On some compressible fluid models: Korteweg, lubrication and shallow water systems

Didier Bresch, Benoît Desjardins, Chi-Kun Lin

Résumé
Dans ce papier, on donne quelques résultats mathématiques concernant un modèle de fluides compressibles capillaires dérivé par J.E. Dunn et J. Serrin dans [1], qui peut être utilisé comme modèle de transition de phase. On considère un domaine périodique \( \Omega = T^d \) \((d = 2 \text{ ou } 3)\) ou de type bande \( \Omega = (0, 1) \times T^{d-1} \). Nous étudions l’influence de la dépendance de la viscosité \( \mu \) et du coefficient de capilarité \( \kappa \) par rapport à la densité \( \rho \). Suivant les choix considérés, on obtient des résultats plus ou moins généraux. On montre par exemple pour une viscosité \( \mu(\rho) = \nu \rho \) et une tension de surface \( \kappa(\rho) = \tilde{\kappa} = \text{cte} \) que le système de Korteweg possède des solutions faibles globales en temps. Ce système d’équations contient un modèle de shallow water mais également un modèle de lubrification. On discute de la validité du résultat dans le cas des équations de shallow water où la densité n’est pas aussi régulière que dans le cas Korteweg.

Abstract
In this paper, we give some mathematical results for an isothermal model of capillary compressible fluids derived by J.E. Dunn and J. Serrin in [1], which can be used as a phase transition model. We consider a periodic domain \( \Omega = T^d \) \((d = 2 \text{ or } 3)\) or a strip domain \( \Omega = (0, 1) \times T^{d-1} \). We look at the dependence of the viscosity \( \mu \) and the capillarity coefficient \( \kappa \) with respect to the density \( \rho \). Depending on the cases we consider, different results are obtained. We prove for instance for a viscosity \( \mu(\rho) = \nu \rho \) and a surface tension \( \kappa(\rho) = \tilde{\kappa} = \text{cte} \) the global existence of weak solutions of the Korteweg system without smallness assumption on the data. This model includes a shallow water model and a lubrication model. We discuss the validity of the result for the shallow water equations since the density is less regular than in the Korteweg case.
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Abstract

In this paper, we give some mathematical results for an isothermal model of capillary compressible fluids derived by J.E. Dunn and J. Serrin in [1], which can be used as a phase transition model. We consider a periodic domain $\Omega = T^d$ ($d = 2$ or $3$) or a strip domain $\Omega = (0,1) \times T^{d-1}$. We look at the dependence of the viscosity $\mu$ and the capillarity coefficient $\kappa$ with respect to the density $\rho$. Depending on the cases we consider, different results are obtained. We prove for instance for a viscosity $\mu(\rho) = \nu \rho$ and a surface tension $\kappa(\rho) = \kappa = $ cte the global existence of weak solutions of the Korteweg system without smallness assumption on the data. This model includes a shallow water model and a lubrication model. We discuss the validity of the result for the shallow water equations since the density is less regular than in the Korteweg case.

Keywords : Phase transition model, Korteweg model, compressible fluids, shallow water equations, lubrication models, existence of global weak solutions.

AMS subject classification : 35D05, 35K65, 35Q30, 35Q35, 76N10.
1 Introduction

The formulation of the theory of capillarity with diffuse interfaces was first introduced by Korteweg a century ago [13], and derived rigorously by Dunn and Serrin [7].

Diffuse interface models have gained renewed interest since the last few years in fluid mechanics applications. From a physical viewpoint, it allows to describe some phase transition phenomena, [1], [21] and contact angle problems [21], [20]. Numerically speaking, diffuse interface models also became popular since sharp interfaces capturing schemes are not always robust enough for the simulation of complex flows, see [12].

The purpose of this work is to study the mathematical behavior of such compressible viscous capillary fluids in a periodic domain $\Omega = T^d$ ($d = 2$ or 3) or in a strip domain $\Omega = (0,1) \times T^{d-1}$. The fluid is characterized by its density $\rho$, internal energy $e$, temperature $\theta$, pressure law $P = P(\rho,e)$, viscosity coefficients $(\lambda, \mu)$ possibly depending on $\rho$, capillary coefficient $\kappa$, thermal diffusivity $\alpha$, and velocity field $u = (u_1, \cdots, u_d)$. The Korteweg model we consider here reads as

$$\begin{cases}
\partial_t \rho + \text{div}(\rho u) = 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) = \text{div}(S + K) + \rho f, \\
\partial_t (\rho(e + \frac{u^2}{2})) + \text{div}(\rho u(e + \frac{u^2}{2})) = \text{div}(\alpha \nabla \theta) + \text{div}((S + K) \cdot u) + \rho f \cdot u,
\end{cases}$$

(1)

where $f$ is a given bulk force. The viscous stress tensor $S$ and the Korteweg stress tensor $K$ are given by

$$\begin{cases}
S_{ij} = (\lambda \text{div} u - P(\rho, e)) \delta_{ij} + 2\mu D_{ij}(u), \\
K_{ij} = \frac{\kappa}{2} (\Delta \rho^2 - |\nabla \rho|^2) \delta_{ij} - \kappa \partial_i \rho \partial_j \rho,
\end{cases}$$

(2)

where $D_{ij}(u) = (\partial_i u_j + \partial_j u_i)/2$ is the strain tensor. In the following, we neglect thermal fluctuations so that the pressure reduces to a function of $\rho$ only. The corresponding model which was also considered in [10], [11], [18] then reads

$$\begin{cases}
\partial_t \rho + \text{div}(\rho u) = 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) - \text{div}(2\mu(\rho) D(u)) - \nabla (\lambda(\rho) \text{div} u) + \nabla P(\rho) = \text{div} K + \rho f,
\end{cases}$$

(3)
supplemented with initial conditions
\[ \rho|_{t=0} = \rho_0, \quad \rho u|_{t=0} = m_0. \] (4)

We consider at first periodic boundary conditions and next a bounded domain \( \Omega \) with boundary conditions that will be defined in Subsection 3.3.

In Section 2, we recall some classical a priori estimates on the Korteweg system and identify the main mathematical difficulties related to the Cauchy problem (3), (4) in the periodic case. Then in Section 3, we investigate the well posedness of the problem assuming a linear dependence of the viscosity with respect to the density. We obtain a global existence result of weak solutions in dimension 2 or 3 using the regularity of the density in a periodic domain \( \Omega = T^d \). The case of a smooth bounded domain will be discussed in Subsection 3.2 depending on the choice of the diffusive term and on the boundary conditions, with a positive result in the case \( \Omega = (0, 1) \times T^{d-1} \). We also indicate in Subsection 3.3 some partial results on the associated shallow water equation that means the system corresponding to \( \kappa = 0 \). In the last section, we define and investigate compressible fluid models of Korteweg type with capillarity coefficients depending on the density and give some existent results.

2 The classical a-priori estimates and some well known results.

The aim of the section is to recall the well known physical energy estimates and identify the main mathematical difficulties arising when trying to prove the existence of solutions in energy spaces in a periodic domain \( \Omega = T^d \).

Multiplying the equation of momentum conservation by \( u \), integrating by parts over \( \Omega \), and using the mass equation together with the definition of the Korteweg tensor, we get the following equality
\[ \frac{d}{dt} \int_\Omega \left( \frac{\kappa |\nabla \rho|^2}{2} + \Pi(\rho) + \rho \frac{|u|^2}{2} \right) + \int_\Omega 2 \mu(\rho) D(u) : D(u) \]
\[ + \int_\Omega \lambda(\rho) |\text{div} u|^2 = \int_\Omega \rho f \cdot u \] (5)

where \( \Pi(\rho) \) denotes the internal energy, \( s \mapsto \Pi(s) \) being given
\[ \Pi(s) = s \int_s^\infty \frac{P(\tau)}{\tau^2} d\tau \]
for some constant reference density \( \bar{\rho} \). It satisfies
\[
s\Pi'(s) - \Pi(s) = P(s)
\]
and hence \( s\Pi''(s) = P'(s) \) which means that the internal energy is a convex function of the density in regions of nondecreasing \( P \). Assuming that the bulk force \( f \) belongs to \( L^1(0, T; (L^\infty(\Omega))^d) \), (5) gives
\[
\Pi(\rho) \in L^\infty(0, T; L^1(\Omega)),
\sqrt{\rho}u \in L^\infty(0, T; (L^2(\Omega))^d),
\nabla \rho \in L^\infty(0, T; (L^2(\Omega))^d),
\sqrt{\mu(\rho)}D(u) \in L^2(0, T; (L^2(\Omega))^{d \times d}),
\sqrt{\lambda(\rho)} \text{div} u \in L^2(0, T; L^2(\Omega)).
\]

Global existence of weak solutions "à la Leray" was proved by P–L. Lions [16], [17] in the non capillary case \( \kappa = 0 \), when \( P \) obeys a gamma type law and \( (\lambda, \mu) \) are constant coefficients such that \( \mu > 0 \) and \( \lambda + 2\mu > 0 \). The main difficulty is to control the oscillations of the density which in that case is only bounded in \( L^q(0, T, L^p(\Omega)) \) type spaces, in order to pass to the limit in the pressure term.

Since when \( \kappa \neq 0 \), \( \nabla \rho \) has global \( L^\infty(0, T; (L^2(\Omega))^d) \) bounds, the only problem when \( \mu \) is bounded from below by a positive constant, is to pass to the limit in the quadratic terms involving \( \nabla \rho \), that means the terms
\[
\text{div}(\nabla \rho \otimes \nabla \rho) \text{ and } \frac{1}{2}\frac{1}{\nabla \rho}.
\]
This is the main difficulty to obtain the global existence of weak solutions.

