Eigenvalues of the Laplacian acting on functions of mean zero with constant boundary values

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Abstract.

We investigate the eigenvalues of the Laplace operator in the space of functions of mean zero and having a constant (unprescribed) boundary value. The first eigenvalue of such problem lies between the first two eigenvalues of the Laplacian with Dirichlet boundary conditions and satisfies an isoperimetric inequality: in the class of open bounded sets of prescribed measure, it becomes minimal for the union of two disjoint balls having the same radius. Existence of an optimal domain in the class of convex sets is also discussed.

1. Introduction.

The classical Faber-Krahn inequality [1, 4, 7, 11] gives a lower bound on the first eigenvalue of the Laplacian with homogeneous Dirichlet boundary condition in the class of bounded open sets of prescribed measure. More precisely, given a bounded open set $\Omega \subset \mathbb{R}^N$, $N \geq 1$, denote by $\lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \ldots$ the eigenvalues of the problem

$$-\Delta u = \lambda u \text{ in } \Omega, \quad u \in H_0^1(\Omega),$$

(1.1)

repeated according to multiplicity. Then, by considering a ball $\Omega^*$ having the same measure as $\Omega$, the following holds:

$$\lambda_1(\Omega^*) = \left( \frac{\omega_N}{|\Omega|} \right)^{\frac{2}{N}} j_{\frac{N-2}{2}, 1}^N \leq \lambda_1(\Omega),$$

(1.2)
where \( \omega_N \) denotes the measure of the unit ball in \( \mathbb{R}^N \), \( \| \cdot \| \) the Lebesgue measure in \( \mathbb{R}^N \) and \( j_{m,k} \) the \( k \)-th positive zero of the Bessel function \( J_m \). Properties of such functions are found in [5, 16]. By applying this inequality to the nodal domains of the second eigenfunction, Krahn [11] also derived a bound from below for the second eigenvalue:

\[
\lambda_2(B) = \left( \frac{2\omega_N}{|\Omega|} \right)^{\frac{2}{N}} j_{N-2,1}^2 \leq \lambda_2(\Omega),
\]

where \( B \) is the union of two disjoint balls of equal radius, such that \( |B| = |\Omega| \). In this paper we investigate the eigenvalues \( \Lambda_k(\Omega) \) of the linear problem:

\[
-\Delta u = \Lambda u \text{ in } \Omega, \quad \int_{\Omega} u = 0, \quad u - c \in H^1_0(\Omega),
\]

where \( c \) is a non-prescribed constant. In particular, we prove that the first eigenvalue \( \Lambda_1(\Omega) \) satisfies an isoperimetric inequality:

**Theorem 1.1.** For every \( N \geq 1 \), the union \( B \) of two disjoint balls of equal radius yields the smallest \( \Lambda_1 \) among all bounded open sets \( \Omega \subset \mathbb{R}^N \) of prescribed Lebesgue measure. More precisely,

\[
\Lambda_1(B) = \left( \frac{2\omega_N}{|\Omega|} \right)^{\frac{2}{N}} j_{N-2,1}^2 \leq \Lambda_1(\Omega).
\]

In the special case \( N = 2 \), inequality (1.5) was proved in [13] in connection with some mean field type equation. More specifically, the nonlinear problem

\[
-\Delta u = \lambda \left( e^u \int_{\Omega} e^u - \frac{1}{|\Omega|} \right), \quad u \in H^1_0(\Omega), \quad \Omega \subset \subset \mathbb{R}^2,
\]

is the Euler-Lagrange equation for \( I^\lambda(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \lambda \log (|\Omega|^{-1} \int_{\Omega} e^u) \). One can check that \( u \equiv 0 \) is a critical point and that at this point the corresponding linearized problem is given by \( -\Delta u = \lambda (u - |\Omega|^{-1} \int_{\Omega} u) \), \( u \in H^1_0(\Omega) \). Such a linear problem is actually equivalent to (1.4) and a simple calculation shows that \( u \equiv 0 \) is a local minimizer of \( I^\lambda \) whenever \( \lambda < |\Omega| \Lambda_1(\Omega) \). This geometry has been exploited in [13] to derive existence of non-trivial solutions when the parameter belongs to the interval \((8\pi, |\Omega| \Lambda_1(\Omega))\). The main feature of this result is that such an interval is never empty because the Faber-Krahn type inequality (1.5) with \( N = 2 \) shows that \((8\pi < 2\pi j_{0,1}^2 \leq |\Omega| \Lambda_1(\Omega))\).

Several other problems of relevance in different fields have led to consider “exotic” eigenvalues. For example, minimizing the Rayleigh’s quotient in \( H^1_{0,T}(\Omega) \), the space of \( H^1_0(\Omega) \)-functions whose integral vanishes, leads to the first twisted eigenvalue \( \lambda^T_1(\Omega) \). This is actually an eigenvalue of the non-local problem \( -\Delta u = \lambda u - |\Omega|^{-1} \int_{\Omega} \Delta u \), with \( u \in H^1_{0,T} \). For such eigenvalues, Freitas and Henrot derived in [8] the following interesting isoperimetric inequality:

\[
\lambda^T_1(B) = \left( \frac{2\omega_N}{|\Omega|} \right)^{\frac{2}{N}} j_{N-2,1}^2 \leq \lambda^T_1(\Omega).
\]
The aim of the present work is twofold. On the one hand, we shall extend the inequality derived in [13] to any dimension and investigate more in details the structure of the eigenvalues of Problem (1.4). Moreover, we will also compare the eigenvalues of (1.4) with the twisted eigenvalues studied in [2]. Denoting by $U(\Omega)$ the space of the admissible functions:

$$U(\Omega) = \left\{ v \in H^1(\Omega) : \int_{\Omega} v = 0 \text{ and } v - c \in H^1_0(\Omega) \text{ for some } c \in \mathbb{R} \right\},$$

we start with the observation that the first eigenvalue $\Lambda_1(\Omega)$ satisfies:

$$\Lambda_1(\Omega) = \min_{v \in U(\Omega)} R(v),$$

where $R(v)$ denotes the Rayleigh’s quotient $R(v) = \int_{\Omega} |\nabla v|^2 / \int_{\Omega} v^2$ of the function $v$. The existence of a minimizer $u$ to (1.9) follows from standard arguments. To check that $u$ indeed solves (1.4) it suffices to apply the classical method of the calculus of variations, taking into account that the admissible variations have constant (not necessarily zero) boundary value. In order to find an open set $\Omega$ that minimizes $\Lambda_1(\Omega)$ among all bounded open set of prescribed measure, we consider a minimizer $u$ of (1.9) in an arbitrary, bounded open set $\Omega$. Then, we apply a Schwarz symmetrization to both open sets $\{ u > c \}$ and $\{ u < c \}$. This procedure transforms $\Omega$ into the union of two disjoint balls, and does not increase $\Lambda_1$. Finally, we prove that two congruent balls yield the least $\Lambda_1$ among all couples of balls of prescribed measure.

Let us explain how our eigenvalue problem is related with the Dirichlet and the twisted eigenvalues. It was proved in [2] that $\lambda^T_k(\Omega) \leq \lambda_{k+1}(\Omega)$. Since $H^1_{0,T}(\Omega)$ is a subset of $U(\Omega)$ we clearly have $\Lambda_1 \leq \lambda^T_1(\Omega)$. Actually our results will show that the following inequalities hold for every $k$:

$$\lambda_k(\Omega) \leq \Lambda_k(\Omega) \leq \lambda^T_k(\Omega) \leq \lambda_{k+1}(\Omega).$$

In particular, our Theorem 1.1 implies both isoperimetric inequalities (1.3) and (1.7). Conditions implying that the strict inequalities $\lambda_1(\Omega) < \Lambda_1(\Omega)$, $\Lambda_k(\Omega) < \lambda_{k+1}(\Omega)$ hold and some examples are also provided.

