MEAN CURVATURE EVOLUTION OF ENTIRE LAGRANGIAN GRAPHS IN $\mathbb{R}^4$

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Abstract. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an area preserving diffeomorphism between $\mathbb{R}^2$.
The graph of $f$ can be viewed as Lagrangian submanifold in $\mathbb{R}^4$. In this paper, we
show that the long time existence for the mean curvature flow of an area preserving
map between $\mathbb{R}^2$. We also have an interior gradient estimate.

1. Introduction

Let $\Sigma_1$ and $\Sigma_2$ are two Riemannian manifolds and $M = \Sigma_1 \times \Sigma_2$ be the product
manifold. One can consider a smooth map $f : \Sigma_1 \to \Sigma_2$. Let $\Sigma$ be the graph of $f$ and
then $\Sigma$ is a submanifold of $M$ by the embedding $F = id \times f$. In this paper we studied
the deformation of $f$ by the mean curvature flow. More precisely, we deform $\Sigma$ along
the direction of its mean curvature vector in $M$ so that $\Sigma$ will remain as a graph. In
particular we consider the mean curvature flow of area-preserving Lagrangian graphs
in $\mathbb{R}^4$. In the [W2], Wang studied the deformation of area-preserving diffeomorphism
between compact surfaces of constant curvature. In particular, he showed that a area
preserving map between sphere can be deformed to an isometry via the mean curva-
ture flow. We will adopt the techniques developed in [W2] to study the deformation
of area preserving map between $\mathbb{R}^2$ by the mean curvature flow. We remark that
the deformation of maps by the mean curvature flow are also studied in the papers
[CLT], [EH1], [EH2], [S1], [S2] and [W3].

Let $f : (\mathbb{R}^2, \omega_1) \to (\mathbb{R}^2, \omega_2)$ be an area-preserving diffeomorphism , i.e. $f^*(\omega_2) =
\omega_1$, where $\omega_1 = dx_1 \wedge dx_2$, $\omega_2 = dx_3 \wedge dx_4$ are the standard symplectic forms for the
reference and target spaces $\mathbb{R}^2$, respectively. Then the graph $\Sigma$ of $f$ is a Lagrangian
surface in $\mathbb{R}^4$ with respect to the symplectic form $\omega_1 - \omega_2$. Let $\omega_1$ denote the Jacobian
of the projection from $\Sigma$ to $\mathbb{R}^2$. Since $\omega_1 \geq C_0 > 0$ if and only if $\Sigma$ is a graph over

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\( \mathbb{R}^2 \) by the implicit function theorem, we say that \( \Sigma \) is an entire Lagrangian graph in \( \mathbb{R}^4 \) if it is a graph of an area-preserving diffeomorphism \( f \) so that \( \ast \omega_1 \geq C_0 > 0 \), where \( \ast \) is the Hodge star-operator on \( \Sigma \). Now we consider the mean curvature flow of \( F_0 : \Sigma \to \mathbb{R}^4 \). That is there exists a family of immersions \( F_t = F(\cdot, t) \) of surfaces with corresponding images \( \Sigma_t = F_t(\Sigma) \) such that

\[
\begin{aligned}
\frac{d}{dt} F(x, t) &= H(x, t), \\
F(x, 0) &= F_0(x),
\end{aligned}
\]

where \( H(x, t) \) is the mean curvature vector of \( \Sigma_t \) at \( F(x, t) \). Then we have the following long time existence of the mean curvature flow (1.1).

**Theorem 1.1.** Let \( \Sigma \) be an \( C^{2, \alpha} \) entire Lagrangian graph in \( \mathbb{R}^4 \) with bounded curvature. Then the mean curvature flow (1.1) has a smooth solution for all time if we have the bounded gradient. Moreover, each \( \Sigma_t \) is an entire Lagrangian graph over \( \mathbb{R}^2 \).