Let us mention here that the existence of strong solutions is known with \( \kappa \) and \( \mu \) constants since the work by H. Hattori and D. Li [10], [11] in the whole space \( R^d \). Notice that high order regularity in Sobolev spaces \( H^s(R^d) \) is required, namely the initial data \((\rho_0, m_0)\) are assumed to belong to \( H^s(R^d) \times (H^{s-1}(R^d))^d \) with \( s \geq d/2 + 4 \). Moreover, the use of general data requires some restrictions on the existence time of strong solutions, and global solutions are obtained only for initial data \((\rho_0, u_0)\) close enough to a stable equilibrium \((\bar{\rho}, 0)\) (where \( \bar{\rho} \) is a constant such that \( P'(\bar{\rho}) > 0 \)). In [5], the well posedness of the problem is studied in critical spaces defined on \( R^2 \) or \( R^3 \) with constant \( \kappa \) and \( \mu \). Such spaces, called Besov spaces, are invariant

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by the scaling of Korteweg’s system. Instead of considering the usual homogeneous Sobolev spaces $\tilde{H}^{d/2}(R^d)$, they use the space $B^{d/2}_{2,1}(R^d)$ which has the following nice property

$$B^{d/2}_{2,1}(R^d) \subset L^\infty(R^d)$$

with $d = 2$ or $3$. They consider at first global existence result for initial data close enough to stable equilibria. Then they give a local in time existence result for initial densities bounded away from zero which does not require any stability assumption on the pressure law.

The goal here is to look at the problem in the classical energy spaces and to analyze the influence of the dependence of $\kappa$ and $\mu$ with respect of $\rho$. We try to find some bounds on $\nabla \rho$ in some $L^p(0, T; (H^a_x(\Omega))^d)$ with $a > 0$, $p \geq 1$ with no restriction on the size of the data and regularity as close as possible to the physical energy bounds. As we shall see later on, viscosity expressed as a power law of $\rho$ provides such bounds for suitable $\kappa$ law. The counterpart is that the $L^2(0, T; (H^1(\Omega))^d)$ control of $u$ is lost when $\rho$ vanishes. Lower bounds on $\rho$ seem to be available only in the framework of strong enough solutions for small time or small data. The definition of weak solutions we shall give takes account of the degeneracy in $\rho$ since suitable test functions are required in the weak formulation.

Dependence of the viscosity with respect to the density on the compressible fluid system has been investigated in a two dimensional square in [24]. They have studied the case

$$\mu = cte, \quad \lambda(\rho) = \rho^\beta, \quad \text{with} \quad \beta \geq 3.$$ 

Here we look at the case where

$$\mu(\rho) = \nu \rho^\beta, \quad \lambda = 0, \quad \text{with} \quad \nu \geq 0, \quad \beta = 0, 1 \text{ or } 2$$

without assuming the flow to be potential as it was done for instance in [23]. Further comments about the consistency of potential flow assumption with density dependent viscosity $\mu$ will be given in Subsection 3.3.
3 An existence result for a general model:  
The case $\mu = \nu \rho$, $\lambda = 0$.

3.1 The Korteweg system

For the sake of simplicity, we shall assume $f = 0$ and drop the factor $2$ in front of $\mu(\rho)$ in the viscosity tensor. The Korteweg system in a periodic box $\Omega = T^d$ with $\mu = \nu \rho$ and $\lambda = 0$ then writes as follows

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla P - \nu \text{div}(\rho D(u)) &= \kappa \rho \nabla \Delta \rho. 
\end{align*}
\tag{6}
\]

This kind of system is similar to the lubrication system derived in [8] except the pressure term $\nabla P$ included in the Korteweg model. Moreover, it includes the shallow water equations mentioned in [17] page 251 where $\kappa = 0$. We will obtain here the global existence of weak solutions for this system without smallness hypothesis in dimension 2 or 3, using the properties of the Korteweg tensor. As a matter of fact, we will strongly use the extra bounds on the density obtained from capillarity effects. The extension to the shallow water equations where $\kappa = 0$ will be discussed in Subsection 3.3.

We assume that $P$ is $C^1([0, \infty))$ and satisfies

\[ P(s) \geq 0, \quad P'(s) \geq 0, \quad \text{and} \quad \Xi(s) \leq A^s \Pi(s) \quad \text{for large enough } s, \]

where $A$ is a positive constant, and $\eta < +\infty$ when $d = 2$ (resp. $\eta < 4$ when $d = 3$), and

\[ \Xi(s) = \int_s^\infty \tau P'(\tau) d\tau. \]

Notice that the assumption for large densities is satisfied in particular when the pressure behaves like a power law at infinity. The non-decreasing hypothesis is assumed at this point mainly for simplicity. In particular, it covers the case of the shallow water model for which $P(\rho) = \rho^2/2$. However, more general pressure laws can be considered, including unstable spinodal regions in which $P$ may be decreasing. This will be explained in a remark after the main proof.

Assuming that the total initial energy is finite

\[ \mathcal{E}_0 = \int_\Omega \left( \kappa \frac{|\nabla \rho_0|^2}{2} + \Pi(\rho_0) + \rho_0 \left| u_0 \right|^2 \right) < +\infty, \tag{7} \]

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and that
\[
\mathcal{F}_0 = 2 \nu^2 \int_\Omega |\nabla \sqrt{\rho_0}|^2 = \frac{1}{2} \int_\Omega \rho_0 |\nu \nabla \log \rho_0|^2 < +\infty,
\]
we are able to prove a global existence result of weak solutions for (6).

For some reasons that will clearly appear in the proof and subsequent
remarks, we need to control the behavior of solutions when \(\rho\) vanishes. Even
though weak solutions both for the mass and momentum equations may
make sense in \(\mathcal{D}'((0, T) \times \Omega)\), we have been unable to prove compactness
of solutions where the limit density vanishes. That’s why we need here to define
test functions for the momentum equations which are somehow supported on
sets of positive \(\rho\). This can be achieved thanks to the regularity of \(\rho\), which
will be proven to belong to \(L^2(0, T; H^2(\Omega))\). Basically, the idea is to consider
test functions of the form \(\rho \varphi\) in the momentum equation, where \(\varphi\) is smooth
in space and time. Indeed, in the complement of the set of vanishing \(\rho\),
the space regularity of the density allows to recover the usual momentum
equation.

Notice that weak solutions “à la Leray” with test functions depending
on the solutions itself were already introduced in [6] when dealing with the
motion of rigid or weakly elastic bodies evolving in viscous compressible or
incompressible fluids.

We shall say that \((\rho, u)\) is a “weak solution” on \((0, T)\) of (6) if and only
if (7), (8) and the following three assumptions are satisfied:

\[
\begin{align*}
\rho &\in L^2(0, T; H^2(\Omega)), \\
\nabla \rho &\quad \text{and} \quad \sqrt{\rho} \in L^\infty(0, T; (L^2(\Omega))^d), \\
\sqrt{\rho} u &\in L^\infty(0, T; (L^2(\Omega))^d), \\
\sqrt{\rho} D(u) &\in L^2(0, T; (L^2(\Omega))^{d \times d}),
\end{align*}
\]

and
\[
\partial_t \rho + \text{div}(\rho u) = 0 \quad \text{in} \quad \mathcal{D}'((0, T) \times \Omega), \quad \rho|_{t=0} = \rho_0 \quad \text{in} \quad \mathcal{D}(\Omega),
\]

and for all \(\varphi \in C^\infty([0, T] \times \Omega)^d\) such that \(\varphi(T, \cdot) = 0\), one has:

\[
\begin{align*}
\int_\Omega \rho_0 u_0 \cdot \rho_0 \varphi(0, \cdot) + \int_0^T \int_\Omega \left(\rho^2 u \cdot \partial_t \varphi + \rho u \otimes \rho u : D(\varphi)
\right. \\
- \rho^2 (u \cdot \varphi) \text{div} u - \nu \rho D(u) : \rho D(\varphi) - \nu \rho D(u) : \varphi \otimes \nabla \rho
\left. + \Xi(\rho) \text{div} \varphi - \tilde{\kappa} \rho^2 \Delta \rho \frac{\nabla \varphi}{\rho} - 2 \kappa (\rho \cdot \nabla \rho) \Delta \rho \right) = 0.
\end{align*}
\]

We will prove the following theorem
Theorem 1 Let $d = 2$ or $3$. Then there exists a global “weak solution” $(\rho, u)$ of (6) that means a solution satisfying (7), (8), (9), (10) and (11).

Before getting into the heart of the proof of Theorem 1, let us give several lemmas that will yield crucial estimates for the proof of existence of global weak solutions.

Lemma 2 We have the following identity

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |\nabla \log \rho|^2 + \int_{\Omega} \nabla \text{div} u \cdot \nabla \rho \ + \int_{\Omega} \rho D(u) : \nabla \log \rho \otimes \nabla \log \rho = 0.$$  \hspace{1cm} (12)

Proof. Differentiating the equation of mass with respect $x_i$, we get

$$\partial_t (\partial_i \log \rho) + \sum_j (u_j \partial_j \partial_i \log \rho + \partial_i \partial_j u_j + \partial_i u_j \partial_j \log \rho) = 0.$$  

Multiplying this equation by $\rho \partial_i \log \rho$ and summing over $i$, this gives

$$\frac{1}{2} \rho \partial_i |\nabla \log \rho|^2 + \rho (u \cdot \nabla) |\nabla \log \rho|^2 + \nabla \text{div} u \cdot \nabla \rho + \rho D(u) : \nabla \log \rho \otimes \nabla \log \rho = 0.$$  

Integrating over the space domain $\Omega$ and using the equation of mass, we get (12). □

Using the relation established in Lemma 2, we prove that

Lemma 3

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} \nu^2 \rho |\nabla \rho|^2 + 4\nu \int_{\Omega} P'(\rho)|\nabla \sqrt{\rho}|^2 + \nu \kappa \int_{\Omega} |\nabla \nabla \rho|^2 \ = \ -\frac{d}{dt} \int_{\Omega} \nu u \cdot \nabla \rho + \int_{\Omega} \nu \rho \nabla u : \nabla u.$$  \hspace{1cm} (13)

Proof. Multiplying the momentum equation by $\nu \nabla \rho / \rho$ and integrating over $\Omega$, we get

$$\int_{\Omega} \nu \rho (\partial_t u + u \cdot \nabla u) \cdot \frac{\nabla \rho}{ho} + \int_{\Omega} \nu^2 D(u) : \rho \nabla \left( \frac{\nabla \rho}{\rho} \right) \ + \int_{\Omega} \nu \kappa |\nabla \nabla \rho|^2 + 4\nu \int_{\Omega} P'(\rho)|\nabla \sqrt{\rho}|^2 \ = \ 0.$$