The paper is organized as follows. In Section 2, we investigate the basic properties of the eigenvalue problem (1.4) and prove the inequalities (1.10). In Section 3 (respectively, 4), we study in detail the case when the domain of the problem is a ball (the union of two disjoint balls). Then, the isoperimetric inequality for the first eigenvalue $\Lambda_1(\Omega)$ is proved in Section 5. In the last section, we show that for each $k = 1, 2, \ldots$ there exists a convex domain $\Omega^k$ that minimizes $\Lambda_k(\Omega)$ in the class of convex domains of prescribed measure.

2. Existence and properties of eigenvalues.

Throughout the paper, $\Omega$ is a bounded open set of $\mathbb{R}^N$ with $N \in \mathbb{Z}^+$ (the set of positive integers). Given a subspace $X$ of $H^1(\Omega)$ and $X'$ its topological dual, we denote by
The bilinear form 

\[
\min - \max
\]

The system 

Homogeneity: 

The eigenvalues of 

Monotonicity: if 

The 

Proposition 2.1. (Completeness of \( \mathcal{U}(\Omega) \)). The bilinear form \( (f, g) = \int_{\Omega} \nabla f \cdot \nabla g \) defines an inner product on \( \mathcal{U}(\Omega) \) whose associated norm is equivalent to the usual \( H^1 \)-norm. With respect to this inner product, the mapping (2.1) is an orthogonal isomorphism and the space \( \mathcal{U}(\Omega), \langle \cdot, \cdot \rangle \) is a Hilbert space.

Proof. Since \( \Omega \) is not necessarily connected, a Poincaré’s inequality is not available for \( H^1 \)-function of mean zero. But it is recovered with the additional condition on the functions to be constant on the boundary. Indeed, let \( u \in \mathcal{U}(\Omega) \) and denote by \( c \in \mathbb{R} \) its boundary value. Then, since \( \int_{\Omega} u = 0 \) and \( u - c \in H^1_0(\Omega) \), we derive:

\[
R(u) = \frac{\int_{\Omega} |\nabla u|^2}{\int_{\Omega} u^2} \geq \frac{\int_{\Omega} |\nabla (u-c)|^2}{\int_{\Omega} (u-c)^2} \geq \lambda_1(\Omega) > 0,
\]

where \( \lambda_1(\Omega) \) stands for the first eigenvalue of \( (-\Delta, H^1_0(\Omega)) \). This implies the first claim. To conclude, we note that \( p \) is clearly an orthogonal isomorphism and therefore \( \mathcal{U}(\Omega) \) inherits automatically the Hilbert structure from \( H^1_0(\Omega) \).

Concerning the natural properties that we may expect from the eigenvalue problem (1.4), it is clear that a necessary condition for a function \( f \in L^2(\Omega) \) to be expressed as sum of eigenfunctions is \( \int_{\Omega} f = 0 \). It turns out that such condition is also sufficient. This and other basic results are collected in the next proposition.

Proposition 2.2. (Basic properties).

(a) The eigenvalues of \( (-\Delta, \mathcal{U}(\Omega)) \) form an unbounded sequence \( 0 < \Lambda_1 \leq \Lambda_2 \leq \ldots \), and there exists a Hilbert basis of \( \dot{L}(\Omega) = \{ f \in L^2(\Omega) : \int_{\Omega} f = 0 \} \) made of corresponding eigenfunctions \( \Phi_k \).

(b) The system \( (\Phi_k)_{k \in \mathbb{Z}^+} \) is also complete in \( \mathcal{U}(\Omega) \).

(c) Min-max property:

\[
\Lambda_k = \min_{V^k \subset \mathcal{U}(\Omega)} \max_{v \in V^k \setminus \{0\}} R(v),
\]

where \( V^k \) is an arbitrary \( k \)-dimensional subspace of \( \mathcal{U}(\Omega) \).

(d) Monotonicity: if \( \Omega' \) is a bounded open subset of \( \mathbb{R}^N \) such that \( \Omega \subset \Omega' \), then \( \Lambda_k(\Omega') \leq \Lambda_k(\Omega) \) for all \( k \in \mathbb{Z}^+ \).

(e) Homogeneity: \( \Lambda_k(t \Omega) = t^{-2} \Lambda_k(\Omega) \) for every \( t > 0 \), \( k \in \mathbb{Z}^+ \).

(f) The \( \Lambda_k \) are related to the eigenvalues \( \lambda_k \) of the Dirichlet-Laplacian \( (-\Delta, H^1_0(\Omega)) \), as well as to the twisted eigenvalues \( \lambda_k^T \), as follows:

\[
\lambda_k(\Omega) \leq \Lambda_k(\Omega) \leq \lambda_k^T(\Omega) \leq \lambda_{k+1}(\Omega), \quad k = 1, 2, \ldots
\]
Proof. (a) Observe that for each \( f \in L^1(\Omega) \) there exists a unique weak solution \( u \in U(\Omega) \) to \(-\Delta u = f\). Moreover, its resolvent \( T: L^1(\Omega) \to L^1(\Omega) \) is given by

\[
T f = T_0 f - |\Omega|^{-1} \int_\Omega T_0 f,
\]

where \( T_0 \) denotes the resolvent of the same equation in \( H_0^1(\Omega) \). So \( T = T_0 \circ p^{-1} \), where \( p^{-1} \) stands for the inverse of the orthogonal isomorphism (2.1). Since \( T_0 \) is compact, self-adjoint and \( p^{-1} \) continuous, we immediately obtain that \( T \) is a compact self-adjoint operator. The claim (a) follows now by applying the Hilbert-Schmidt theorem.

(b) Take \( u \in U(\Omega) \) such that \( \int_\Omega \nabla u \cdot \nabla \Phi_k = 0 \) for every \( k \), and denote by \( c \) its boundary value. Since \( \int_\Omega \Phi_k = 0 \), we have \( \int_\Omega \nabla u \cdot \nabla \Phi_k = \int_\Omega \nabla (u - c) \cdot \nabla \Phi_k = \Lambda_k \int_\Omega (u - c) \Phi_k = \Lambda_k \int_\Omega u \Phi_k \). Since the \( \Phi_k \) are complete in \( L^1(\Omega) \), \( u \) must be identically zero and the conclusion follows.

(c) Consider \( V_0 = \text{span}(\Phi_1, \ldots, \Phi_k) \), and observe that the Rayleigh’s quotient of an arbitrary \( v = \sum v_i \Phi_i \in V_0 \setminus \{0\} \) is the convex combination \( R(v) = (\sum v_i^2)^{-1} \sum v_i^2 \Lambda_i \). Hence, \( \max_{v \in V_0 \setminus \{0\}} R(v) = \Lambda_k \). Furthermore, if \( V^k \neq V_0 \) then there exists a non-trivial \( v_0 \in V^k \cap V_0^\perp \). Since the first \( k \) coefficients of the Fourier expansion of \( v_0 \) with respect to the system \( \{ \Phi_j \}_{j \in \mathbb{N}} \) vanish, we deduce that \( \max_{v \in V^k \setminus \{0\}} R(v) \geq R(v_0) \geq \Lambda_{k+1} \), and (2.3) follows.

(d) Consider the linear extension operator \( L \) that takes \( v \in U(\Omega) \), whose boundary value is denoted by \( c \), onto \( w \in U(\Omega') \),

\[
w(x) = \begin{cases} 
  v(x) - c + c |\Omega'|^{-1} |\Omega| & \text{if } x \in \Omega, \\
  c |\Omega'|^{-1} |\Omega| & \text{if } x \in \Omega' \setminus \Omega,
\end{cases}
\]

and observe that \( w \in U(\Omega) \). Furthermore, if \( v \neq 0 \) then \( w \neq 0 \) and by computation we find \( R(w) \leq R(v) \). Let \( V_0 \) be as in the proof of part (c). We have:

\[
\max_{w \in L(V_0) \setminus \{0\}} R(w) \leq \max_{v \in V_0 \setminus \{0\}} R(v) = \Lambda_k(\Omega)
\]

and monotonicity follows from the min-max property.