We prove the long time existence result by using the methods of Wang's blow-up analysis as in[W1, W3]. As a contrast, if we assume \( \ast \omega_1 \geq C_0 > 1 / \sqrt{2} \) on \( \Sigma \) ([W1]) or there exists a holomorphic 2-form \( \Omega \) such that its real part \( \text{Re} \Omega \) restricts to \( \Sigma \) is positive ([S2]) so that the maximum principle can be applied, then one can have the uniformly bounded estimate on the second fundamental form \( |A|^2 \). And therefore the long time existence follows. Under suitable initial conditions, one can obtain an asymptotic convergence for the solutions of (1.1). However, in the general cases, the solution may blow-up as time goes to infinity.

In this paper we also get the interior gradient estimate by using a method of Korevaar [Ko], see section 5. And in the appendix we show that planes are the only entire Lagrangian graphs in \( \mathbb{R}^4 \) which are contracting self-similar solutions of the mean curvature flow.

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2. Preliminary

In this section, we recall some evolution equations and relations along the mean curvature flow. First, we follow the same definitions and notations as in [W1, W2]. We suppose that $F(x, t)$ satisfies the mean curvature flow equation (1.1). Let $H$ be the mean curvature vector of the surface $\Sigma_t = F(\Sigma, t)$ in $\mathbb{R}^4$, $A$ is the second fundamental form and $g = \{g_{ij}\}$ is the induced metric on $\Sigma_t$. Let $\Delta$ and $\nabla$ be the Laplace operator and the covariant derivative for the induced metric on $\Sigma_t$, respectively. We choose an orthonormal frame $\{e_1, e_2, \nu_1, \nu_2\}$ of $\mathbb{R}^4$ such that $\{e_1, e_2\}$ is an orthonormal frame for the tangent bundle $T\Sigma_t$, and $\{\nu_1, \nu_2\}$ is an orthonormal frame for the normal bundle $N\Sigma_t$.

Now we fixed the complex structure $J$ in $\mathbb{R}^4$ which corresponds to $\omega_1 - \omega_2$ and choose an orthonormal frame $\{e_1, e_2, \nu_1, \nu_2\}$ so that $\nu_1 = Je_1$ and $\nu_2 = Je_2$. Then the components of the second fundamental form of $\Sigma_t$ is defined by $h_{ijk} = \langle \nabla_{e_i} e_j, Je_k \rangle$. So $h_{ijk}$ is symmetric with respect to any pair if $\Sigma_t$ is Lagrangian. We can write

$$A(e_i, e_j) = h_{ijk} \nu_k \text{ and } H = -h_k \nu_k,$$

where $h_k = g^{ij} h_{ijk}$. Now we have the following evolution equation for $\ast \omega_1$, which is derived in equation (2.2) in [W2].

**Proposition 2.1.** Let $\eta = \ast \omega_1$, it satisfies the following equation

\begin{equation}
\left( \frac{d}{dt} - \Delta \right) \eta = \left( 2 \lvert A \rvert^2 - \lvert H \rvert^2 \right) \eta.
\end{equation}

Next for $X = F(p, t)$, we have

\begin{equation}
\left( \frac{d}{dt} - \Delta \right) \lvert X \rvert^2 = -4.
\end{equation}

To simplify our calculation at any $p \in \Sigma_t$, we can choose a coordinate system $\{x^1, x^2\}$ on the reference plane $\mathbb{R}^2$ and $\{y^1, y^2\}$ on the target $\mathbb{R}^2$ so that $\partial f^i / \partial x^j$ is diagonalized at $p$. That is $\partial f^i / \partial x^j = \lambda_i \delta_{ij}$ where $\lambda_i \geq 0$ are the singular values of
$df$, or eigenvalues of $\sqrt{(df)^\top df}$. Then we have in terms of the singular values $\lambda_i$,

\[ g_{ij} = (1 + \lambda_i^2)\delta_{ij}, \quad g^{ij} = \frac{1}{1 + \lambda_i^2}\delta_{ij}, \quad \text{and} \]

\[ *\omega_1 = \frac{1}{\sqrt{\det((df)^\top df)}} \frac{1}{\sqrt{(1 + \lambda_i^2)(1 + \lambda_j^2)}} \]