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That means
\[
\int_\Omega \nu \rho (\partial_t u + u \cdot \nabla u) \cdot \frac{\nabla \rho}{\rho} + \int_\Omega \nu^2 D(u) : \left( \nabla \nabla \rho - \frac{\nabla \rho \otimes \nabla \rho}{\rho} \right) \\
+ \int_\Omega \nu \tilde{\kappa} |\nabla \nabla \rho|^2 + 4\nu \int_\Omega P'(\rho)|\nabla \sqrt{\rho}|^2 = 0.
\]
Adding this equation to (12) multiplied by \( \nu^2 \), this gives
\[
4\nu \int_\Omega P'(\rho)|\nabla \sqrt{\rho}|^2 + \nu \tilde{\kappa} \int_\Omega |\nabla \nabla \rho|^2 + \frac{1}{2} \frac{d}{dt} \int_\Omega \nu^2 \rho |\nabla \log \rho|^2 \\
= -\int_\Omega \nu \partial_t u \cdot \nabla \rho - \int_\Omega \nu^2 \nabla \text{div} u \cdot \nabla \rho \\
- \int_\Omega \nu (u \cdot \nabla u) \cdot \nabla \rho - \int_\Omega \nu^2 D(u) : \nabla \nabla \rho = I.
\]
(14)
Let us now rewrite the right-hand side \( I \), as follows
\[
I = -\nu \frac{d}{dt} \int_\Omega u \cdot \nabla \rho + \nu \int_\Omega u \cdot \nabla \partial_t \rho \\
- \nu^2 \int_\Omega \nabla \text{div} u \cdot \nabla \rho - \nu \int_\Omega (u \cdot \nabla u) \cdot \nabla \rho - \nu^2 \int_\Omega D(u) : \nabla \nabla \rho.
\]
Therefore, using the mass equation
\[
I = -\nu \frac{d}{dt} \int_\Omega u \cdot \nabla \rho - \nu \int_\Omega u \cdot \nabla \text{div}(\rho u) - \nu^2 \int_\Omega \nabla \text{div} u \cdot \nabla \rho \\
- \nu \int_\Omega (u \cdot \nabla u) \cdot \nabla \rho - \nu^2 \int_\Omega D(u) : \nabla \nabla \rho.
\]
(15)
Integrating by parts, we get
\[
- \int_\Omega u \cdot \nabla \text{div}(\rho u) - \int_\Omega (u \cdot \nabla u) \cdot \nabla \rho = \int_\Omega \rho \nabla u \cdot \nabla u
\]
(16)
and using \( \nabla \text{div} = \text{curl curl} - \Delta \), we obtain
\[
- \int_\Omega \nabla \text{div} u \cdot \nabla \rho - \int_\Omega D(u) : \nabla \nabla \rho = 0.
\]
(17)
Therefore, using (15), (16) and (17), Equation (14) gives (13). □
Now we give an interesting estimate on $u + \nu \nabla \log \rho$. More precisely, we have

**Lemma 4** The following energy dissipation holds:

$$4\nu \int_\Omega P'(\rho)|\nabla \sqrt{\rho}|^2 + \nu \tilde{\kappa} \int_\Omega |\nabla \nabla \rho|^2 + \frac{d}{dt} \left( \int_\Omega \left( \frac{1}{2} \rho u + \nu \nabla \log \rho \right)^2 + \Pi(\rho) + \frac{\kappa}{2} |\nabla \rho|^2 \right) \leq \int_\Omega \nu \rho D(u) : D(u).$$

**Proof.** Estimate (13) reads

$$4\nu \int_\Omega P'(\rho)|\nabla \sqrt{\rho}|^2 + \nu \tilde{\kappa} \int_\Omega |\nabla \nabla \rho|^2 + \frac{1}{2} \frac{d}{dt} \int_\Omega \rho |u + \nu \nabla \log \rho|^2 = \frac{1}{2} \frac{d}{dt} \int_\Omega \rho |u|^2 + \int_\Omega \nu \rho \nabla u : \nabla u.$$

Since

$$\int_\Omega \nu \rho \nabla u : \nabla u \leq 2 \int_\Omega \nu \rho D(u) : D(u),$$

and using the energy estimate (5), we easily conclude. □

We remark that $\nu \nabla \log \rho$ has the dimension of a velocity. It means that we get some information on an auxiliary velocity $v = u + \nu \nabla \log \rho$.

**Proof of Theorem 1**

Notice that the key point to prove the existence of global weak solutions is the regularity on $\rho$, which allows to ignore the degeneracy of the model when $\rho$ vanishes. First, we assume that a sequence $(\rho_n, u_n)_{n \in \mathbb{N}}$ of approximate weak solutions has been constructed by a mollifying process, which have suitable regularity to justify the formal estimates of the previous lemmas. Such a sequence can be for instance constructed by using a Galerkin-like approximation on the velocity $u$, and smoothing out the initial momentum $\rho_0 u_0$ and density $\rho_0$ (in such way that $\rho^0_n$ and hence $\rho^n(t, .)$ remain bounded away from zero for all time, the advecting velocity being smooth). For examples of construction of approximate sequences for compressible Navier–Stokes equations, we refer to [17].
Thus, using the classical physical \textit{a priori} estimates given in Section 2, we obtain the following uniform bounds

\[
\begin{align*}
\| \nabla \rho_n \|_{L^\infty(0,T; (L^2(\Omega))^d)} &\leq c, \\
\| \sqrt{\rho_n} u_n \|_{L^\infty(0,T; (L^2(\Omega))^d)} &\leq c, \\
\| \sqrt{\rho_n} D(u_n) \|_{L^2(0,T; (L^2(\Omega))^{d \times d})} &\leq c,
\end{align*}
\]

(18)

On the other hand, (13) yields

\[
\| \rho_n \|_{L^2(0,T; H^2(\Omega))} \leq c, \quad \| \nabla \sqrt{\rho_n} \|_{L^\infty(0,T; (L^2(\Omega))^d)} \leq c.
\]

(19)

Now since we have

\[
\partial_t \rho_n = -\text{div}(\rho_n u_n) \text{ bounded in } L^2(0,T; H^{-1}(\Omega))
\]

and

\[
\rho_n \text{ bounded in } L^2(0,T; H^2(\Omega)) \cap L^\infty(0,T; H^1(\Omega)),
\]

then up to the extraction of a subsequence, there exists \( \rho \in L^2(0,T; H^2(\Omega)) \) such that \( \nabla \rho \) and \( \nabla \sqrt{\rho} \) belong to \( L^\infty(0,T; (L^2(\Omega))^d) \) and

\[
\rho_n \rightharpoonup \rho \text{ in } L^{2/s}(0,T; H^{1+s}(\Omega)) \cap C([0,T]; H^s(\Omega)) \text{ for all } s \in (0,1).
\]

In particular, \( \rho_n^2 \) and \( \rho_n \nabla \rho_n \) converge strongly in \( L^2(0,T; L^2(\Omega)) \) respectively to \( \rho^2 \) and \( \rho \nabla \rho \), which combined to the weak convergence of \( \Delta \rho_n \) to \( \Delta \rho \) in \( L^2(0,T; L^2(\Omega)) \) allows to pass to the limit in the last two terms of (11). As far as the pressure term is concerned, we first deal with large densities \( \rho > C \) for which the property \( \Xi(\rho) \leq A \rho^2 \Pi(\rho) \) holds. Thus, we have

\[
\int_0^T \int_\Omega |\Xi(\rho_n)| 1_{\rho_n > C} \leq A \int_0^T \int_\Omega \rho_n^2 \Pi(\rho_n) 1_{\rho_n > C},
\]

\[
\leq A \Pi(\rho_n) \| \| L^\infty(0,T; L^4(\Omega)) \| \| C^{\frac{\eta}{4}} \int_0^T \| \rho_n (t, \cdot) \|_{L^{\eta+\eta}(\Omega)}^{\| \eta+\eta \| / 2},
\]

where \( q = 2\eta \) when \( d = 2 \), and \( q = 4 \) when \( d = 3 \). From the uniform \( L^\infty(0,T; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega)) \) bounds on \( \rho_n \), we deduce that the last term of the above estimate is bounded. Taking \( C \) large enough, we are reduced to prove that \( \Xi(\rho_n) 1_{\rho_n < C} \) converges strongly in \( L^1(0,T; L^1(\Omega)) \) to \( \Xi(\rho) 1_{\rho < C} \). It is a straightforward consequence of the dominated convergence Theorem, since \( \rho_n \) converges to \( \rho \) almost everywhere in \( (0,T) \times \Omega \).
Next, using the uniform boundedness of $\sqrt{\rho_n} u_n$ in $L^2(0, T; (L^2(\Omega))^d)$, we deduce that it converges weakly to some $g \in L^2(0, T; (L^2(\Omega))^d)$. It allows to define a limit velocity $u$ as follows

$$u(t, x) = \begin{cases} 
\frac{g(t, x)}{\sqrt{\rho(t, x)}} & \text{if } \rho(t, x) > 0, \\
0 & \text{otherwise}.
\end{cases}$$

In particular, since $\rho_n u_n$ converges weakly to $\sqrt{\rho} g = \rho u$ in $L^2(0, T; (L^2(\Omega))^d)$, we have already proven that

$$\begin{aligned}
&\partial_t \rho + \text{div}(\rho u) = 0 \text{ in } D'((0, T) \times \Omega), \\
&\rho_{|t=0} = \rho_0 \text{ in } D'(\Omega).
\end{aligned}$$

In addition, we are able to pass to the limit in the first two terms of (11): for the first term, it follows from the strong convergence property of the initial data. For the second one, it suffices to use once again the strong convergence of $\rho_n$ in $C([0, T]; L^3(\Omega))$ to prove the weak convergence of $\rho_n^3 u_n$ to $\rho^2 u$.