(e) For every \( v \in U(\Omega) \), let \( v_t \in U(t \Omega) \) be defined by \( v_t(x) = v(t^{-1} x) \). Since the equation \(-\Delta v = \Lambda_k v \) in \( U(\Omega) \) is equivalent to \(-\Delta v_t = t^{-2} \Lambda_k v_t \) in \( U(t \Omega) \), the claim follows.

(f) To prove that \( \lambda_k(\Omega) \leq \Lambda_k(\Omega) \), consider the projection \( p \) defined by (2.1). If \( v \neq 0 \) then using \( \int_\Omega v = 0 \) we find:

\[
R(p(v)) = \frac{\int_\Omega |\nabla (v - c)|^2}{\int_\Omega (v - c)^2} = \frac{\int_\Omega |\nabla v|^2}{|\Omega| c^2 + \int_\Omega v^2} \leq R(v), \tag{2.6}
\]

and the inequality \( \lambda_k(\Omega) \leq \Lambda_k(\Omega) \) follows from the min-max property. To go further, observe that every \( k \)-dimensional subspace of \( H_0^1(\Omega) \) is also a subspace of \( U(\Omega) \) (still
of dimension $k$). This and the min-max property imply $\Lambda_k(\Omega) \leq \lambda_k^T(\Omega)$. As observed in [2], the inequality $\lambda_k^T(\Omega) \leq \lambda_{k+1}(\Omega)$ is also a consequence of the min-max property: indeed, it suffices to observe that every $k+1$-dimensional subspace $V^{k+1} \subset H_0^1(\Omega)$ has a $k$-dimensional subspace $V^k$ included in $H_0^1, T$, and the conclusion follows. \hfill $\Box$

Let us investigate some cases when the strict inequalities $\lambda_1(\Omega) < \Lambda_1(\Omega)$ and $\Lambda_k(\Omega) < \lambda_{k+1}(\Omega)$ hold.

**Proposition 2.3. (Strict inequalities).** If $\Omega$ is connected then $\lambda_1(\Omega) < \Lambda_1(\Omega)$. If, in addition, the eigenspace of $(-\Delta, H^1_0(\Omega))$ associated to $\lambda_{k+1}(\Omega)$ contains an eigenfunction whose mean does not vanish, then $\Lambda_k(\Omega) < \lambda_{k+1}(\Omega)$.

**Proof.** Let $u$ be a minimizer of (1.9), and denote by $c$ its boundary value. Letting $v = u$ in (2.6) we see that if $\lambda_1(\Omega) = \Lambda_1(\Omega)$ then $R(p(u)) = R(u)$. Hence, $c = 0$ and $u \in H^1_0(\Omega)$ is an eigenfunction to (1.1) corresponding to the eigenvalue $\lambda_1(\Omega)$. Since $\int_\Omega u = 0$, the function $u$ changes sign and therefore $\Omega$ is not connected.

To complete the proof, assume that $\Omega$ is connected. If $\lambda_{k+1}$ is a multiple eigenvalue, then, without altering its value, we reduce the index $k$, if necessary, until $\lambda_k < \lambda_{k+1}$. This is always possible because $\lambda_1$ is simple. Choose orthogonal eigenfunctions $\varphi_1, \ldots, \varphi_{k+1}$ to $(-\Delta, H^1_0(\Omega))$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_{k+1}$, respectively, and such that $\int_\Omega \varphi_{k+1} \neq 0$. Let $\varphi_i$ be normalized in the sense that $\|\varphi_i\|_{L^2} = 1$, $i = 1, \ldots, k+1$. Consider the $k+1$-dimensional subspace $V^{k+1} = \text{span}(\varphi_1, \ldots, \varphi_{k+1}) \subset H^1_0(\Omega)$, and observe that $\max_{v \in V^{k+1}\{0\}} R(v) = \lambda_{k+1}$. Furthermore, since $\lambda_k < \lambda_{k+1}$, we have $R(v) < \lambda_{k+1}$ for every $v \in V^{k+1}\{0\}$ not proportional to $\varphi_{k+1}$. Since $\int_\Omega \varphi_{k+1} \neq 0$, the $k$-dimensional subspace $V^k = \{ v \in V^{k+1} : \int_\Omega v = 0 \} \subset \mathcal{U}(\Omega)$ does not contain any multiple of $\varphi_{k+1}$ apart from $v \equiv 0$, and therefore $\max_{v \in V^{k+1}\{0\}} R(v) < \lambda_{k+1}$. The conclusion follows from the min-max principle. \hfill $\Box$

**Remark 2.4.**

i) The case $\Omega = \text{ball}$, treated in detail in the next section, shows that the first eigenvalue $\lambda_1(\Omega)$ is in general not simple.

ii) The assumption $\int_\Omega \varphi_2 \neq 0$ cannot be dropped when proving $\Lambda_1 < \lambda_2$: for instance, if $\Omega = \text{ball}$ then $\int_\Omega \varphi_2 = 0$ and $\Lambda_1 = \lambda_2$.

iii) A connected domain $\Omega$ possessing a second eigenfunction $\varphi_2$ with non-vanishing integral is the circular sector $\Omega = \{ (\rho, \theta) \mid 0 < \rho < 1, 0 < \theta < \pi/8 \}$, where $(\rho, \theta)$ are polar coordinates. In such a case $\lambda_2(\Omega)$ is simple and the corresponding eigenfunctions are multiple of $J_4(j_{4,2}\rho) \sin 4\theta$, whose integral over $\Omega$ is positive.

iv) The strict inequalities $\lambda_1(\Omega) < \Lambda_1(\Omega) < \lambda_2(\Omega)$ may hold even if $\Omega$ is disconnected: an example is $\Omega = B_1 \cup B_2$ where $B_1, B_2$ are disjoint balls of radii $R_1 \neq R_2$. If, instead, $R_1 = R_2$ then the equality $\lambda_1(\Omega) = \Lambda_1(\Omega) = \lambda_2(\Omega)$ is attained (see Section 4).
3. Eigenvalues in a ball.

In this section we determine the eigenvalues of Problem (1.4) when \( \Omega \) is a ball centered at the origin and of radius \( R > 0 \) that will be denoted by \( B \). It is remarkable that in such case the spectrum of Problem (1.4) can be recovered from the Neumann as well as the Dirichlet spectrum of the Laplacian. In particular, the eigenspace associated to \( \Lambda_1(B) \) is spanned by the first radial Neumann eigenfunction and the eigenfunctions associated to the second Dirichlet eigenvalue.

If \( N = 1 \), we check easily that Problem (1.4) is equivalent to finding the periodic eigenfunctions of the Dirichlet-Laplacian. Then the system of eigenfunctions \((\sigma_i, \varphi_i)\) in \( (-\Delta, H^0(B)) \) of period 2 in the functions \( \cos(\frac{k\pi}{R} x) \) and \( \sin(\frac{k\pi}{R} x) \). So what follows is of real interest for \( N \geq 2 \). To handle higher dimensions, we introduce the following spaces of radial functions:

\[
L^r(B) := \{ u \in L^2(B) : u \text{ radial} \}, \quad \check{L}^r(B) := \{ u \in L^r(B) : \int_B u = 0 \}. \tag{3.1}
\]

Furthermore by setting \( \check{H}(\Omega) \) to be the space of \( H^1(\Omega) \)-functions of mean zero, we also define

\[
H^r_0(B) := H^1_0(B) \cap L^r(B), \quad \check{H}^r(B) := \check{H}(B) \cap L^r(B). \tag{3.2}
\]

The spaces defined in (3.1) and respectively (3.2) are Hilbert spaces when endowed with the scalar product \( (u, v) \mapsto \int_B uv \) and respectively \( (u, v) \mapsto \int_B \nabla u \cdot \nabla v \). We shall denote by \( L^{n-r} : (\check{H}^r(B), \check{H}^r(B)) \) the orthogonal of \( L^r(B) \) in \( L^2(B) \) and by \( H^r_0(B) \) the orthogonal of \( H^r_0(B) \) in \( H^r_0(B) \). Both of these spaces are obviously made of non-radial functions. Classical arguments, already used to derive parts (a) and (b) of Proposition 2.2, yield

**Lemma 3.1.** Let \( B = B(0, R) \subset \mathbb{R}^N \). Denote by \((X, Y)\) any pair of the spaces \((H^r_0(B), L^r(B)), (H^{n-r}_0(B), L^{n-r}(B))\) or \((\check{H}^r(B), \check{L}^r(B))\). Then the eigenvalues of \((-\Delta, X)\) form an unbounded sequence \( 0 < \rho_1 \leq \rho_2 \leq \ldots \) and the corresponding eigenfunctions provide an orthogonal basis of \( X \) and also of \( Y \).