Let

\[ e_1 = \frac{1}{\sqrt{1 + \lambda_1^2}}\left( \frac{\partial}{\partial x^1} + \lambda_1 \frac{\partial}{\partial y^1} \right), \quad e_2 = \frac{1}{\sqrt{1 + \lambda_2^2}}\left( \frac{\partial}{\partial x^2} + \lambda_2 \frac{\partial}{\partial y^2} \right) \]

be an orthonormal frame for the tangent bundle $T\Sigma_t$. And let

\[ \nu_1 = \frac{1}{\sqrt{1 + \lambda_1^2}}\left( \frac{\partial}{\partial y^1} - \lambda_1 \frac{\partial}{\partial x^1} \right), \quad \nu_2 = \frac{1}{\sqrt{1 + \lambda_2^2}}\left( \frac{\partial}{\partial y^2} - \lambda_2 \frac{\partial}{\partial x^2} \right) \]

be an orthonormal frame for the normal bundle $N\Sigma_t$.

Now we state the relation between parametric and nonparametric solutions to the mean curvature flow equation. The higher codimension case was derived in [W4], which extended the codimension one case [EH2].

Let $D$ be a bounded domain in $\mathbb{R}^2$ and $f_0 : D \to \mathbb{R}^2$ be a vector-valued function. The graph of $f_0$ is then given by the embedding $I \times f_0 : D \to \mathbb{R}^{2+2}$ where $I$ is the identity map on $D$.

Let $F : D \times [0, T) \to \mathbb{R}^{2+2}$ be a parametric solution to the Dirichlet problem of the mean curvature flow, i.e.

\[
\begin{align*}
\frac{d}{dt} F &= H, \\
F \mid_{\partial D} &= I \times f_0 |_{\partial D}.
\end{align*}
\]

Then we have the following relation.

**Proposition 2.2.** Suppose $F$ is a solution to the Dirichlet problem for mean curvature flow (2.3) and each $F(D, t)$ can be written as a graph over $D \subset \mathbb{R}^2$. There exists a family of diffeomorphism $\zeta_t$ of $D$ s.t. $\bar{F}_t = F_t \circ \zeta_t$ is of the form

\[ \bar{F}(x^1, x^2) = (x^1, x^2, f^1, f^2), \]

\[ f = (f^1, f^2) : D \times [0, T) \to \mathbb{R}^2 \]

satisfies

\[
\begin{align*}
\frac{d}{dt} f^k &= g^{ij} \frac{\partial^2}{\partial x^i \partial x^j} f^k, & k = 1, 2 \\
f \mid_{\partial D} &= f_0 |_{\partial D}.
\end{align*}
\]
where \( g^{ij} = (g_{ij})^{-1} \) and \( g_{ij} = \delta_{ij} + \sum_k \frac{\partial u^k}{\partial x^i} \frac{\partial u^k}{\partial x^j} \). Conversely, if \( f = (f^1, f^2) : D \times [0, T) \rightarrow \mathbb{R}^2 \) satisfies (2.4), then \( \vec{F} = I \times f \) is a solution to
\[
\frac{d}{dt} \vec{F}(x, t) = \vec{H}(x, t).
\]

3. Short time existence

In this section, based on [W4], we show the short time existence by the standard parabolic theory.

First we have the short time existence on any bounded domain \( D \) of \( \mathbb{R}^2 \). By the Schauder fixed point theorem, the solvability of equation (2.4) reduces to the estimates of the solution \( (f^k) \) to
\[
\begin{cases}
\frac{d}{dt} f^k = g^{ij} \frac{\partial^2}{\partial x^i \partial x^j} f^k, & k = 1, 2 \\
f \big|_{\partial D} = f_0|_{\partial D}.
\end{cases}
\]

where \( g^{ij} = (g_{ij})^{-1} \) and \( g_{ij} = \delta_{ij} + \sum_k \frac{\partial u^k}{\partial x^i} \frac{\partial u^k}{\partial x^j} \), for any \( u = (u^k) \) with uniform \( C^{1, \alpha} \) bound.