The next step consists in proving the strong convergence of $\rho_n u_n$ to $\rho u$ in $L^2(0, T; (L^2(\Omega))^d)$. The main two arguments rely upon a uniform $L^2(0, T; (L^{3/2}(\Omega))^{d \times d})$ bound on $\nabla(\rho_n^{3/2} u_n)$, and space mollifying arguments combined with weak time continuity of the momentum. First, we remark that

$$D(\rho_n^{3/2} u_n) = \rho_n \sqrt{\rho_n} D(u_n) + \frac{3}{2} \sqrt{\rho_n} u_n \otimes \nabla \rho_n,$$

where $a \otimes b$ is defined as $(a \otimes b + b \otimes a)/2$. Using the uniform $L^\infty(0, T; L^6(\Omega))$ bound on $\rho_n$, the uniform $L^2(0, T; (L^2(\Omega))^{d \times d})$ bound on $\sqrt{\rho_n} D(u_n)$, the uniform $L^\infty(0, T; (L^2(\Omega))^d)$ bound on $\sqrt{\rho_n} u_n$, and the uniform $L^2(0, T; (L^6(\Omega))^d)$ bound on $\nabla \rho_n$, we deduce that

$$D(\rho_n^{3/2} u_n) \text{ is uniformly bounded in } L^2(0, T; (L^{3/2}(\Omega))^{d \times d}).$$

Observing that for any $f = (f_1, \cdots, f_d)$ satisfying $D(f) \in (L^{3/2}(\Omega))^{d \times d}$, one has

$$\partial_i f_j = \sum_k \partial_i \Delta^{-1} \partial_k (2D(f)_{kj}) - \partial_i \partial_j \Delta^{-1} \text{div} f,$$

so that

$$\|\nabla f\|_{(L^{3/2}(\Omega))^{d \times d}} \leq C\|D(f)\|_{(L^{3/2}(\Omega))^{d \times d}}.$$

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Consequently, \( \nabla (\rho_n^{3/2} u_n) \) is uniformly bounded in \( L^2(0, T; (L^{3/2}(\Omega))^{d \times d}) \), and from Sobolev embeddings \( \rho_n^{3/2} u_n \) is bounded in \( L^2(0, T; (L^{3}(\Omega))^{d}) \) (the exponent is limited by the case \( d = 3 \)).

In order to prove that \( \rho_n u_n \) converges strongly in \( L^2(0, T; (L^{2}(\Omega))^d) \) to \( \rho u \), we introduce a mollifying kernel \( \phi \in C^\infty(\Omega) \) such that \( \phi \geq 0 \), \( \int_{\Omega} \phi = 1 \) and denote \( \phi_m(\cdot) = m^d \phi(m \cdot) \) for all \( m \in N \). For any \( v \in \mathcal{D}(\Omega) \), \( v \ast \phi_m \) will be defined by
\[
v \ast \phi_m(x) = \int_{\Omega} v(y) \phi_m(x - y) dy.
\]
First, using the weak \( L^2(0, T; (L^{2}(\Omega))^d) \) convergence of \( \sqrt{\rho_n} u_n \) to \( \sqrt{\rho} u \) combined with
\[
\rho_n^2 |u_n|^2 = \rho_n^{3/2} u_n \ast \rho_n^{1/2} u_n,
\]
we deduce that it suffices to prove that \( \rho_n^{3/2} u_n \) converges strongly to \( \rho^{3/2} u \) in \( L^2(0, T; (L^{2}(\Omega))^d) \). Let us write for all \( m \in N^* \)
\[
\| \rho_n^{3/2} u_n - \rho^{3/2} u \|_{L^2(0,T;(L^2(\Omega))^d)} \leq \| \rho_n^{3/2} u_n - (\rho_n^{3/2} u_n) \ast \phi_m \|_{L^2(0,T;(L^2(\Omega))^d)}
+ \| (\rho_n^{3/2} u_n) \ast \phi_m - (\rho^{3/2} u) \ast \phi_m \|_{L^2(0,T;(L^2(\Omega))^d)}
+ \| \rho^{3/2} u - (\rho^{3/2} u) \ast \phi_m \|_{L^2(0,T;(L^2(\Omega))^d)}.
\]
On the one hand, one has
\[
\| \rho_n^{3/2} u_n - (\rho_n^{3/2} u_n) \ast \phi_m \|_{L^2(0,T;(L^2(\Omega))^d)} \leq C \| \rho_n^{3/2} u_n - (\rho_n^{3/2} u_n) \ast \phi_m \|_{L^2(0,T;(L^{3/2}(\Omega))^d)} \| \nabla (\rho_n^{3/2} u_n) \|_{L^2(0,T;(L^{3/2}(\Omega))^d)}^{1/2}
\leq C \frac{1}{\sqrt{m}} \| \nabla (\rho_n^{3/2} u_n) \|_{L^2(0,T;(L^{3/2}(\Omega))^d)}^{1/2},
\]
and similarly for the limit function \( \rho^{3/2} u \). On the other hand, we write for all \( \ell \in N^* \)
\[
\| (\rho_n^{3/2} u_n) \ast \phi_m - (\rho^{3/2} u) \ast \phi_m \|_{L^2(0,T;(L^1(\Omega))^d)} \| \nabla (\rho_n^{3/2} u_n) \|_{L^2(0,T;(L^1(\Omega))^d)}
\leq C_m \left( \frac{1}{\ell} + \| (\rho_n^{3/2} u_n) \ast \phi_m - (\rho^{3/2} u) \ast \phi_m \|_{L^2(0,T;(L^1(\Omega))^d)} \right).
\]
Let us introduce \( \beta \in C^\infty(R) \) such that \( \beta(s) = 1 \) for \( s \geq 2 \), \( \beta(s) = 0 \) for \( s \leq 1 \), and \( 0 \leq \beta(\cdot) \leq 1 \). For any \( \alpha > 0 \), \( \beta_\alpha \) is defined by \( \beta_\alpha(\cdot) = \beta(\cdot/\alpha) \).
This function allows to deal separately with density close to zero and density bounded from below:
\[
\| (\rho_n^{3/2} u_n (1 - \beta_a (\rho_n))) \ast \phi_\ell \|_{L^2 (0, T; (L^1 (\Omega))^d)} \\
\leq \| \sqrt{\rho_n} u_n \|_{L^\infty (0, T; (L^1 (\Omega))^d)} \| \rho_n (1 - \beta_a (\rho_n)) \|_{L^2 (0, T; L^2 (\Omega))} \\
\leq C \alpha.
\]
For the other term corresponding to densities bounded away from zero, we write
\[
\| (\rho_n^{3/2} u_n \beta_a (\rho_n)) \ast \phi_\ell - \rho_n^{-1/2} \beta_a (\rho_n) (\rho_n^2 u_n) \ast \phi_\ell \|_{L^2 (0, T; (L^1 (\Omega))^d)} \\
\leq C \frac{1}{\ell} \| \rho_n^2 u_n \|_{L^2 (0, T; (L^2 (\Omega))^d)} \| \nabla (\rho_n^{-1/2} \beta_a (\rho_n)) \|_{L^\infty (0, T; (L^2 (\Omega))^d)} \\
\leq C \alpha \frac{1}{\ell} \| \sqrt{\rho_n} \|_{L^\infty (0, T; (L^2 (\Omega))^d)} \| \rho_n^{3/2} u_n \|_{L^2 (0, T; (L^3 (\Omega))^d)} \| \nabla \rho_n \|_{L^\infty (0, T; (L^2 (\Omega))^d)}
\]
As a result, we have
\[
\| \rho_n^{3/2} u_n - \rho^{3/2} u \|_{L^2 (0, T; (L^2 (\Omega))^d)} \leq C \left( \frac{1}{m^{1/2}} + \alpha \right) + C_{m, a} \frac{1}{\ell} \\
+ \| \rho_n^{-1/2} \beta_a (\rho_n) (\rho_n^2 u_n) \ast \phi_\ell - \rho^{-1/2} \beta_a (\rho) (\rho^2 u) \ast \phi_\ell \|_{L^2 (0, T; (L^1 (\Omega))^d)}.
\]
Given \( \alpha, m \) and \( \ell \), we already know that \( \rho_n^{-1/2} \beta_a (\rho_n) \) converges strongly in \( C ([0, T]; L^2 (\Omega)) \) to \( \rho^{-1/2} \beta_a (\rho) \), so that it only remains to prove that \( \rho_n^2 u_n \ast \phi_\ell \) converges strongly to \( \rho^2 u \ast \phi_\ell \) in \( L^2 (0, T; L^2 (\Omega)) \).

Using the fact that \( \rho_n^2 u_n \) is bounded in \( L^2 (0, T; (L^2 (\Omega))^d) \), it suffices to prove that \( \phi_\ell (\rho_n^2 u_n) \) is bounded in \( L^2 (0, T; (H^{-s} (\Omega))^d) \) for some positive \( s \). Let \( \varphi \in C^\infty_0 ((0, T) \times \Omega) \) such that \( \varphi (T, \cdot) = 0 \). Writing \( \rho_n u_n \otimes \rho_n u_n \) as \( \rho_n^{3/2} u_n \otimes \rho_n^{1/2} u_n \), we infer that \( \rho_n u_n \otimes \rho_n u_n \) is bounded in \( L^2 (0, T; (L^{6/5} (\Omega))^{d \times d}) \). From both the \( L^2 (0, T; (L^3 (\Omega)^d)) \) and \( L^\infty (0, T; (L^{3/2} (\Omega)^d)) \) bound on \( \rho_n^{3/2} u_n \), we deduce that \( \rho_n^{3/2} u_n \) is bounded in \( L^{5/2} (0, T; (L^{5/2} (\Omega)^d)) \), which yields a uniform \( L^{10/9} (0, T; (L^{10/9} (\Omega))^d) \) bound on \( \rho_n^{3/2} u_n \). The uniform \( L^\infty (0, T; L^4 (\Omega)) \) bound on \( \rho_n^{3/2} u_n \) yields a uniform bound on \( \rho_n^2 D(u_n) \) in \( L^2 (0, T; (L^{1/3} (\Omega))^{d \times d}) \), whereas the bound of \( \sqrt{\rho_n} \nabla \rho_n \) in \( L^4 (0, T; (L^2 (\Omega))^d) \) yields a uniform bound of \( \rho_n D(u_n)_{ij} \partial_j \rho_n \) in \( L^{4/3} (0, T; L^4 (\Omega)) \). Finally, as mentioned before for the compactness of the pressure induced contribution, \( \Xi (\rho_n) \) is uniformly bounded in \( L^p (0, T; L^1 (\Omega)) \) for some \( p > 1 \). The last two terms associated with capillarity effects, namely \( \rho_n^2 \Delta \rho_n \) and \( \rho_n \Delta \rho_n \nabla \rho_n \), are respectively bounded in \( L^2 (0, T; L^1 (\Omega)) \) and \( L^{4/3} (0, T; (L^4 (\Omega))^d) \). Then, it follows from (11) that
there exists $r < +\infty$ such that
\[
\left| \int_0^T \int_{\Omega} \partial_t (\rho_n^2 u_n) \cdot \varphi \right| \leq c \left( \| \varphi \|_{L^r(0,T;(L^\infty(\Omega)))} + \| \nabla \varphi \|_{L^r(0,T;(L^\infty(\Omega)))} \right).
\] (20)

It means that $\partial_t (\rho_n^2 u_n)$ is bounded in $L^q(0,T; (H^{-s}(\Omega))^d)$ for some $s > 1 + d/2$ and $q > 1$, and hence $(\rho_n^2 u_n) \ast \phi \times \phi$ converges strongly in $L^2(0,T; (L^2(\Omega))^d)$. We have proven at this point that $\rho_n^{3/2} u_n$ converges to $\rho^{3/2} u$ in $L^2(0,T; (L^2(\Omega))^d)$ strongly, which suffices to obtain the strong $L^2(0,T; (L^2(\Omega))^d)$ convergence of $\rho_n u_n$ to $\rho u$.