In order to give a complete system of eigenfunctions of \((-\Delta, \mathcal{U}(B))\), consider:

(1) the sequence of eigenvalues \( \mu^r = (\mu^r_k)_{k \in \mathbb{Z}^+} \) and corresponding complete system of eigenfunctions \((\psi^r_k)_{k \in \mathbb{Z}^+}\) of \((-\Delta, H^r(B))\) ("radial spectrum" of the Neumann-Laplacian);

(2) the sequence of eigenvalues \( \lambda^{n-r} = (\lambda^{n-r}_k)_{k \in \mathbb{Z}^+} \) and corresponding complete system of eigenfunctions \((\varphi^{n-r}_k)_{k \in \mathbb{Z}^+}\) of \((-\Delta, H^{n-r}_0(B))\) ("non-radial spectrum" of the Dirichlet-Laplacian).

Define the direct sum \( \sigma^1 \oplus \sigma^2 \) of two sequences \( \sigma^i = (\sigma^i_1, \sigma^i_2, \ldots) \), \( i = 1, 2 \), as the non-decreasing rearrangement of the sequence \( (\sigma^i_1, \sigma^i_2, \sigma^i_3, \ldots) \). We have:

**Proposition 3.2. (Value of \( \Lambda_k(B) \)).** Let \( B = B(0, R) \subset \mathbb{R}^N \). Then

\[
\check{H}^r(B), H^{n-r}_0(B) \subset \mathcal{U}(B) \quad \text{and} \quad \check{H}^r(B) \oplus H^{n-r}_0(B) = \mathcal{U}(B), \tag{3.3}
\]
where $\oplus$ denote the orthogonal sum in the Hilbert space $\mathcal{U}(B)$. In particular, the full sequence of eigenvalues and a complete system of eigenfunctions of $(-\Delta, \mathcal{U}(B))$ is given by:

$$\mu^{ra} \oplus \lambda^{n-ra} \quad \text{and} \quad (\psi_1^{ra}, \psi_2^{ra}, \ldots, \varphi_1^{n-ra}, \varphi_2^{n-ra}, \ldots). \quad (3.4)$$

**Proof.** Any element of $\mathcal{H}^{ra}(B)$ has average zero (by definition) and is constant on $\partial B$ (because it is radial), so clearly $\mathcal{H}^{ra}(B) \subset \mathcal{U}(B)$. Let then $\varphi \in H_0^{n-ra}(B)$. Such a function is obviously constant on the boundary. To see that it has mean zero, we note that the eigenfunctions of $(-\Delta, \mathcal{H}^{ra}(B))$ span $L^{ra}(B)$ (see Lemma 3.1), and in particular they also span the constant function $u \equiv 1$. By orthogonality it follows that $\int_B \varphi = \int_B 1 \cdot \varphi = 0$. This prove the first claim in (3.3).

Consider now the isomorphism map $p : \mathcal{U}(B) \to H_0^{ra}(B)$ defined by (2.1). On the one hand, from the definition of $p^{-1}$ we get

$$p^{-1}(H_0^{n-ra}(B)) = H_0^{n-ra}(B) \quad \text{and} \quad p^{-1}(H_0^{ra}(B)) = H^{ra}(B). \quad (3.5)$$

We just emphasize that first of above equality follows from the property that any element of $H_0^{n-ra}(B)$ has mean zero. On the other hand, since $p^{-1}$ is an orthogonal map (see Prop. 2.1), we deduce that both spaces in (3.5) are closed and mutually orthogonal in $\mathcal{U}(B)$. So (3.3) follows and by applying Lemma 3.1 we also deduce (3.4).

**Remark 3.3.** Above proposition is important for the two-dimensional nonlinear problem (1.6) when the domain is a ball, since it allows to calculate the Leray-Schauder index of the trivial solution $u \equiv 0$ whenever $\lambda \neq \Lambda_k(B)$.

The sequences of eigenfunctions in (3.4) can be expressed explicitly, by using the Bessel functions $J_\nu$ as well as the hyperspherical harmonics, i.e., homogeneous polynomials of degree $d = 0, 1, \ldots$, that are harmonic on $\mathbb{R}^N$. It is well-known (see, for instance, [6, Vol. II, p. 237-240] and [14]) that the hyperspherical harmonics of degree $d$ form a linear space $\mathcal{Y}_d$ whose dimension is

$$\dim(\mathcal{Y}_0) = 1, \quad \dim(\mathcal{Y}_d) = \begin{cases} 
2 & \text{if } N = 2, d \geq 1, \\
\frac{(2d + N - 2)(d + N - 3)!}{(N - 2)! d!} & \text{if } N \geq 3, d \geq 1.
\end{cases} \quad (3.6)$$

For a fixed value of the dimension $N$, let us construct a sequence $(Y_m)$ of spherical harmonics as follows. For each value of the degree $d$, choose an orthonormal basis $(Y_j^{(d)})$, $j = 1, \ldots, \dim(\mathcal{Y}_d)$ in the linear space $\mathcal{Y}_d$. When the index $j$ ranges in the set of all admissible values, the elements $Y_j^{(d)}$ form a sequence that is denoted by $(Y_m)$, for shortness. The eigenvalues $\lambda_{m,k} = \lambda_{m,k}(B)$ to the Dirichlet-Laplacian in $B$, and a complete system of corresponding eigenfunctions $\varphi_{m,k}(x) = \varphi_{m,k}(B; x)$ may be given as follows:

$$\lambda_{m,k} = \left(\frac{J_{\deg(Y_m) + \frac{N-2}{2}, k}}{R}\right)^2, \quad \varphi_{m,k}(x) = \frac{J_{\deg(Y_m) + \frac{N-2}{2}}(\sqrt{\lambda_{m,k}} |x|)}{|x|^\frac{N-2}{2}} Y_m(x/|x|), \quad (3.7)$$
where \( \deg(Y_m) \) denotes the degree of the polynomial \( Y_m \), and \( k = 1, 2, \ldots \). Similarly, the positive eigenvalues \( \mu_{m,k} \) to the Neumann-Laplacian in \( B \), and a system \( \psi_{m,k}(x) = \psi_{m,k}(B; x) \) of corresponding eigenfunctions are:

\[
\mu_{m,k} = \left( \frac{J_{\deg(Y_m)+\frac{N}{2},k}}{R} \right)^2, \quad \psi_{m,k}(x) = \frac{J_{\deg(Y_m)+\frac{N}{2},k} \left( \sqrt{\mu_{m,k}} |x| \right)}{|x|^{\frac{N-2}{2}}} Y_m(x/|x|).
\]  (3.8)

From the above formula we deduce: The expressions (3.7) with \( \deg(Y_m) \geq 1 \) and (3.8) with \( m = 0 \) give explicitly both sequences in (3.4).