The equation (3.1) is a decoupled system of linear parabolic equations. This equation is uniform parabolic and the required estimate follows from the linear theory for scalar equations.

In order to extend the short time solution to the entire \( \mathbb{R}^2 \) with initial \( C^{2, \alpha} \) entire Lagrangian graph with bounded curvature, we need the following boundary gradient estimate derived in Theorem 3.2 of [W4]. We include the proof for reader’s convenience.

**Theorem 3.1.** Let \( D \) be a bounded \( C^2 \) convex domain in \( \mathbb{R}^2 \). Suppose the flow (2.4) exists smoothly on \( D \times [0, T) \), then we have
\[
|\nabla f| < 8(1 + \xi) \sup_D |\nabla^2 f_0| + \sqrt{2} \sup_{\partial D} |\nabla f_0| \quad \text{on } \partial D \times [0, T)
\]

where \( \xi = \sup_{D \times [0, T]} |\nabla f| \) and \( \delta = \) the diameter of \( D \).

**Proof.** Let \( L \) be the supporting line at a boundary point \( p \in \partial D \) and \( d_L(\cdot) \) be the distance function to \( L \). For each \( k = 1, 2 \), we define the following function on \( \mathbb{R}^2 \)
\[
S(x^1, x^2) = \nu \log(1 + \mu d_L) - (f^k - f^k_0)
\]
where $\nu, \mu > 0$ are to be determined. We easily compute show that $S$ satisfies the evolution equation

$$
(\frac{d}{dt} - \Delta)S = \nu \mu \frac{\partial d_L}{\partial x^i} \frac{\partial d_L}{\partial x^j} - \Delta f^k_0,
$$

there we used $d_L$ is a linear function, so $\Delta d_L = 0$.

Because the eigenvalues of $g^{ij}$ are between $\frac{1}{1+\xi}$ and $1$ and $|\nabla d_L| = 1$, we have

$$
g^{ij} \frac{\partial d_L}{\partial x^i} \frac{\partial d_L}{\partial x^j} \geq \frac{1}{1+\xi},
$$
on $[0, T)$. Therefore

$$
\nu \mu^2 \frac{1}{(1+\mu \delta)^2} \frac{1}{1+\xi} \geq \nu \mu \frac{1}{(1+\mu \delta)^2} \frac{1}{1+\xi},
$$
because $d_L(y) \leq |y - p| \leq \delta$ for any $y \in D$. On the other hand, we have

$$
|\Delta f^k_0| = \left| g^{ij} \frac{\partial^2 f_0}{\partial x^i \partial x^j} \right| \leq 2 |\nabla^2 f_0|.
$$

Now we require

$$
\frac{\nu \mu^2}{(1+\mu \delta)^2} \frac{1}{1+\xi} \geq 2 \sup_D |\nabla^2 f_0|.
$$

In view of (3.2), the condition (3.3) implies $(\frac{d}{dt} - \Delta)S \geq 0$ on $[0, T)$. Notice that on the boundary of $D$, we have $S > 0$ except $S = 0$ at $p$ by convexity. On the other hand, $S \geq 0$ on $D$ at $t = 0$. It follows from the strong maximum principle that $S > 0$ on $D \times (0, T)$. Likewise we can apply this procedure to $S' = \nu \log(1+\mu d_L) + (f^k - f^k_0)$.

Therefore at the boundary point $p$, the normal derivative satisfies

$$
\left| \frac{\partial (f^k - f^k_0)}{\partial n} \right| (p) \leq \lim_{d_L(x) \to 0} \left| \frac{f^k(x) - f^k_0(x)}{d_L(x)} \right| \leq \lim_{d_L(x) \to 0} \frac{\nu \log(1+\mu d_L(x))}{d_L(x)} = \nu \mu.
$$

We may assume $\frac{\partial f^2}{\partial n} = 0$ by changing coordinates of $\mathbb{R}^2$. Thus we obtain

$$
\left| \frac{\partial f}{\partial n} \right| < \nu \mu + \left| \frac{\partial f_0}{\partial n} \right|.
$$

For $x \in \partial D$, we define $|\nabla^{\partial D} f| = \sup_v |\nabla f(x)(v)|$ where the sup is taken over all unit vectors $v$ tangent to $\partial D$. The Dirichlet condition implies

$$
|\nabla^{\partial D} f| = |\nabla^{\partial D} f_0|
$$
on $\partial D$, therefore

$$
|\nabla f| < \sqrt{2} |\nabla f_0|.$$
on $\partial D$.