In order to conclude, it remains to pass to the limit in the nonlinear terms of (11), $\varphi \in C^\infty([0,T) \times \Omega)$ such that $\varphi(T, \cdot) = 0$ being given. The first three and the last three terms have already been treated previously. Writing again $\rho_n^2 |u_n|^2 = \rho_n^{3/2} u_n \cdot \rho_n^{1/2} u_n$, estimating the left hand side in $L^{6/5}(\Omega)$ and the two terms of the product respectively in $(L^3(\Omega))^d$ and $(L^2(\Omega))^d$, we deduce that
\[
\| \rho_n u_n \|_{L^4(0,T;(L^{12/5}(\Omega)))^d} \leq \| \rho_n^{3/2} u_n \|_{L^2(0,T;(L^3(\Omega)))^d}^{1/2} \| \rho_n^{1/2} u_n \|_{L^\infty(0,T;(L^2(\Omega)))^d}^{1/2}.
\] (21)

From estimate (21), it follows that for all $\alpha > 0$,
\[
\| (1 - \beta_\alpha(\rho_n)) \rho_n^2 u_n \|_{L^1(0,T;(L^1(\Omega)))^d} \leq \| \sqrt{\rho_n} \|_{L^2(0,T;(L^2(\Omega)))} \| \rho_n u_n \|_{L^2(0,T;(L^{12/5}(\Omega)))} \| (1 - \beta_\alpha(\rho_n)) \sqrt{\rho_n} \|_{L^\infty(0,T;(L^2(\Omega)))}
\leq C \sqrt{\alpha}.
\]

Similarly, the low density part of the other two terms is estimated as follows
\[
\| (1 - \beta_\alpha(\rho_n)) \rho_n D(u_n) \|_{L^1(0,T;(L^1(\Omega)))^d} \leq \| \sqrt{\rho_n} D(u_n) \|_{L^2(0,T;(L^2(\Omega)))^d} \| (1 - \beta_\alpha(\rho_n)) \rho_n^{3/2} \|_{L^2(0,T;(L^2(\Omega)))}
\leq C \alpha^{3/2},
\]
and
\[
\| (1 - \beta_\alpha(\rho_n)) \rho_n D(u_n) \cdot \partial_j \rho_n \|_{L^1(0,T;(L^1(\Omega)))^d} \leq \| \sqrt{\rho_n} D(u_n) \|_{L^2(0,T;(L^2(\Omega)))^d} \| (1 - \beta_\alpha(\rho_n)) \sqrt{\rho_n} \|_{L^\infty(0,T;L^\infty(\Omega))}
\| \nabla \rho_n \|_{L^2(0,T;(L^2(\Omega)))^d}
\leq C \sqrt{\alpha},
\]

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so that we are reduced to study the three sequences

$$
\beta_a(\rho_n)\rho_n^2 u_n \text{div} u_n, \quad \beta_a(\rho_n)\rho_n^2 D(u_n), \quad \text{and} \quad \beta_a(\rho_n)\rho_n D(u_n) \partial_j \rho_n,
$$

for a given positive $\alpha$. First, we observe that $\beta_a(\rho_n)\sqrt{\rho_n} D(u_n)$ converges weakly in $L^2(0, T; (L^2(\Omega))^{d \times d})$ to $\beta_a(\rho)\sqrt{\rho} D(u)$. Indeed, there exists $h_a \in L^2(0, T; (L^2(\Omega))^{d \times d})$ such that $\beta_a(\rho_n)\sqrt{\rho_n} D(u_n)$ converges to $h_a$ weakly in $L^2(0, T; (L^2(\Omega))^{d \times d})$. Let us reformulate the above sequence in terms of compact quantities

$$
\beta_a(\rho_n)\sqrt{\rho_n} D(u_n) = D(\beta_a(\rho_n)\sqrt{\rho_n} u_n) - \sqrt{\rho_n} u_n \nabla \rho_n (\beta_a'(\rho_n) + \frac{\beta_a(\rho_n)}{2_\rho_n}).
$$

Now using the strong convergence of $\beta_a(\rho_n)$ and $\beta_a'(\rho_n) + \beta_a(\rho_n)/2\rho_n$ in $C([0, T]; L^p(\Omega))$ for all $p < +\infty$, together with the strong convergence of $\nabla \rho_n$ in $L^2(0, T; (L^1(\Omega))^d)$ and the weak convergence of $\sqrt{\rho_n} u_n$ to $\sqrt{\rho} u$ in $L^2(0, T; (L^2(\Omega))^{d \times d})$, we deduce the following identity in $D'((0, T) \times \Omega)^{d \times d}$

$$
h_a = D(\beta_a(\rho)\sqrt{\rho} u_n) - \sqrt{\rho} u \nabla \rho (\beta_a'(\rho) + \frac{\beta_a(\rho)}{2\rho_n}) = \beta_a(\rho)\sqrt{\rho} D(u),
$$

which proves the claimed weak convergence.

Finally, the strong convergence of $\rho_n^{3/2} u_n$, $1_{\rho_n \geq \alpha} \rho_n^{3/2}$ and $1_{\rho_n \geq \alpha} \sqrt{\rho_n} \nabla \rho_n$ in $L^2(0, T; L^2(\Omega))$ to $\rho^{3/2} u$, $1_{\rho \geq \alpha} \rho^{3/2}$ and $1_{\rho \geq \alpha} \sqrt{\rho} \nabla \rho$ allows to pass to the limit in the remaining three terms. This ends the proof of the Theorem. \qed

Remark. The assumption that the pressure is a nondecreasing function of the density may be replaced by $P'(\rho) \geq -c_0$ for some positive $c_0$. Indeed the positivity of $P$ is the only assumption that ensures the control of the internal energy $\Pi(\rho)$. On the other hand, the estimates of Lemma 3 and 4 are slightly modified: an extra term appears on the right-hand side of the physical energy

$$
\nu \int_{x/P'(\rho(x) \leq 0)} \frac{-P'(\rho)}{\rho} |\nabla \rho|^2 \leq 4\nu c_0 \int |\nabla \sqrt{\rho}|^2
$$

which can be absorbed on any interval $[0, T]$ by Gronwall’s arguments. This assumption covers in particular the case of equations of states like Van der Waal’s and may be useful for tracking phase transition phenomena with the above model \cite{1}. \qed
Remark. In all this section, we have considered $f = 0$. It is possible to generalize the previous result to $f \in L^1_{\text{loc}}(R^+; (L^\infty(\Omega))^d)$ by the use of Gronwall’s arguments. Furthermore, Theorem 1 does not seem to extend to non zero $\lambda$ coefficient, because the estimates of Lemma 4 no longer hold. A different diffusive term like $-\text{div}(\rho \nabla u)$ has been proposed by P.-L. Lions [17] page 251 in the framework of shallow water modeling. Our analysis still holds in this case when capillarity effects are included. We will study in the next Subsection 3.3 the non-capillary case, namely the so-called Saint-Venant model for shallow water, showing some consistency of the model. □

Remark. We can find the following system expressed in a 1-d setting in [4]

$$
\begin{align*}
\partial_t h + \partial_x (hu) &= 0, \\
h\partial_t u + hu\partial_x u - 4\nu\partial_x (h\partial_x u) + \frac{\gamma}{\theta} h\partial_x C &= 0
\end{align*}
$$

(22)

where $u$ is the velocity and $C = -\partial_x^2 h (1 + (\partial_x h)^2)^{-3/2}$ is the mean curvature of the film. This kind of equation has been derived in [8] and is used in particular to modelize the retraction of thin films. The reader interested by related problems is refered to [3], [4], [8].

The Korteweg model for zero pressure law includes the above lubrication model when the curvature is linearized, that means $C = -\partial_x^2 h$. The existence result in Theorem 1, which does not make use of the bounds induced by the pressure, therefore extends to the curvature linearized lubrication model. □

### 3.2 The boundary value problem

This subsection is devoted to the boundary value problem in a smooth bounded domain $\Omega$. In order to obtain the physical energy estimate, we have to require at first

$$
u \cdot n = 0,
$$

(23)

$$
\partial_n \rho = 0 \quad \text{or} \quad \rho = \text{cte on } \partial \Omega,
$$

(24)

for the convection by $u$ and for the capillary contribution to the energy, where $n$ denotes the outward normal to the boundary $\partial \Omega$. Indeed

$$(K \cdot n)_{\text{tang}} = -\kappa (\nabla \rho)_{\text{tang}} (n \cdot \nabla \rho) = 0$$
if (24) is satisfied, where we define \( f_{\text{tang}} = f - (f \cdot n)n \) for any vector \( f \). Then, multiplying the momentum equation by \( u \) and the mass equation by \(-\tilde{\kappa}\Delta \rho\), we get the following extra boundary term in (5):

\[
- \int_{\partial\Omega} (2\mu(\rho)D(u) \cdot n)|_{\text{tang}} \cdot u|_{\text{tang}}.
\]

It remains to determine \( d - 1 \) scalar conditions for the velocity. We can choose

\[
u|_{\text{tang}} = 0 \quad \text{or} \quad (D(u) \cdot n)|_{\text{tang}} = 0.
\]

In both cases, we obtain the physical energy estimate. However in the proof of Lemma 3 when \( \mu(\rho) = \nu\rho \) and \( \lambda = 0 \), three extra terms appear

\[
-\nu^2 \int_{\partial\Omega} (D(u) \cdot n) \cdot \nabla \rho, \tag{25}
\]

\[
\nu^2 \int_{\partial\Omega} (\nabla \rho \cdot \nabla) u \cdot n, \tag{26}
\]

\[
\nu \int_{\partial\Omega} (u \cdot \nabla) u \cdot n. \tag{27}
\]

If we take \( u|_{\text{tang}} = 0 \), (25) does not seem to be controlled by the available bounds. Consequently, \( (D(u) \cdot n)|_{\text{tang}} = 0 \) has to be chosen. Similarly (26) and (27) do not vanish, unless \( \Omega \) has no curvature that means for instance \( \Omega = (0, 1) \times T^{d-1} \).