Note that \( \lambda_{m+1,k} = \mu_{m,k} \) for every \( m, k \). As a consequence, the eigenspace of \((-\Delta, \mathcal{U}(B))\) associated to the eigenvalue \( \mu_{0,k} \) is made of both “non-radial Dirichlet eigenfunctions” and a “radial Neumann eigenfunction”. This holds in particular for the first eigenvalue \( \Lambda_1(B) \), and leads to the following

**Proposition 3.4.** The first eigenvalue \( \Lambda_1(B) \) of Problem (1.4) coincides with the second eigenvalue \( \lambda_2(B) \) of the Dirichlet-Laplacian \((-\Delta, H^1_0(B))\) and also with the first positive eigenvalue \( \mu_1(B) \) of the Neumann-Laplacian \((-\Delta, H^1(B))\). Its explicit value is

\[
\Lambda_1(B) = \left( \frac{J_{\frac{N}{2},1}}{R} \right)^2,
\]  (3.9)

it has multiplicity \( N + 1 \), and the associated eigenspace is spanned by

\[
\frac{J_{\frac{N}{2}}(\sqrt{\Lambda_1} |x|)}{|x|^{-\frac{N-2}{2}}} \frac{x_i}{|x|}, \quad i = 1, \ldots, N, \quad \frac{J_{\frac{N-2}{2}}(\sqrt{\Lambda_1} |x|)}{|x|^{-\frac{N-2}{2}}}.
\]

### 4. The union of two disjoint balls.

We now turn our attention to Problem (1.4) when \( \Omega \) is the union of two disjoint balls \( B_1 = B(x_1, R_1) \) and \( B_2 = B(x_2, R_2) \) of \( \mathbb{R}^N \) with \( 0 < R_1 \leq R_2 \). In order to prove Theorem 1.1, we study in particular the first eigenvalue of \((-\Delta, \mathcal{U}(B_1 \cup B_2))\). Unlike the case of a single ball, we shall see that \( \Lambda_1(B_1 \cup B_2) \) is strictly less than the second Dirichlet eigenvalue, provided the two balls have different radii.

It is well known that if \( \Omega_1, \Omega_2 \) are two disjoint, bounded open sets of \( \mathbb{R}^N \), then the sequence \( \sigma(\Omega_1 \cup \Omega_2) \) of the eigenvalues to the Dirichlet-Laplacian \((-\Delta, H^1_0(\Omega_1 \cup \Omega_2))\) is given by \( \sigma(\Omega_1 \cup \Omega_2) = \sigma(\Omega_1) \oplus \sigma(\Omega_2) \), where \( \oplus \) has the meaning defined in the previous section. However, a similar result does not hold for Problem (1.4). Indeed, if we take an eigenfunction \( u_1 \) to \((-\Delta, \mathcal{U}(B_1))\), then setting \( u_2 \equiv 0 \) in \( B_2 \) yields an eigenfunction \((u_1, u_2)\) to \((-\Delta, \mathcal{U}(B_1 \cup B_2))\) if and only if \( u_1 \) is not radial, because the boundary values of \( u_1, u_2 \) have to agree. Furthermore, there are eigenfunctions \( u = (u_1, u_2) \) to \((-\Delta, \mathcal{U}(B_1 \cup B_2))\) such that neither \( u_1 \) is an eigenfunction to \((-\Delta, \mathcal{U}(B_1))\), nor \( u_2 \) is an eigenfunction to \((-\Delta, \mathcal{U}(B_2))\). An example is the first eigenfunction given below. Let us introduce the space:

\[
\mathcal{U}^a(B_1 \cup B_2) = \{ v \in \mathcal{U}(B_1 \cup B_2) : x \mapsto v_i(x + x_i) \text{ is radial in } B(0, R_i), \ i = 1, 2 \}.
\]
and consider also $H^{\text{ra}}_0(B_i)$, $H^{\text{ra}}_{n-\text{ra}}(B_i)$ ($i = 1, 2$) as defined in previous section. Similarly to what has been done in the previous section, let us consider

(i) the complete sequence of eigenvalues $\lambda^{\text{ra}}(B_i)$ with a corresponding system of eigenfunctions $\varphi^{\text{ra}}_k(B_i; \cdot)_{k \in \mathbb{Z}^+}$ of $(-\Delta, H^{\text{ra}}_0(B_i))$;

(ii) the complete sequence of eigenvalues $\lambda^{n-\text{ra}}(B_i)$ and corresponding system of eigenfunctions $\varphi^{n-\text{ra}}_k(B_i; \cdot)_{k \in \mathbb{Z}^+}$ of $(-\Delta, H^{n-\text{ra}}_0(B_i))$.

Furthermore, The space of functions $v_1 \times v_2 : B_1 \times B_2 \to \mathbb{R}$ with $v_1$ belonging to $H^{n-\text{ra}}_{0}(B_1)$ and $v_2 \equiv 0$ will still be denoted by $H^{n-\text{ra}}_{0}(B_1)$ (and similarly for $H^{n-\text{ra}}_{0}(B_2)$).

As a consequence of (3.3), we deduce that:

\textbf{Proposition 4.1.}

$$\mathcal{U}^{\text{ra}}(B_1 \cup B_2) \oplus H^{\text{ra}}_{0}(B_1) \oplus H^{n-\text{ra}}_{0}(B_2) = \mathcal{U}(B_1 \cup B_2).$$  \hspace{1cm} (4.1)

By above proposition, the study of the spectrum of $(-\Delta, \mathcal{U}(B_1 \cup B_2))$ is reduced to searching the eigenvalues $\Lambda$ whose corresponding eigenfunction $v = (v_1, v_2)$ is such that both $v_1$ and $v_2$ are radial functions. To this end, let us introduce the function

$$\psi_N(t) = \frac{J_{N-2}(t)}{t^{N-2}}, \quad t \neq 0. \hspace{1cm} (4.2)$$

By the theory of Bessel functions it follows that $\psi_N$ admits an analytic extension (still denoted by $\psi_N$) to the whole real line, and is a solution of the equation

$$-(t^{N-1} f')' = t^{N-1} f \quad \text{for} \quad t \neq 0. \hspace{1cm} (4.3)$$

Although degenerate at $t = 0$, the ODE (4.3) with the initial conditions $f(0) = f_0$, $f'(0) = 0$ is known to possess a unique, classical solution $f \in C^2([0, +\infty))$ for every $f_0 \in \mathbb{R}$. This and the preceding remark implies that for each prescribed $\Lambda > 0$ and $N \geq 1$, the function $u(x) = \psi_N(\sqrt{\Lambda} |x|)$ is (up to a multiplicative constant) the unique radial solution to the equation $-\Delta u = \Lambda u$ in $\mathbb{R}^N$. Finally, recall that the positive values of $t$ such that $\psi_N'(t) = 0$ are the positive zeros of $J_{N/2}$: this is a consequence of the following formula:

$$\frac{d}{dt} \frac{J_{\frac{N}{2}}(t)}{t^{\frac{N}{2}}} = - \frac{J_{\frac{N}{2}}(t)}{t^{\frac{N}{2}}}.$$
Furthermore, observe that by integrating in polar coordinates we find
\[ \int_B u = -N \omega_N R^{N-1} \Lambda^{-1/2} \psi_N(\sqrt{\Lambda} R). \]  
(4.4)
As a consequence, \( \int_B u = 0 \) if and only if the outer derivative \( \partial u / \partial n \) vanishes on \( \partial B \). Then, apart from a multiplicative constant, we must have
\[ v_1(x) = \psi_N(\sqrt{\Lambda} |x-x_1|) \quad \text{and} \quad v_2(x) = k \psi_N(\sqrt{\Lambda} |x-x_2|), \]
with \( k \in \mathbb{R} \setminus \{0\} \) chosen appropriately to guarantee that \( v \) has vanishing integral and also constant boundary value. If \( \Lambda, R_1, R_2 \) are such that \( \psi_N(\sqrt{\Lambda} R_1) = \psi_N(\sqrt{\Lambda} R_2) = 0 \), then, in view of (4.4), \( v \) is an eigenfunction if and only if
\[ k = -R_1^{N-1} R_2^{1-N} \psi_N'(\sqrt{\Lambda} R_1) / \psi_N'(\sqrt{\Lambda} R_2), \]
which expresses the condition \( \int_{B_1 \cup B_2} v = 0 \). If, instead, the boundary value is different from zero, then we must take
\[ k = \psi_N(\sqrt{\Lambda} R_1) / \psi_N(\sqrt{\Lambda} R_2) \]
(4.7)
to ensure \( v_1|_{\partial B_1} = v_2|_{\partial B_2} \). With this value of \( k \), we are led to look for \( \Lambda \) such that \( \int_{B_1} v_1 + \int_{B_2} v_2 = 0 \). Taking (4.4) into account, the condition on \( v \) to be of mean zero is equivalent to the following equation in the unknown \( \Lambda \):
\[ R_1^{N-1} \frac{\psi_N'(\sqrt{\Lambda} R_1)}{\psi_N(\sqrt{\Lambda} R_1)} + R_2^{N-1} \frac{\psi_N'(\sqrt{\Lambda} R_2)}{\psi_N(\sqrt{\Lambda} R_2)} = 0. \]
Thus, we are led to define
\[ q(t) := t^{N-1} \psi_N'(t) / \psi_N(t) \quad \text{and} \quad \Phi(z) := q(\sqrt{z} R_1) + q(\sqrt{z} R_2), \]
and to investigate the positive zeros of the function \( \Phi \). We have:

**Lemma 4.2. (Zeros of \( \Phi \)).** The positive zeros of the function \( \Phi(z) \) form an increasing, unbounded sequence \( \phi = \phi(R_1, R_2) \) whose first element \( \phi_1 \) satisfies \( \phi_1 > \lambda_1(B_2) \). Moreover, the (distinct) points of the sequence \( \lambda'^a(B_1) \oplus \lambda'^a(B_2) \) are the endpoints of a countable family of intervals, each containing exactly one zero of \( \Phi \).

**Proof.** Observe, firstly, that \( q(t) \) is negative for \( t \in (0, j_{\frac{N-2}{2},1}) \). Next, observe that \( q(t) \) is strictly decreasing on each interval free of singular points because by (4.3) we have
\[ q'(t) = -t^{N-1} - t^{N-1} \left( \frac{\psi_N'(t)}{\psi_N(t)} \right)^2 < 0. \]
(4.8)
Furthermore, \( q(t) \to \pm \infty \) when \( t \to j_{\frac{N-2}{2},k} \), \( k \in \mathbb{Z}^+ \). The conclusion follows from the definition of \( \Phi \). \( \square \)

Proposition 4.1 and the above construction of eigenfunctions in \( U'^a(B_1 \cup B_2) \) give a complete description of the spectrum of \( -\Delta \) on \( U(B_1 \cup B_2) \).
**Proposition 4.3. (Spectrum of \(-\Delta\) in \(\mathcal{U}(B_1 \cup B_2)\)).** The sequence of the eigenvalues to \((-\Delta, \mathcal{U}(B_1 \cup B_2))\) is

\[
(\lambda^{ra}(B_1) \cap \lambda^{ra}(B_2)) \oplus \phi(R_1, R_2) \oplus \lambda^{n-ra}(B_1) \oplus \lambda^{n-ra}(B_2).
\]

A complete system of corresponding eigenfunctions is the following.

(a) To each eigenvalue \(\lambda_k \in \lambda^{n-ra}(B_1)\) we associate \(v_1(x) = \varphi^{n-ra}_k(B_1; x), v_2 \equiv 0\).

(b) Similarly, to each eigenvalue \(\lambda_k \in \lambda^{n-ra}(B_2)\) we associate \(v_1 \equiv 0, v_2(x) = \varphi^{n-ra}_k(B_2; x)\).

(c) To each \(\Lambda \in \lambda^{ra}(B_1) \cap \lambda^{ra}(B_2)\) we associate \(\nu\) as in (4.5), with \(k\) as in (4.6).

(d) Finally, to each positive \(\phi_k\) of \(\Phi\) we associate \(\nu\) as in (4.5), with \(\Lambda = \phi_k\) and \(k\) as in (4.7).

**Proof.** From the preceding discussion it is clear that the functions in the statement are eigenfunctions to \((-\Delta, \mathcal{U}(B_1 \cup B_2))\). In particular, case (c) is distinct from case (d) because \(\Phi(z)\) is undefined whenever \(z \in \lambda^{ra}(B_1) \cup \lambda^{ra}(B_2)\). We conclude by applying Proposition 4.1. \(\square\)

Now, let us consider in particular the first eigenvalue \(\Lambda_1(B_1 \cup B_2)\).

**Proposition 4.4. (On the first eigenvalue \(\Lambda_1(B_1 \cup B_2)\)).** The eigenvalue \(\Lambda_1 = \Lambda_1(B_1 \cup B_2)\) has multiplicity 1 and a corresponding (not normalized) eigenfunction \(u = (u_1, u_2): B_1 \times B_2 \to \mathbb{R}\) is given by:

\[
u_1(x) = \psi_N(\sqrt{\Lambda_1} |x - x_1|),
\]

\[
u_2(x) = \begin{cases} 
-\psi_N(\sqrt{\Lambda_1} |x - x_2|), & \text{if } R_1 = R_2, \\
\psi_N(\sqrt{\Lambda_1} R_1) \psi_N(\sqrt{\Lambda_1} |x - x_2|), & \text{if } R_1 < R_2.
\end{cases}
\]

If \(R_1 = R_2 = R\), then \(\Lambda_1(B_1 \cup B_2)\) coincides with \(\lambda_1(B_i), i = 1, 2\), and its value is the following:

\[
\Lambda_1(B_1 \cup B_2) = \left(\frac{j_{N-2,1}^2}{R}\right)^2.
\]

(4.9)

If, instead, \(R_1 < R_2\) then \(\Lambda_1(B_1 \cup B_2) = \phi_1\) and we have:

\[
\frac{2^{2/N} j_{N-2,1}^2}{(R_1^N + R_2^N)^{2/N}} < \Lambda_1(B_1 \cup B_2) < \lambda_2(B_1 \cup B_2).
\]

(4.10)

**Proof.** If \(R_1 = R_2\), on the one hand \(\phi_1(R_1, R_2) > \lambda_1(B_1)\) (by Lemma 4.2) and on the other hand \(\lambda^{n-ra}_1(B_1) > \lambda^{ra}_1(B_1)\). The conclusion follows now from Proposition 4.3. Thus, assume \(R_1 < R_2\). In this case, if \(\lambda_1(B_1) \leq \lambda_2(B_2)\) then Proposition 4.3 implies \(\Lambda_1(B_1 \cup B_2) = \phi_1\). If, instead, \(\lambda_2(B_2) < \lambda_1(B_1)\), then the same conclusion follows because \(\Phi(\lambda_2(B_2)) < 0\), and therefore \(\phi_1 < \lambda_2(B_2)\). In both cases the multiplicity of
$\Lambda_1(B_1)$ is 1, the associated eigenfunction is as claimed and the second inequality in (4.10) holds.

To prove the first inequality in (4.10), we now solve a constrained minimization problem. Observe, firstly, that by the implicit function theorem the equation

$$q(\sqrt{\Lambda_1 R_1}) + q(\sqrt{\Lambda_1 R_2}) = 0 \quad (4.11)$$

defines $\Lambda_1$ as a smooth function of $R_1, R_2$ in the octant $0 < R_1 < R_2$. It is natural to extend its definition to the whole quadrant $R_1, R_2 > 0$: if $R_1 > R_2$ then by symmetry we let $\Lambda_1(R_1, R_2) = \Lambda_1(R_2, R_1)$; furthermore, by (4.9) we define

$$\Lambda_1(R, R) = \left(\frac{j_{N-2,1}}{R}\right)^2. \quad (4.12)$$

The function $\Lambda_1(R_1, R_2)$ defined in this way is continuous also at $(R, R)$ for every positive $R$: indeed, the results obtained so far imply

$$\lambda_1(B_2) < \Lambda_1(R_1, R_2) < \lambda_2(B_1 \cup B_2) \quad \text{for } R_1 < R_2. \quad (4.13)$$

Since $\lambda_1(B_2), \lambda_2(B_1 \cup B_2) \to \Lambda_1(R, R)$ as $R_1, R_2 \to R > 0$, the continuity of $\Lambda_1(R_1, R_2)$ follows. Such a function has to be minimized over the plane curve $\gamma = \{(R_1, R_2) : R_1, R_2 > 0, \ R_1^N + R_2^N = \text{const.} > 0\}$, representing the family of all couples of balls