Now we can minimize $\nu \mu$ subject to the constraint (3.3). The minimum is achieved when $\mu = \delta^{-1}$ and $\nu k = \delta(1 + \xi) \sup_D |\nabla^2 f_0|$. Then this implies the boundary gradient estimate.

By choosing an exhaustion of $\mathbb{R}^2$ by a family of open convex subsets $\{D_k\}$ such that $D_k \subset D_{k+1}$, $\bar{D}_k$ is a compact subset of $\mathbb{R}^2$ and $\mathbb{R}^2 = \bigcup_{k=1}^{\infty} D_k$, then one can solve a suitable Dirichlet problem (2.4) for each $D_k$ and let $k \to \infty$. Then we have the short time existence of the mean curvature flow on the entire $\mathbb{R}^2$.

**Theorem 3.2.** Let $\Sigma$ be an $C^{2,\alpha}$ entire Lagrangian graph in $\mathbb{R}^4$ with bounded curvature. Then there exists a $T > 0$ such that (1.1) have a smooth solution on the time interval $[0, T)$ if $f$ has bounded gradient.

4. **Preserve Lagrangian Condition**

In this section we show that the Lagrangian condition is preserved along the mean curvature flow as long as the solution exists smoothly. We can apply the maximum principle in [EH1] by using the backward heat kernel on complete manifold. However, we will use a method of Hamilton [Ha] to get this result. First we need the following Lemma.

**Lemma 4.1.** ([Ha, Lemma 5.2.]) Given any constant $C_3 > 0$, any $\tau > 0$, and any compact set $K$ in space-time we can find a function $\psi = \psi(x, t)$ depending on space and time such that

1. $\psi \leq \tau$ on the set $K$, and $\psi \geq \gamma$ for some $\gamma > 0$,

2. $\psi(x, t) \to \infty$ as $x \to \infty$,

3. $(\frac{d}{dt} - \Delta)\psi > C_3 \psi$.

**Theorem 4.2.** $\Sigma_\tau$ is an entire Lagrangian graph as long as the solution of (1.1) exists.

**Proof.** It suffices to show that the Lagrangian condition is preserved. We fix a time interval $[0, T]$ and so there exists a constant $C_4$ so that $|A|^2 \leq C_4$ on $[0, T]$. Let $\zeta = *(\omega_1 - \omega_2)$, then $\zeta$ satisfies the evolution equation $(\frac{d}{dt} - \Delta)\zeta = (2 |A|^2 - |H|^2) \zeta$. 

We show that $\zeta = 0$ on $[0, T]$. Consider the function $f_\varepsilon := \zeta - \varepsilon e^{2C_1} \psi$, where $\varepsilon$ is an arbitrary positive constant, $C$ is a positive constant large than $C_4$ and $\psi$ is as in Lemma 4.1. Obviously $f_\varepsilon(0) \leq -\varepsilon \gamma < 0$ and

$$\frac{d}{dt} f_\varepsilon \leq \Delta f_\varepsilon + 2 |A|^2 - |H|^2 \zeta - (2C + C_3) \varepsilon e^{2C_1} \psi < \Delta f_\varepsilon + C_5 f_\varepsilon.$$ 

Thus by the parabolic maximum principle we get that $f_\varepsilon < 0$ for all $t \in [0, T]$ and all positive $\varepsilon$. Then we let $\varepsilon$ tend to zero and obtain the inequality $\zeta \leq 0$ on $[0, T]$. Since $\zeta \geq 0$ this shows that $\zeta = 0$ on $[0, T]$.