The analysis in the case of a diffusion tensor of the form \(-\nu\rho\nabla u\) with the supplementary condition

\[
(\nabla u \cdot n)|_{\text{tang}} = 0 \tag{28}
\]

is exactly similar. The same case with \( \tilde{\kappa} = 0 \) will be treated in Subsection 3.3 related to viscous Saint-Venant model.

### 3.3 The shallow water equations.

The general purpose of such models is to describe vertically averaged flows in three dimensional shallow domains in terms of the mean velocity field \( u \) and the variation of depth \( h \) due to the free surface. Therefore, the domain will be two dimensional \( \Omega = T^2 \) or a channel \( \Omega = (0, 1) \times S^1 \).

A particular model reads as

\[
\begin{cases}
\partial_t h + \text{div}(hu) = 0, \\
\partial_t (hu) + \text{div}(hu \otimes u) + h\nabla h - \nu \text{div}(h \nabla u) = 0.
\end{cases} \tag{29}
\]
supplemented by the initial and boundary conditions

\[
\begin{align*}
  h|_{t=0} &= h_0, \\
  h u|_{t=0} &= h_0 u_0, \\
  u \cdot n &= 0, \\
  (\nabla u \cdot n)|_{\text{tang}} &= 0 \text{ on } \partial \Omega.
\end{align*}
\] (30)

Let us mention here that the existence of weak solutions with small enough data has been proven by P. Oprea [19] for a diffusion like (8.72) [16] page 251. This diffusion, namely \(-\nu h \Delta u\) instead of \(-\nu \text{div}(h \nabla u)\), allows to simplify by \(h\) the momentum equation. The underlying assumption is of course that \(h\) stays away from a vanishing depth. Then in the low Reynolds number regime, it is possible to assume potential flow as in [24], [15]. This is not possible with \(-\nu \text{div}(h \nabla u)\) as diffusion term, since an equation on the velocity potential cannot be written.

We also refer to [14], [17], [2] and [22] for various results related to low Reynolds number regime, local in time results and semi-empirical derivation of the viscous stress.

The initial data are taken in such way that

\[
h_0 \in L^2(\Omega), \quad \nabla \sqrt{h_0} \in (L^2(\Omega))^2, \quad \sqrt{h_0} u_0 \in (L^2(\Omega))^2.
\] (31)

We remark that Theorem 1 uses strongly the regularity coming from the stress tensor \(K\), that means \(\rho \in L^2(0,T; H^2(\Omega))\). It seems to be impossible to obtain such kind of regularity on \(h\) for weak solutions of (29). As we shall see, the depth \(h\) is only \(L^2(0,T; H^1(\Omega))\), which is not enough to obtain an existence result as in Theorem 1.

Even if the extra regularity of the density is not available, using the classical energy estimate and Lemma 4 in the limit of a vanishing \(\kappa\), the following \textit{a priori} regularities remain true

\[
\begin{align*}
  \nabla \sqrt{h} &\in L^\infty(0,T; (L^2(\Omega))^2), \\
  \sqrt{h} u &\in L^\infty(0,T; (L^2(\Omega))^2), \\
  \sqrt{h} \nabla u &\in L^2(0,T; (L^2(\Omega))^4), \\
  \nabla h &\in L^2(0,T; (L^2(\Omega))^2).
\end{align*}
\]

Let us now turn to the main stability result of this section.

We assume that a sequence \((h_n, u_n)\) of suitably smooth approximate solutions of (29) exist for example constructed from mollified Korteweg equations. The following uniform bounds are obtained

\[
\begin{align*}
  \|\nabla \sqrt{h_n}\|_{L^\infty(0,T; (L^2(\Omega))^2)} &\leq c, \\
  \|\sqrt{h_n} u_n\|_{L^\infty(0,T; (L^2(\Omega))^2)} &\leq c, \\
  \|\sqrt{h_n} \nabla u_n\|_{L^2(0,T; (L^2(\Omega))^4)} &\leq c, \\
  \|\nabla h_n\|_{L^2(0,T; (L^2(\Omega))^2)} &\leq c
\end{align*}
\] (32)
for initial data \((h_0^n, u_0^n)\) belonging and converging strongly in the spaces introduced in (31).

**Theorem 5** Let \((h_n, u_n)\) be a sequence of approximate solutions of (29) satisfying (32) and (30) with approximate initial data \((h_0^n, u_0^n)\). Assume in addition that there exists \(\beta > 0\) such that for all \(n \in \mathbb{N}\)

\[
h_n(t, x) \geq \beta \quad \text{for a.e. } (t, x) \in (0, T) \times \Omega,
\]

then up to the extraction of a subsequence, \((h_n, u_n)\) converges to some \((h, u)\) in \(L^2(0, T; H^s(\Omega)) \times (L^2(0, T; (H^1(\Omega))^2)\text{ weakly })\) for \(s < 1\). Moreover, \((h, u)\) is a weak solution of (29) in \(\mathcal{D}'((0, T) \times \Omega)^3\).

**Proof.** Even if the additional uniform lower bound assumed in (33) can not be proven globally in time for approximate solutions, it allows to put emphasis on the necessary bounds to obtain solutions in the sense of distributions. Using the continuity equation on \(h_n\), we deduce that \(\partial_t h_n\) is bounded in \(L^2(0, T; H^{-1}(\Omega))\), which combined with the uniform \(L^2(0, T; (L^2(\Omega))^2)\) bound on \(\nabla h_n\) gives the strong compactness of \(h_n\) to some \(h\) in \(L^2(0, T; H^s(\Omega))\) for all \(s < 1\). Since \(h_n\) satisfies (33), then \(u_n\) is uniformly bounded in \(L^2(0, T; (H^1(\Omega))^2)\). Using the momentum equation, we get that \(\partial_t(h_n u_n)\) is uniformly bounded in \(L^2(0, T; (W^{-1,p}(\Omega))^2)\) for all \(p < 2\). Moreover, \(h_n u_n\) is uniformly bounded in \(L^2(0, T; (W^{1,p}(\Omega))^2)\) for all \(p < 2\). Therefore, we deduce that \(h_n u_n\) is compact in \(L^2(0, T; (W^{s,p}(\Omega))^2)\) for all \(s < 1, p < 2\). Using the above compactness, we can pass to the limit in \(h_n u_n, h_n u_n \otimes u_n, h_n^2\). It remains to prove the weak convergence of \(h_n \nabla u_n\) which is done by using the weak convergence in \(L^2(0, T; (L^2(\Omega))^4)\) of \(\nabla u_n\) and the strong convergence of \(h_n\) in \(L^2(0, T; L^2(\Omega))\). \(\square\)

4 Stokes like models.

4.1 The Korteweg momentum equations with density dependent surface tension coefficients.

Let us assume in this section that \(\mu\) or \(\kappa\) may depend on the density \(\rho\), recalling that \(f = 0\). We will prove some global existence results in different cases for low Reynolds number models. We introduce a function \(\beta\) such that

\[
\kappa(\rho) = \beta(\rho)^2 \tilde{\kappa}.
\]

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Let us give the new momentum equation that we get using this expression in the tensor corresponding to Korteweg. The surface tension energy is given by

\[ E_\kappa = \int \kappa(\rho) \frac{|\nabla \rho|^2}{2}. \]

Thus differentiating with respect to time, we get

\[
\frac{dE_\kappa}{dt} = \int -\partial_t \beta(\rho) \Delta \beta(\rho)
\]

\[
= - \int u \cdot \left( \beta(\rho) \nabla \Delta \beta(\rho) + \nabla((\rho \beta'(\rho) - \beta(\rho)) \Delta \beta(\rho)) \right)
\]

\[
= - \int u \cdot \left( \nabla (\rho \beta'(\rho) \Delta \beta(\rho)) - \text{div}(\nabla \beta(\rho) \otimes \nabla \beta(\rho)) + \nabla \left( \frac{|\nabla \beta(\rho)|^2}{2} \right) \right)
\]

\[
= - \int u \cdot \left( \nabla \text{div}(\rho \beta'(\rho) \nabla \beta(\rho)) - \text{div}(\nabla \beta(\rho) \otimes \nabla \beta(\rho))
\]

\[
- \nabla \left( \left( \rho \beta'(\rho) \beta''(\rho) + \frac{\beta'(\rho)^2}{2} \right) |\nabla \rho|^2 \right). \]

Then if \( \kappa \) depends on \( \rho \), we get the following momentum equation

\[
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla P - \text{div}(2\mu(\rho) D(u)) - \nabla (\lambda(\rho) \text{div} u)
\]

\[
= \kappa \left( \nabla \text{div}(\rho \beta'(\rho) \nabla \beta(\rho)) - \text{div}(\nabla \beta(\rho) \otimes \nabla \beta(\rho))
\]

\[
- \nabla \left( \left( \rho \beta'(\rho) \beta''(\rho) + \frac{\beta'(\rho)^2}{2} \right) |\nabla \rho|^2 \right) \equiv \text{RHS}. \]

that means, in terms of \( \kappa \) only

\[
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla P - \text{div}(2\mu(\rho) D(u)) - \nabla (\lambda(\rho) \text{div} u)
\]

\[
= \kappa \left( \nabla \text{div}(\rho \kappa(\rho) \nabla \rho) - \text{div}(\kappa(\rho) \nabla \rho \otimes \nabla \rho)
\]

\[
- \nabla \left( \left( \rho \kappa'(\rho) + \kappa(\rho) \right) \frac{|\nabla \rho|^2}{2} \right) \equiv \text{RHS}. \]