Figure 2: The first eigenfunction in the union of two discs

whose union has prescribed measure. To minimize $\Lambda_1$ over $\gamma$, let us compute the
partial derivatives \( \partial \Lambda_1/\partial R_1, \partial \Lambda_1/\partial R_2 \) in the case \( R_1 < R_2 \) by differentiating equality (4.11). Denoting by \( \alpha = \alpha(R_1, R_2) \) a convenient, positive function, we find:

\[
\begin{cases}
\frac{\partial \Lambda_1}{\partial R_1} = \alpha' (\sqrt{\Lambda_1} \, R_1), \\
\frac{\partial \Lambda_1}{\partial R_2} = \alpha' (\sqrt{\Lambda_1} \, R_2).
\end{cases}
\] (4.14)

Observe that the vector \( \tau = (-R_2^{N-1}, R_1^{N-1}) \) is tangent to \( \gamma \). By plugging first the expression (4.8) of \( q' \) into (4.14), and using then the equation \( \Phi(\Lambda_1) = 0 \), we obtain:

\[
\tau \cdot \text{grad} \Lambda_1 = \alpha (\sqrt{\Lambda_1} \, R_1 R_2)^{N-1} \left[ \frac{\psi_N' (\sqrt{\Lambda_1} \, R_1)}{\psi_N (\sqrt{\Lambda_1} \, R_1)} - \frac{\psi_N' (\sqrt{\Lambda_1} \, R_2)}{\psi_N (\sqrt{\Lambda_1} \, R_2)} \right]
\]

which shows that \( \tau \cdot \text{grad} \Lambda_1 > 0 \) whenever \( R_1 < R_2 \). Hence, \( \Lambda_1(R_1, R_2) > \Lambda_1(R, R) \), where \( R = (\frac{1}{2}(R_1^N + R_2^N))^{1/N} \). By (4.12), the first inequality in (4.10) follows, and the proof is complete. \( \Box \)

Let us now drop the assumption \( 0 < R_1 \leq R_2 \). By collecting the results of Corollary 3.4 and Proposition 4.4, we obtain:

**Proposition 4.5. (Isoperimetric inequality for \( \Lambda_1(B_1 \cup B_2) \)).** Let \( B_1 = B(x_1, R_1), B_2 = B_2(x_2, R_2) \) be two (possibly empty) disjoint balls of \( \mathbb{R}^N \) such that \( |B_1 \cup B_2| > 0 \). Then:

\[
\left( \frac{2 \omega_N}{|B_1 \cup B_2|} \right)^{\frac{2}{N}} j_{\frac{N}{2}, 1}^2 \leq \Lambda_1(B_1 \cup B_2),
\] (4.15)

and equality holds if and only if \( R_1 = R_2 \).

**Proof.** If both \( B_1 \) and \( B_2 \) are nonempty, then (4.15) follows immediately from (4.9) and (4.10). In order to extend the result to the case when one ball is empty, it suffices to recall (3.9) and to use the inequality

\[
2^{1/N} j_{\frac{N}{2}, 1} \leq j_{\frac{N}{2}, 1},
\] (4.16)

which can be recovered from (4.10) as follows. Recall that when \( 0 < R_1 \leq R_2 \), we have

\[
\lambda_2(B_1 \cup B_2) = \min\{ \lambda_1(B_1), \lambda_2(B_2) \} = \min\{ j_{(N-2)/2}^2 / R_1^2, j_{N/2}^2 / R_2^2 \}.
\]

By inserting this expression into (4.10), and letting \( R_1 \to 0 \), we obtain (4.16). \( \Box \)

**Remark 4.6.**

i) Classical numerical methods allow to compute \( \Lambda_1(R_1, R_2) \) by solving equation (4.11) for \( \Lambda_1 \) in the interval \((a, b)\). It is also possible to prescribe \( \Lambda_1 \) and to plot \( R_2 \) as a function of \( R_1 \) (see Figure 3).

ii) Actually, inequality (4.16) is well-known: it is usually derived by letting \( \Omega = B \) and plugging \( \lambda_2(B) = j_{N/2}^2 / R^2 \) in the classical Faber-Krahn inequality (1.3).
5. An isoperimetric inequality for the eigenvalue $\Lambda_1$.

Using Schwarz symmetrization (see [1, 4, 10]) and Lemma 4.5, we prove in this section the isoperimetric inequality for $\Lambda_1(\Omega)$ stated in Theorem 1.1. Given a bounded open set $\Omega$ and a function $u \in H^1(\Omega)$, denote by $\Omega^*$ the ball centered at 0 having same measure as $\Omega$, and let $u^* : \Omega^* \to \mathbb{R}$ be the (decreasing) Schwarz symmetrization of $u$. If $u \in H^1_0(\Omega)$ and is non-negative, then $u$ belongs to $H^1_0(\Omega^*)$ and the Polya-Szeg"{o} inequality holds:

$$
\int_{\Omega^*} |\nabla u^*|^2 \leq \int_{\Omega} |\nabla u|^2.
$$

If the function $u$ takes also negative values, simple examples show that in general $u^* \neq H^1(\Omega)$. For instance, consider in the disk $B = B(0, 1) \subset \mathbb{R}^2$ the function $u_0$ defined by $u_0(x) = |x| - 1$. Then $u_0 \in H^1_0(B)$ and a straight calculation shows that $u_0^*(x) = \sqrt{1 - |x|^2} - 1$, which does not belong to $H^1(B)$. Further examples are found in [3]. Similarly, we cannot expect (5.1) to hold without restriction in $U(\Omega)$. As an example we may take $u_0 + \frac{1}{2} \in U(\Omega)$, whose Schwartz symmetrization $(u_0 - \frac{1}{4})^* = u_0^* - \frac{1}{4}$ does not even belong to $H^1(B)$.

As a consequence, the classical argument used to prove the Faber-Krahn inequality (1.2) cannot be applied to any eigenfunction $u$ to $(-\Delta, U(\Omega))$ because $u$ must change sign. Neither can be used the replacement $u \mapsto |u|$, although preserving the Rayleigh’s quotient and carrying $H^1_0$ into itself, because it takes $u$ out of the space of admissible functions. In order to overcome these difficulties, we proceed as follows.

**Proof of Theorem 1.1.** The value of $\Lambda_1(\mathcal{B})$, where $\mathcal{B}$ is the union of two disjoint congruent balls, is given by (4.9). It remains to prove the inequality in (1.5). To this aim, let $u$ be a minimizer of (1.9) and denote by $c$ its boundary value. Define $\Omega_+ = \{ x \in \Omega : u(x) - c > 0 \}$, $\Omega_- = \{ x \in \Omega : u(x) - c < 0 \}$, and denote by $(u - c)_+, (u - c)_-$ the positive and the negative part of $u - c$, as usual. Consider the Schwartz symmetrization $\Omega^*_+, \Omega^*_-$, $(u - c)^*_+, (u - c)^*_-$ of the preceding sets and functions. Choose two points $x_1, x_2 \in \mathbb{R}^N$ so far from each other that the balls $B_1 = \Omega^*_+ + x_1$ and $B_2 = \Omega^*_+ + x_2$ are disjoint. Finally, define $\tilde{u}$ as follows:

$$
\tilde{u} : B_1 \cup B_2 \to \mathbb{R}, \quad x \mapsto \begin{cases} 
  c + (u - c)^*_+(x - x_1), & x \in B_1, \\
  c - (u - c)^*_-(x - x_2), & x \in B_2.
\end{cases}
$$

By Cavalieri’s principle, we have

$$
\int_{\Omega} u^2 = \int_{B_1 \cup B_2} \tilde{u}^2 \quad \text{and} \quad \int_{B_1 \cup B_2} \tilde{u} = \int_{\Omega} u = 0.
$$