5. Interior gradient estimate

In this section, we shall adapt the proof of Korevaar [Ko] to get the interior gradient estimate in higher codimension case. The same method has been applied for mean curvature flow of hypersurfaces by Ecker-Huisken [EH2].

**Theorem 5.1.** Suppose that $\Sigma_t = \text{graph} f(\cdot,t)$ is a solution of the mean curvature flow (2.4) over a ball $B_R(x_0) \subset \mathbb{R}^2$ for $0 \leq t \leq T$. Then the gradient of $f = (f^1, f^2)$ satisfies the estimate

$$\sqrt{\text{det}\left[I + (df)^T(x_0,t)df(x_0,t)\right]} \leq C_6 \sup_{B_R(x_0)} \sqrt{\text{det}\left[I + (df_0)^T df_0\right]} \times \exp\left[C_7 R^{-2} \max_i \left( \sup_{B_R(x_0) \times [0,T]} \sup_{B_R(x_0)(x,t)} f^i(x,t) - f^i(x_0,t)\right)^2 \right],$$

where $0 \leq t \leq T$, and $f_0$ denotes the initial area-preserving diffeomorphism.

**Proof.** Let us first assume $f = (f^1, f^2) : B_1 \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$ satisfies $f^i \leq 0$ for each $i$ and for $t \in [0, T]$. Let $\xi = \xi(x,t)$ be a nonnegative smooth function which for each $t \in [0, T]$ vanishes outside some compact subset of $B_1 \times \{ y^i < 0 \}$. Now we consider the function $u\xi$, where $u = \eta^{-1}$ and $\eta = \ast \omega$. Suppose the function reaches a positive maximum value, which is large than $\max u\xi$ in $B_1$ at $t = 0$, in $B_1$ for the first time at $0 < t_0 \leq T$. At the maximum point $p$ at time $t_0$, we have

$$\nabla (u\xi) = 0$$

(5.1)
and

\begin{equation}
\left(\frac{d}{dt} - \Delta\right)(v \xi) \geq 0.
\end{equation}

In view of the evolution equation for $v$ (2.1), we obtain from (5.1) and (5.2) the inequality

\begin{equation}
0 \leq v \left(\frac{d}{dt} - \Delta\right)\xi - \left[2|A|^2 - |H|^2\right] v \xi,
\end{equation}

where we used the relation

\[ -2\nabla v \cdot \nabla \xi = -2v^{-1} \nabla v \cdot \nabla (v \xi) + 2v^{-1} |\nabla v|^2 \xi = 2v^{-1} |\nabla v|^2 \xi. \]

Since $v$ and $\xi$ are nonnegative we therefore conclude

\begin{equation}
\left(\frac{d}{dt} - \Delta\right)\xi \geq 0
\end{equation}

at a maximum point $p$ of $v \xi$.

Let now $\xi = -1 + \exp(\mu \phi)$ where the function $\phi$ will be chosen later and where $\mu > 0$. From (5.3) we infer

\begin{equation}
\left(\frac{d}{dt} - \Delta\right)\phi \geq \mu |\nabla \phi|^2.
\end{equation}

By choosing coordinates, we may assume at $p$, $\lambda_2 \leq \lambda_1 = \partial f^1/\partial x^1$. Set

\[ \phi = \left(\frac{1}{2\beta} y^1 + 1 - |X|^2\right)^+, \]

where $\beta > 0$ is to be determined. On the set where $\phi$ is positive we compute

\begin{equation}
\nabla \phi = \frac{1}{2\beta} \nabla y^1 - \nabla |X|^2
\end{equation}

\begin{equation}
|\nabla \phi|^2 = \frac{1}{4\beta^2} |\nabla y^1|^2 + |\nabla |X|^2|^2 - \frac{1}{\beta} \nabla y^1 \cdot \nabla |X|^2
\end{equation}

\begin{equation}
\left(\frac{d}{dt} - \Delta\right)\phi = \frac{1}{2\beta} \left(\frac{d}{dt} - \Delta\right) y^1 - \left(\frac{d}{dt} - \Delta\right) |X|^2
\end{equation}