Some examples:

- If \( \kappa(\rho) \propto 1 \), then the right-hand side reads

\[
\text{RHS} = \kappa \rho \nabla \Delta \rho = \kappa \left( \nabla \Delta \rho^2 / 2 - \text{div}(\nabla \rho \otimes \nabla \rho) - \nabla \frac{|\nabla \rho|^2}{2} \right). \]

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- If $\kappa(\rho) \propto 1/\rho^2$, then the right-hand side reads
\[
RHS = \tilde{\kappa} \left( \nabla \Delta \log \rho - \text{div}(\nabla \log \rho \otimes \nabla \log \rho) + \frac{1}{2} \nabla |\nabla \log \rho|^2 \right)
\]
\[
= \tilde{\kappa} (\nabla \Delta \log \rho - \nabla \log \rho \Delta \log \rho).
\]
- If $\kappa(\rho) \propto \rho$, then the right-hand side reads
\[
RHS = \tilde{\kappa} \left( \nabla \text{div}(\rho^2 \nabla \rho) - \text{div}(\rho \nabla \rho \otimes \nabla \rho) - \nabla (\rho |\nabla \rho|^2) \right). \quad \Box
\]

4.2 Existence and stability results.

The case $\kappa(\rho) \propto 1/\rho^2$.

We investigate the compressible semi-stationary ($\bar{p} = 0$) or nonstationary ($\bar{p} \neq 0$) Stokes-like model in $\Omega = T^2$, i.e. the following system
\[
\begin{aligned}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\bar{p} \partial_t u + \nabla P - \text{div}(2\mu(\rho) D(u)) - \nabla (\lambda(\rho) \text{div} u) &= \text{div} K
\end{aligned}
\]
\[
= \tilde{\kappa} \left( \nabla \Delta \log \rho - \text{div}(\nabla \log \rho \otimes \nabla \log \rho) + \frac{1}{2} \nabla |\nabla \log \rho|^2 \right)
\]
\[
= \tilde{\kappa} (\nabla \Delta \log \rho - \nabla \log \rho \Delta \log \rho),
\]

where the viscosity coefficients $\lambda$ and $\mu$ are $C^1$ functions such that
\[
\underline{\mu} \leq \mu(\cdot) \leq \bar{\mu}, \quad \underline{\lambda} \leq \lambda(\cdot) \leq \bar{\lambda},
\]

for some positive constants $\underline{\mu}, \bar{\mu}, \underline{\lambda}, \bar{\lambda}$. In addition, the pressure $P$ satisfies for simplicity a $\gamma$-type law $P(\rho) = \alpha \rho^\gamma$, for some $\alpha > 0$ and $\gamma > 1$. Such system has been studied in [17] page 236 for $\mu$ and $\lambda$ constant, and $\tilde{\kappa} = 0$.

We get the following energy conservation
\[
\frac{d}{dt} \int_\Omega \left( \bar{p} |u|^2 + \Pi(\rho) + \frac{\tilde{\kappa} |\nabla \log \rho|^2}{2} \right) + \int_\Omega \left( 2\mu(\rho) D(u) : D(u) + \lambda(\rho) \text{div} u |\text{div} u|^2 \right) = 0,
\]

where $\Pi(s) = \alpha s^\gamma / (\gamma - 1)$. The total initial energy is denoted $\mathcal{E}_0$ and is assumed to be finite. More precisely, we assume that
\[
\mathcal{E}_0 = \int_\Omega \left( \bar{p} |u_0|^2 + \Pi(\rho_0) + \frac{\tilde{\kappa} |\nabla \log \rho_0|^2}{2} \right) \leq +\infty.
\]
As emphasized in Section 2, the problem to obtain the existence of global in time weak solutions is to get some regularity bounds on the density. It turns out that in the case \( \kappa(\rho) \propto 1/\rho^2 \), such bounds can be derived on \( \Delta \log \rho \), assuming that the space dimension is \( d = 2 \) and that the data are small enough. No lower or upper bounds on the density are needed. The obtained regularity will be sufficient to pass to the limit in the non-linear terms.

More precisely, the following result holds

**Theorem 6** We assume \( d = 2 \). Then there exists \( \eta > 0 \) such that if

\[ \mathcal{E}_0 < \eta \]

then there exists a solution \( (\rho, u) \) of (34) in the sense of distributions such that

\[
\begin{align*}
\rho^{1/2} u &\in L^\infty(0, T; (L^2(\Omega))^2) \\
u &\in L^2(0, T; (H^1(\Omega))^2), \\
\log \rho &\in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)), \\
\Pi(\rho) &\in L^\infty(0, T; L^2(\Omega)), \\
\nabla \rho^2 &\in L^2(0, T; (L^2(\Omega))^2).
\end{align*}
\]

**Proof.** The only problem when trying to pass to the limit for a sequence of approximate solutions is to deal with quadratic terms involving \( \nabla \log \rho \), the viscosity terms \( 2\mu(\rho)D(u) \) and \( \lambda(\rho)\text{div} u \), and the pressure term. It suffices to find an estimate related to \( \Delta \log \rho \) to obtain the required compactness.

Indeed, multiplying the momentum equation (34) by \( \nabla \log \rho \) and integrating over \( (0, T) \times \Omega \), we get

\[
\begin{align*}
\int_0^T \int_\Omega \frac{4\rho}{\gamma} |\nabla \rho^2|^2 + \int_0^T \int_\Omega \kappa |\Delta \log \rho|^2 &
\leq c \int_0^T \int_\Omega \left( |\nabla D(u)|^2 + |\lambda \text{div} u|^2 \right) + c \int_0^T \int_\Omega |\nabla \log \rho|^4 \\
&+ c \int_0^T \int_\Omega |\text{div} u| |u \nabla \log \rho + \text{div} u| \\
&+ c\bar{p} \int_\Omega |u(T) \cdot \nabla \log \rho(T)| + c\bar{p} \int_\Omega |u(0) \cdot \nabla \log \rho(0)| \\
&\leq c + c \int_0^T \| \nabla \log \rho \|_{L^4(\Omega)}^4 \\
&\leq c + c \int_0^T \| \nabla \log \rho \|^2_{L^2(\Omega)} \| \Delta \log \rho \|^2_{L^2(\Omega)}.
\end{align*}
\]
This gives the claimed $L^2(0, T; L^2(\Omega))$ estimate on $\Delta \log \rho$ as soon as

$$\|\nabla \log \rho\|_{L^\infty(0, T; (L^2(\Omega))^2)} \ll 1,$$

that means assuming small enough initial data, since $\nabla \log \rho$ is controlled by the energy conservation.

Let us now consider a sequence $(\rho_n, u_n)$ of approximate solutions such that the following uniform bounds hold for all $T < +\infty$

$$\|\Delta \log \rho_n\|_{L^2(0, T; L^2(\Omega))} \leq C_T, \quad \|\nabla \log \rho_n\|_{L^\infty(0, T; (L^2(\Omega))^2)} \leq C_T, \quad \|\rho_n\|_{L^\infty(0, T; L^\gamma(\Omega))} \leq C_T, \quad \|\nabla \rho_n\|_{L^2(0, T; (L^2(\Omega))^2)} \leq C_T \quad \|\bar{\rho}^{1/2} u_n\|_{L^\infty(0, T; (L^2(\Omega))^2)} \leq C_T, \quad \|\nabla u_n\|_{L^2(0, T; (L^2(\Omega))^2)} \leq C_T.$$

Next observing that $\partial_t \log \rho_n = -u_n \cdot \nabla \log \rho_n - \text{div}u_n$ is uniformly bounded in $L^{4/3}(0, T; L^2(\Omega))$, we deduce that $\log \rho_n$ is compact in $L^2(0, T; H^s(\Omega))$ for all $s < 2$. Therefore, up to the extraction of a subsequence, $\log \rho_n(t, x)$ converges to some $z(t, x)$ for a.e. $(t, x) \in (0, T) \times \Omega$. It allows to define $\rho(t, x) = \exp(z(t, x))$ a.e. in $(0, T) \times \Omega$. In particular, we can pass to the limit in all the terms of $\text{div}K_n$.

Moreover, there exists $u \in L^2(0, T; (H^1(\Omega))^2)$ such that $\nabla u_n$ converges to $\nabla u$ weakly in $L^2(0, T; (L^2(\Omega))^2)$. It remains to pass to the limit in the pressure and the viscosity terms. First, we remark that $\bar{\rho}^{1/2}$ satisfies

$$\partial_t \bar{\rho}^{1/2} + \text{div}(u_n \bar{\rho}^{1/2}) + \left(\frac{\gamma}{2} - 1\right) \bar{\rho}^{1/2} \text{div}u_n = 0,$$

so that $\partial_t \bar{\rho}^{1/2}$ is bounded in $L^{4/3}(0, T; H^{-1}(\Omega))$ uniformly, hence $\bar{\rho}^{1/2}$ is compact in $L^2(0, T; H^s(\Omega))$ for all $s < 1$. This allows to deal with the pressure term. For the viscosity terms, it suffices to show that $\mu(\rho_n)$ and $\lambda(\rho_n)$ converge strongly in $L^2(0, T; L^2(\Omega))$ to $\mu(\rho)$ and $\lambda(\rho)$. For all $\eta \in C^1(R) \cap L^\infty(R)$, we deduce from the a.e. convergence of $\rho_n$ to $\rho$ that $\eta(\rho_n)$ converges in $L^2(0, T; L^2(\Omega))$ to $\eta(\rho)$, which ends up the proof. □

**The case $\mu = \text{cte}, \lambda = \text{cte},$ and $\kappa = \text{cte}$.**

We now look at the Stokes system with constant $\mu$ and $\kappa$, that means independent on $\rho$. More precisely, we consider the following system in the torus $\Omega = T^d$

$$
\begin{cases}
\partial_t \rho + \text{div}(\rho u) = 0, \\
- \text{div}(2\mu D(u)) - \lambda \nabla \text{div}u + \nabla P = \kappa \rho \nabla \Delta \rho, 
\end{cases}
$$

(35)
with initial data as in Subsection 3.1, and viscosity coefficients $\mu > 0$ and $\lambda + 2\mu > 0$. This kind of model in the case $\tilde{\kappa} = 0$ has been studied for example in [17].

