) and \( h \) are continuous on \( [a, b] \), we have

\[
\int_a^b \vec{r}(t) \, dt = \left( \int_a^b f(t) \, dt \right) \hat{i} + \left( \int_a^b g(t) \, dt \right) \hat{j} + \left( \int_a^b h(t) \, dt \right) \hat{k}.
\]

Homework: 6, 12, 16, 20, 26, 32, 36, 40

Notes:
10 Arc Length and Curvature

The arc length of a space curve is defined (parametrically) in the same manner that we defined said quantity in Chapter 10. Specifically, for the vector equation $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ where the derivatives of $f$, $g$ and $h$ are continuous on $[a, b]$, the arc length of said curve, if the curve is traversed exactly once as $t$ increases from $a$ to $b$, is

$$L = \int_a^b \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2} \, dt = \int_a^b |\mathbf{r}'(t)| \, dt.$$

The arc length computation is independent of the parametrization that is used for the curve in question (Stewart, page 863).

More generally, we have the arc length function

$$s(t) = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2} \, du = \int_a^t |\mathbf{r}'(u)| \, du.$$

By the Fundamental Theorem of Calculus, we have $ds/dt = |\mathbf{r}'(t)|$. As arc length, and thus curvature (see below) are determined independently of the coordinate system that one uses, it is helpful to parametrize a curve with respect to arc length (page 863).

The unit tangent vector $\mathbf{T}(t)$ to a curve at a point where $\mathbf{r}'(t) \neq \mathbf{0}$ is given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|},$$

and indicates the curve’s direction. As the rate of change of direction of the unit tangent vector is a function of the curve’s tendency to “twist” or “bend”, we may define the curve’s curvature as

$$\kappa = \left| \frac{dT}{ds} \right|,$$

or, in terms of the parameter $t$,

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|},$$

where the expression on the right comes from use of the Chain Rule and the observation that $ds/dt = |\mathbf{r}'(t)|$.

Notes:
The following formula for the curvature (see page 865 for the proof) is often convenient to implement:

\[ \kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3} \]

In the case of a plane curve with \( y = f(x) \) (so that \( x \) is the parameter), we have \( \vec{r}'(x) \times \vec{r}''(x) = <0, 0, f''(x)> \) and \( |\vec{r}'(x)| = \sqrt{1 + [f'(x)]^2} \), and thus (page 866)

\[ \kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}. \]

Observing that \( \vec{T}(t) \cdot \vec{T}'(t) = 0 \) (Example 5, Section 13.2), we define the unit normal vector \( \vec{N}(t) \) as

\[ \vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}. \]

From this we define the binormal vector \( \vec{B}(t) \) as \( \vec{B}(t) = \vec{T}(t) \times \vec{N}(t) \). Observe that not only is \( \vec{B} \) perpendicular to both \( \vec{T} \) and \( \vec{N} \), but it is also a unit vector.

From the text (page 867): “The plane determined by the normal and binormal vectors \( \vec{N} \) and \( \vec{B} \) at a point \( P \) on a curve \( C \) is called the normal plane of \( C \) at \( P \). It consists of all lines that are orthogonal to the tangent vector \( \vec{T} \). The plane determined by the vectors \( \vec{T} \) and \( \vec{N} \) is called the osculating plane of \( C \) at \( P \). The name comes from the Latin osculum, meaning ‘kiss.’ It is the plane that comes closest to containing the part of the curve near \( P \)... The circle that lies in the osculating plane of \( C \) at \( P \), has the same tangent as \( C \) at \( P \), lies on the concave side of \( C \) (toward which \( \vec{N} \) points), and has radius \( \rho = 1/\kappa \) (the reciprocal of the curvature) is called the osculating circle (or the circle of curvature) of \( C \) at \( P \). It is the circle that best describes how \( C \) behaves near \( P \); it shares the same tangent, normal, and curvature at \( P \).”

**Homework:** 4, 10, 16, 18, 24, 26, 40, 42, 52

**Notes:**
11 Motion in Space: Velocity and Acceleration

We study physical applications of space curves in this section.

First, for an object whose position function is given by \( \vec{r}(t) \), it is obvious that the velocity of said object is given by \( \vec{v}(t) = \vec{r}'(t) \), and thus the speed of said object is given by \( v(t) = |\vec{r}'(t)| \). Hence, the acceleration of the object is given by \( \vec{a}(t) = \vec{r}''(t) \). From this we have Newton’s Second Law of Motion, given by \( \vec{F}(t) = m\vec{a}(t) \), where \( \vec{F}(t) \) is the force applied to an object of mass \( m \) at time \( t \). Example 5 (page 873), having to do with projectile motion, is a good example that I recommend to your attention.

It is often useful to separate the acceleration vector into two components, one tangential and the other normal. Use of the definitions of unit tangent vector and curvature (see Stewart, page 874) permit us to write \( \vec{a}(t) = v'(t)\vec{T}(t) + \kappa(t)v^2(t)\vec{N}(t) \). Thus, the acceleration vector always lies in the plane defined by \( \vec{T} \) and \( \vec{N} \). Further calculations (page 875) show that 

\[
v(t) = \frac{\vec{r}''(t) \cdot \vec{r}'''(t)}{|\vec{r}'(t)|}
\]

while

\[
\kappa(t)v^2(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|}
\]

The section culminates with the derivation of Kepler’s First Law of Planetary Motion. See pages 876-878 of Stewart’s text for said derivation.

**Homework:** 6, 12, 16, 20, 24, 34

**Notes:**