From (2.2) and (2.4) we get

\begin{equation}
\left(\frac{d}{dt} - \Delta\right)\phi = 4.
\end{equation}

Substituting (5.5) and (5.6) into (5.4) we arrive at the inequality

\begin{equation}
\mu \left(\frac{1}{4\beta^2} |\nabla y^1|^2 - \frac{1}{\beta} \nabla y^1 \cdot \nabla |X|^2\right) \leq 4.
\end{equation}
Now we derive
\[
\frac{1}{4\beta^2} |\nabla y^j|^2 - \frac{1}{\beta} \nabla y^j \cdot \nabla |x|^2
= \frac{1}{4\beta^2} \sum_{i,j} \frac{g^{ij}}{2} \frac{\partial y^k}{\partial x^i} \frac{\partial y^l}{\partial x^j} - \frac{1}{\beta} \sum_{i,j} g^{ij} \frac{\partial |x|^2}{\partial x^i} \frac{\partial |x|^2}{\partial x^j}
\geq \frac{1}{4\beta^2} \frac{\lambda_1^2}{1 + \lambda_1^2} - \frac{2}{\beta} \frac{\lambda_1}{1 + \lambda_1^2},
\]
where we used $g^{ij} = \frac{1}{1 + \lambda_1^2} \delta_{ij}$ at $p$. Therefore we deduce from (5.7) that
\[
\frac{1}{4\beta^2} \lambda_1^2 - \frac{2}{\beta} \lambda_1 \leq \frac{4}{\mu} (1 + \lambda_1^2).
\]
Choose $\mu = 32\beta^2$. Then at the maximum point $p$ we have
\[
\lambda_1 \leq 1 + 16\beta
\]
and so
\[
v = \sqrt{(1 + \lambda_1^2)(1 + \lambda_2^2)} \leq 1 + (1 + 16\beta)^2.
\]
Hence
\[
v \xi(0,0) \leq \max_{B_1} v \xi \leq (2 + 16\beta)^2 e^\mu = (2 + 16\beta)^2 e^{32\beta^2}
\]
where we used the fact that $\phi \leq 1$. This proves that for all $t \in [0, T]$ we have the estimate
\[
v \xi \leq \sup_{B_1} v \xi(\cdot, 0) + (2 + 16\beta)^2 e^{32\beta^2}.
\]
At the point $x = 0$ in the reference plane $\mathbb{R}^2$ we obtain in equivalent to (5.9) for $t \in [0, T]$ and arbitrary $\beta > 0$
\[
[e^{32\beta^2}(\max_{B_1(0)} y^1(0,t) + 1)^4 - 1] \sqrt{\det[I + (df)^t(0,t)df(0,t)]} 
\leq e^{32\beta^2} \left( \sup_{B_1(0)} \sqrt{\det[I + (df_0)^t df_0]} + (2 + 16\beta)^2 \right).
\]
Now choose $\beta = \sup_{t \in [0, T]} - y^j(0, t)$. We then infer from (5.10)
\[
\sqrt{\det[I + (df)^t(0,t)df(0,t)]} \leq C_6 \sup_{B_1(0)} \sqrt{\det[I + (df_0)^t df_0]} 
\cdot \exp[C_7 \sup_{t \in [0, T]} (-y^j(0,t))^2].
\]
To achieve the condition $y^j < 0$ on $B_1(0)$ we replace $y^j$ by $y^j - \sup_{B_1(0) \times [0, T]} y^j(x,t)$ and the estimate on $B_R(x_0)$ is then obtained by scaling and translating.

6. Long time existence

In this section we prove the long time existence by using the Wang’s blow-up analysis as in [W1, W3].

**Theorem 6.1.** Let $\Sigma$ be an $C^{2,\alpha}$ entire Lagrangian graph in $\mathbb{R}^4$ with the same assumptions as in Theorem 3.2. Then the mean curvature flow (1.1) has a smooth solution for all time if we have the bounded gradient.