The following energy conservation stems from classical arguments

$$\frac{d}{dt} \int_{\Omega} \left( \Pi(\rho) + \tilde{\kappa} \frac{\left| \nabla \rho \right|^2}{2} \right) + \int_{\Omega} \left( 2\mu D(u) : D(u) + \lambda |\text{div} u|^2 \right) = 0,$$

where the pressure law is assumed to satisfy the same hypothesis as in Theorem 1, and $\Pi(s) = s \int_0^s P(\tau) / \tau^2 d\tau$. Once again, the total initial energy is denoted $E_0$ and is assumed to be finite. More precisely, we assume that

$$E_0 = \int_{\Omega} \left( \Pi(\rho_0) + \tilde{\kappa} \frac{\left| \nabla \rho_0 \right|^2}{2} \right) < +\infty.$$

We will obtain the stability of sequences of approximate weak solutions $(\rho_n, u_n)_{n \in \mathbb{N}}$ without smallness assumptions on the data, but assuming that $1/\rho_n$ is uniformly bounded in $L^\infty((0, T) \times \Omega)$. In other words, we will assume that there exists $\beta > 0$ such that

$$\rho_n(t, x) \geq \beta > 0 \text{ a.e. in } (0, T) \times \Omega.$$  \hspace{1cm} (36)

Unfortunately, we have been unable to derive estimate (36). Nevertheless, it allows to prove the convergence of $(\rho_n, u_n)$ to some $(\rho, u)$ which satisfies (35).

Assuming that the sequence of approximate weak solutions has uniformly bounded initial energy, we first obtain the following uniform bounds

$$\| \Pi(\rho_n) \|_{L^\infty(0, T; L^1(\Omega))} \leq C_T, \quad \| \nabla \rho_n \|_{L^\infty(0, T; (L^2(\Omega))^d)} \leq C_T$$

$$\| \nabla u_n \|_{L^2(0, T; (L^2(\Omega))^{d \times d})} \leq C_T$$ \hspace{1cm} (37)

Next multiplying the momentum equation by $\nabla \rho_n / \rho_n$ and using (36), we get

$$\| \Delta \rho_n \|_{L^2(0, T; L^2(\Omega))}^2 \
\leq c_\beta \| D(u_n) \|_{L^2(0, T; (L^2(\Omega))^{d \times d})} \left( \| \Delta \rho_n \|_{L^2(0, T; L^2(\Omega))} + \| \nabla \rho_n \|_{L^4(0, T; (L^4(\Omega))^d)} \right).$$

All the terms in the right-hand side are bounded by using the Cauchy–Schwarz inequality and interpolation arguments. Therefore, we get an uniform estimate on $\rho_n$ in $L^2(0, T; H^2(\Omega))$.

This yields the following result
Theorem 7 We assume $d = 2$ or $d = 3$ and consider a sequence of approximate weak solutions $(\rho_n, u_n)$ of (35) such that $\rho_n$ is uniformly bounded in $L^2(0, T; H^2(\Omega))$ and (36), (37) hold. Then, there exists a weak solution $(\rho, u)$ of (35) such that $\rho_n$ converges to $\rho$ in $L^2(0, T; H^1(\Omega))$ for all $s < 2$, $u_n$ converges weakly in $L^2(0, T; (H^1(\Omega))^d)$ to $u$ and

$$u \in L^2(0, T; (H^1(\Omega))^d),$$

$$\rho \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)),$$

$$\Pi(\rho) \in L^\infty(0, T; L^1(\Omega)).$$

Proof. Once the preceding bounds are derived as previously sketched, the convergence arguments follow from classical compactness analysis. \qed

The case $\mu \propto \rho^2$ and $\kappa = \text{cte.}$

We will consider in this case the semi-stationary Stokes version of the Korteweg model with a damping term $\zeta\rho u$ in the torus $\Omega = T^d$. It means that we consider the following system

$$\begin{cases}
\partial_t \rho + \text{div}(\rho u) = 0, \\
\zeta \rho u - \mu \text{div}(\rho^2 D(u)) - \tilde{\kappa}\rho \nabla \rho + \rho \nabla \rho = 0,
\end{cases} \quad (38)$$

with initial condition $\rho|_{t=0} = \rho_0$ and where $\zeta$ is a positive constant. A more general pressure $P$ may have been chosen as in the end of Subsection 3.1

Assuming that the total initial energy is finite

$$E_0 = \int_\Omega \left( \frac{1}{2} |\nabla \rho_0|^2 + \Pi(\rho_0) \right) < +\infty, \quad (39)$$

we claim that an existence result of a weak solutions can be proven for (38).

At first let us give the definition of a weak solution of (38). We will say that $(\rho, u)$ is a weak solution of (38) if and only if:

$$\begin{cases}
\rho \in L^2(0, T; H^2(\Omega)), \quad \nabla \rho \in L^\infty(0, T; (L^2(\Omega))^d), \\
\rho D(u) \in L^2(0, T; (L^2(\Omega))^{d \times d}), \quad \sqrt{\rho} u \in L^2(0, T; (L^2(\Omega))^d),
\end{cases} \quad (40)$$

and

$$\partial_t \rho + \text{div}(\rho u) = 0 \text{ in } \mathcal{D}'((0, T) \times \Omega), \quad \rho|_{t=0} = \rho_0 \text{ in } \mathcal{D}'(\Omega), \quad (41)$$
and for all \( \varphi \in C^\infty([0, T] \times \Omega) \) such that \( \varphi(T, \cdot) = 0 \), one has:
\[
\begin{align*}
\int_\Omega \left( \zeta \rho u \cdot \varphi + \mu \rho^2 D(u) : D(\varphi) - \rho \text{div} \varphi \right.
&+ \kappa \Delta \text{div} \varphi \cdot \rho^2 - \frac{\kappa}{2} |\nabla \rho|^2 \text{div} \varphi - \kappa \nabla \rho \otimes \nabla \rho : D(\varphi) \bigg) = 0.
\end{align*}
\]
(42)

Lemma 8 We have the following relation
\[
- \int_\Omega \text{div}(\rho^2 D(u)) \frac{\nabla \rho}{\rho} = \frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla \rho|^2 + \int_\Omega u \cdot \nabla \rho \Delta \rho.
\]

Proof. We have, using the mass equation,
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla \rho|^2 = \int_\Omega \partial_t \nabla \rho \cdot \nabla \rho
= - \int_\Omega \nabla \text{div}(\rho u) \cdot \nabla \rho = \int_\Omega \nabla (\rho u) : \nabla \nabla \rho. \tag{43}
\]
Moreover, integrating by parts, we get
\[
\int_\Omega \nabla \cdot (\rho^2 D(u)) \frac{\nabla \rho}{\rho} = - \int_\Omega \rho \nabla u : \nabla \nabla \rho + \int_\Omega \nabla u : \nabla \rho \otimes \nabla \rho
= - \int_\Omega \nabla (\rho u) : \nabla \nabla \rho - \int_\Omega u \cdot \nabla \rho \Delta \rho. \tag{44}
\]
We conclude using (43) and (44). \( \square \)

Now we are able to prove the following result

Theorem 9 Let \( d = 2 \) or 3 and (39) be satisfied. Then there exists a global weak solution \((\rho, u)\) satisfying (40), (41) and (42).

Proof. At first we multiply the momentum equation by \( u \) and the conservation of mass by \( \Delta \rho \). We obtain the classical equality
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \left( \kappa |\nabla \rho|^2 + \Pi(\rho) \right) + \int_\Omega \mu \rho^2 |D(u)|^2 + \int_\Omega \zeta \rho |u|^2 = 0. \tag{45}
\]

Then we test the momentum equation against \( \nabla \log \rho \). Using the relation proved in Lemma 8, we get
\[
\frac{\mu}{2} \frac{d}{dt} \int_\Omega |\nabla \rho|^2 + \kappa \int_\Omega |\nabla \nabla \rho|^2 + \int_\Omega |\nabla \rho|^2
= -\mu \int_\Omega u \cdot \nabla \rho \Delta \rho + \zeta \frac{d}{dt} \int_\Omega \rho + \zeta \int_\Omega \rho \text{div} u.
\]
The mass equation gives the following equivalent form
\[
\frac{\mu}{2} \frac{d}{dt} \int_\Omega |\nabla \rho|^2 + \tilde{\kappa} \int_\Omega |\nabla \nabla \rho|^2 + \int_\Omega |\nabla \rho|^2
= \int_\Omega \mu \rho \text{div} \Delta \rho + \mu \int_\Omega \partial_t \rho \Delta \rho + \zeta \frac{d}{dt} \int_\Omega \rho + \zeta \int_\Omega \rho \text{div} u.
\]
Finally we get
\[
\mu \frac{d}{dt} \int_\Omega |\nabla \rho|^2 + \tilde{\kappa} \int_\Omega |\nabla \nabla \rho|^2 + \int_\Omega |\nabla \rho|^2
= \int_\Omega \mu \rho \text{div} u \Delta \rho + \zeta \frac{d}{dt} \int_\Omega \rho + \zeta \int_\Omega \rho \text{div} u.
\]
Since we control $\rho \text{div} u$ in $L^2(0, T; L^2(\Omega))$ by (40), this gives uniform estimates on $\nabla^2 \rho$ without any assumptions on the data, nor on the space dimension or on some bounds of the density.

Now we have to pass to the limit in the different terms especially in diffusive terms (the proof is in fact easier, since strong convergence of the momentum is not required to pass to the limit). We just use the following identity
\[
\rho^2 D(u) = D(\rho^2 u) - 2 \rho u \otimes \nabla \rho,
\]
and the compactness of $\rho_n$ in
\[
C([0, T]; H^s(\Omega)) \cap L^{2/s}(0, T; H^{1+s}(\Omega)) \quad \text{for all } s < 1.
\]
The damping term has been added in order to obtain sufficient regularity on $\rho_n u_n$. □

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