Furthermore, since $(u - c)_\pm \in H^1_0(\Omega)$ and is non-negative, by (5.1) we get

$$
\int_{\Omega^*_\pm} |\nabla (u - c)_\pm|^2 \geq \int_{\Omega^*_\pm} |\nabla (u - c)^*_\pm|^2.
$$


Using (5.2) and (5.3), we deduce:

\[
\Lambda_1(\Omega) = \frac{\int_{\Omega} |\nabla u|^2}{\int_{\Omega} u^2} = \frac{\int_{\Omega^+} |\nabla u|^2 + \int_{\Omega^-} |\nabla u|^2}{\int_{\Omega} u^2} \geq \frac{\int_{\Omega^+} |\nabla (u - c)^+|^2 + \int_{\Omega^-} |\nabla (u - c)^-|^2}{\int_{\Omega} u^2} = \frac{\int_{B_1 \cup B_2} |\nabla \tilde{u}|^2}{\int_{B_1 \cup B_2} \tilde{u}^2}.
\]

Since \(\tilde{u} \in \mathcal{U}(B_1 \cup B_2)\), we get \(\Lambda_1(\Omega) \geq \Lambda_1(B_1 \cup B_2)\). By applying (4.15) and noting that \(|\Omega| = |B_1 \cup B_2|\), the proof of inequality (1.5) is concluded. \(\square\)

6. Optimal shape domain.

Consider the set of all convex domains in \(\mathbb{R}^N\) of prescribed (positive) measure. In such a class, the existence of a minimizing domain for each eigenvalue \(\lambda_k(\Omega), k = 1, 2, \ldots\), of the Laplace operator with Dirichlet boundary conditions is known (see [9, Theorem 4]). In this section, we use similar arguments to prove the corresponding result for \(\Lambda_k\):

**Theorem 6.1.** For every \(k = 1, 2, \ldots\) there exists a domain \(\Omega_k^1\) that minimizes \(\Lambda_k(\Omega)\) among all convex domains of prescribed measure.

In order to prove the theorem, we need the following properties of the Hausdorff distance (see [12, Appendix]):

- The function

\[
\text{HD}(S, T) = \max \left\{ \sup_{s \in S} \text{dist}(s, T), \sup_{t \in T} \text{dist}(S, t) \right\}
\]

is a metric in the class of (non-empty) compact subsets of \(\mathbb{R}^N\), called the Hausdorff distance, a notion actually introduced by Pompeiu in his thesis [15].

- Every sequence of compact sets \(C_k\) satisfying \(C_k \subset B(0, R)\) for some \(R > 0\) has a subsequence that converges to some compact set \(C\) in the Hausdorff distance.

- Convergence in the Hausdorff distance preserves convexity.

Observe that the function HD is well defined even for non-closed, bounded arguments. We also put into evidence that convergence of convex bounded sets in the Hausdorff distance implies convergence of the boundaries in the following sense. For a bounded set \(T \subset \mathbb{R}^N\) and \(\varepsilon > 0\), let \(U_\varepsilon(\partial T)\) be the \(\varepsilon\)-neighborhood of \(\partial T\), i.e., \(U_\varepsilon(\partial T) = \{ x \in \mathbb{R}^N : \text{dist}(x, \partial T) < \varepsilon \}\). We have:

**Lemma 6.2.** Let \(C, T\) be bounded subsets of \(\mathbb{R}^N\) with \(C\) convex. If \(\text{HD}(C, T) < \varepsilon\) then \(\partial C \subset U_\varepsilon(\partial T)\).
Proof. Consider an arbitrary \( x \in \partial C \). If \( x \notin T \) then \( \text{dist}(x, \partial T) = \text{dist}(x, T) < \varepsilon \). If, instead, \( x \in T \), then, since \( C \) is convex, consider a supporting hyperplane through \( x \) and denote by \( n \) the corresponding outer normal. Choose \( y \in \partial T \) such that \( y = x + tn \) for some \( t \geq 0 \): if \( x \in \partial T \) then we may take \( y = x \); otherwise \( x \) is interior to \( T \) and the existence of such a \( y \) follows from the boundedness of \( T \). Since \( \text{dist}(x, \partial T) \leq \text{dist}(x, y) = \text{dist}(C, y) < \varepsilon \), the conclusion follows. \( \square \)

The assumption that \( C \) is convex cannot be removed from the preceding lemma, as the following example shows.

Example. Define \( T = B(0,1) \) and \( C_\vartheta = T \setminus S_\vartheta \), where \( S_\vartheta \) is the cone \( S_\vartheta = \{ x \in \mathbb{R}^N : |x|^{-1} x_1 > \cos \vartheta \} \), \( \vartheta \in (0, \pi/2) \). Then, \( \text{HD}(C_\vartheta, T) = \sin \vartheta \) and therefore \( \lim_{\vartheta \to 0} \text{HD}(C_\vartheta, T) = 0 \). However, for every \( \vartheta \) we have \( \partial C_\vartheta \notin U_{1/2}(\partial T) \).

Finally, we need the following continuity property:

Lemma 6.3. (Continuity). Let \( \Omega \) and \( \Omega_n \), \( n \in \mathbb{N} \), be convex bounded domains in \( \mathbb{R}^N \). If \( \text{HD}(\Omega_n, \Omega) \to 0 \) then \( \Lambda_k(\Omega_n) \to \Lambda_k(\Omega) \).

Proof. By the preceding lemma, for every \( t \in (0,1) \) there exists an integer \( n_0 \) such that \( t \Omega \subset \Omega_n \subset t^{-1} \Omega \) for every \( n > n_0 \). By monotonicity and homogeneity (see Prop. 2.2) we deduce \( t^2 \Lambda_k(\Omega) \leq \Lambda_k(\Omega_n) \leq t^{-2} \Lambda_k(\Omega) \) and the conclusion follows. \( \square \)

We are now in a position to prove Theorem 6.1.

Proof of Theorem 6.1. Observe that a convex domain of finite measure must be bounded. Let \( (\Omega^k_n)_{n \in \mathbb{N}} \) be a minimizing sequence for \( \Lambda_k(\Omega) \). Since problem (1.4) is invariant under translation, we may assume that \( 0 \in \Omega^k_n \) for every \( n \). Following [9, Theorem 4], we claim that there exists \( R > 0 \) such that \( \Omega^k_n \subset B(0, R) \) for every \( n \in \mathbb{N} \). Indeed, if this were not the case, then by convexity and the condition \( |\Omega^k_n| = \text{const.} \) there would be a sequence of parallelepipeds \( (P_j)_{j \in \mathbb{N}} \) with at least one dimension tending to zero, and a subsequence \( (\Omega^k_{n_j})_{j \in \mathbb{N}} \) such that \( \Omega^k_{n_j} \subset P_j \) for all \( j \). But then \( \Lambda_k(\Omega^k_{n_j}) \geq \lambda_1(\Omega^k_{n_j}) \to +\infty \) by (2.4), contradicting the fact that \( (\Omega^k_n)_{n \in \mathbb{N}} \) is a minimizing sequence for \( \Lambda_k(\Omega) \). Hence, there exists \( R > 0 \) such that \( \Omega^k_n \subset B(0, R) \) for every \( n \in \mathbb{N} \). By compactness in the Hausdorff distance (see above), and since convexity is preserved, there exists a subsequence \( (\Omega^k_{n_j})_{j \in \mathbb{N}} \) such that the closure \( \overline{\Omega^k_{n_j}} \) converges to some convex, closed \( C_k \subset \overline{B}(0, R) \). By Lemma 6.2, Lebesgue measure is also preserved in the limit, and the interior of \( C_k \) is a convex domain \( \Omega^\dagger_k \) such that \( \lim_{j \to +\infty} \text{HD}(\Omega^k_{n_j}, \Omega^\dagger_k) = 0 \).

This and the continuity of \( \Lambda_k \) with respect to the Hausdorff distance imply that \( \Omega^\dagger_k \) is an optimal domain. \( \square \)

The problem of characterizing the optimal domains \( \Omega^\dagger_k \) in the class of convex domains of prescribed measure is left open.

References.