**Proof.** We begin from the inequality

$$\left(\frac{d}{dt} - \Delta\right)\eta = \eta \left[2 |A|^2 - |H|^2 \right] \geq \frac{2}{3} |A|^2 \eta,$$

since $|H|^2 \leq \frac{4}{3} |A|^2$. Then the maximum principle for parabolic equations implies $\min_{\Sigma_t} \eta$ is nondecreasing in time. In particular, $\eta$ has a positive low bound $C_0$. Since $\eta$ is the Jacobian of the projection map from $\Sigma_t$ to $\mathbb{R}^2$, by the implicit function theorem, $\Sigma_t$ remains the graph of a map $f_t : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ whenever the flow exists and each graph of $f_t$ with uniformly bounded $|df_t|$. This is because $\eta = 1/\sqrt{(1 + \lambda_1^2)(1 + \lambda_2^2)}$ has a positive low bound, so $(1 + \lambda_1^2)(1 + \lambda_2^2)$ is bounded and $(1 + \lambda_1^2)(1 + \lambda_2^2) \geq 1 + \lambda_1^2 + \lambda_2^2 = 1 + |df_t|^2$.

Now we apply the inequality

$$\left(\frac{d}{dt} - \Delta\right)\eta \geq \delta |A|^2,$$

for some constant $\delta > 0$, and the uniformly bounded $|df_t|$ to the argument in Proposition 6.1 of [W1] (see also Theorem A of [W3]) shows that $|A|^2$ vanishes on a parabolic blow-up limit, so White’s regularity theorem [Wh] gives $C^{2,\alpha}$ bound. All the higher order estimate of $|A|^2$ can be also obtained (see [S2]). Then the long time existence result holds.

\[\Box\]

**Appendix A.**

In [CLT] Chen, Li and Tian proved that, if the initial surface $\Sigma$ in $\mathbb{R}^4$ under the assumptions $\ast \omega_1 \geq C_0 > 1/\sqrt{2}$, then the mean curvature flow has a long time solution and the scaled surfaces converge to a expanding self-similar solution as time goes to
infinity. Here we show that in the case of entire Lagrangian graphs in $\mathbb{R}^4$ the only contracting self-similar solutions of the mean curvature flow are planes.

**Proposition A.1.** If $\Sigma$ is an entire Lagrangian graph in $\mathbb{R}^4$ of at most polynomial growth satisfying the equation

(A.1) \[ H = \langle X, \nu_k \rangle \nu_k, \]

then $\Sigma$ is a plane.

**Proof.** Let $\Sigma$ be the entire Lagrangian graph of an area-preserving diffeomorphism $f$ from $\mathbb{R}^2$ to $\mathbb{R}^2$. Let $\eta = \ast \omega_1$, where $\omega_1$ be the standard area-form on the reference plane $\mathbb{R}^2$. From (A.1) we have $h_k = - \langle X, \nu_k \rangle$, then we compute to yield

$$\nabla_{e_i} h_k = h_{kij} \langle X, e_j \rangle$$

and thus

$$\Delta \eta = -\eta \left[ 2 \left| A \right|^2 - \left| H \right|^2 \right] + \nabla_k \eta \langle X, e_k \rangle \cdot$$

Let $v = \eta^{-1}$ we have

$$\Delta v = v \left[ 2 \left| A \right|^2 - \left| H \right|^2 \right] + \nabla_k v \langle X, e_k \rangle + 2v^{-1} \left| \nabla v \right|^2.$$

We multiply this equation by $\rho = \exp(-|X|^2/2)$ which after integration by parts leads to

$$\int_{\Sigma} \left[ 2 \left| A \right|^2 - \left| H \right|^2 \right] \rho d\mu + 2 \int_{\Sigma} v^{-1} \left| \nabla v \right|^2 \rho d\mu = 0,$$

thus implying the result.

\[ \blacksquare \]

**References**


[W4] ——, *The Dirichlet problem for the minimal surface system in arbitrary codimension*, CPAM.


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