Frobenius Difference Equations and Difference Galois Groups

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1. Introduction

This is a survey article on recent progress concerning transcendence problems over function fields in positive characteristic. We are interested in some special values that occur in the following two ways. One is the special values of certain special transcendental functions, e.g., Carlitz \( \zeta \)-values at positive integers, which are specialization of Goss’ two-variable \( \zeta \)-function, arithmetic (resp. geometric) \( \Gamma \)-functions at proper fractions (resp. proper rational functions), which are specialization of Goss’ two-variable \( \Gamma \)-function, and Drinfeld logarithms at algebraic points, etc. The other is from algebro-geometric objects that are defined over algebraic function fields. The suitable geometric objects here are Drinfeld modules and the special values are the entries of the period matrix of a Drinfeld module that is related to the comparison between the de Rham and Betti cohomologies of the given Drinfeld module. A natural question concerns the transcendence of these special values.

In the 1980s and 1990s, Yu successfully developed methods of Gelfond-Schneider-Lang type which can be applied to prove many important results on transcendence of the special values mentioned above. The breakthrough from transcendence of single values to linear independence of several special values is Yu’s sub-\( t \)-module theorem [37], which is an analogue of Wüstholz’ subgroup theorem [32]. Here \( t \)-modules are higher-dimension analogues of Drinfeld modules introduced by Anderson [1] and they play the analogous role of commutative algebraic groups in classical transcendence theory. The key ingredient when applying Yu’s sub-\( t \)-module theorem is to relate the special values in question to periods of certain \( t \)-modules. For more details we refer the readers to [38].

In 2004, Anderson-Brownawell-Papanikolas [3] developed a linear independence criterion over function fields, the so-called ABP criterion. It results from a system of Frobenius difference equations which are analogous to classical first-order linear differential equations. Passing from rigid analytically trivial abelian \( t \)-modules
to rigid analytically trivial dual $t$-motives in the terminology of [3], we naturally
have Frobenius difference equations which parameterize the dual $t$-motives in ques-
tion. As remarked in [3, §1.3.4], following this direction the ABP criterion may be
regarded as a $t$-motivic translation of Yu’s sub-$t$-module theorem.

In [20], Papanikolas developed a Tannakian formulation for certain kinds of
Frobenius difference modules, which are called rigid analytically trivial pre-$t$-
motives. Note that there is a fully faithful functor $\mathcal{F}$ from the category of rigid
analytically trivial dual $t$-motives up to isogeny to the Tannakian category $\mathcal{R}$ of
rigid analytically trivial pre-$t$-motives (cf. [20, Thm. 3.4.9]). We mention that
the category of $t$-motives in the terminology of [20] is the strictly full Tannakian
subcategory of $\mathcal{R}$ generated by the essential image of $\mathcal{F}$.

Given a system of Frobenius difference equations occurring from a rigid an-
alytically trivial pre-$t$-motive $M$, Papanikolas further developed a Picard-Vessiot
theory for this system and constructed its difference Galois group explicitly. He
further proved that the difference Galois group in question is isomorphic to the
Galois group of $M$ from Tannakian duality.

Using the ABP criterion and Picard-Vessiot theory, Papanikolas achieved an
analogue of Grothendieck’s periods conjecture for abelian varieties: the dimension
of the Galois group of a rigid analytically trivial pre-$t$-motive that is an image of
$\mathcal{F}$ is equal to the transcendence degree of the period matrix of the pre-$t$-motive.
We say that this pre-$t$-motive has the $\text{GP}$ (Grothendieck periods) property. This
property is the central spirit of our $t$-motivic transcendence program. We will
review this $t$-motivic transcendence theory in §2.

From §3 to §5, we will review the recent algebraic independence results on
special $\zeta$-values, $\Gamma$-values, and periods and logarithms of Drinfeld modules by using
these $t$-motivic techniques. In §6, we will review a refined version of the ABP
criterion investigated by the author of the present article. We will see that not
only rigid analytically trivial pre-$t$-motives that are images of $\mathcal{F}$ have the $\text{GP}$
property, but that there is a bigger class of pre-$t$-motives that have the property.
We will also review its application to transcendence problems concerning Carlitz
$\zeta$-values with varying finite constant fields.

Finally, we mention that in this article we will use the terminology of rigid
analytically trivial pre-$t$-motives that have the $\text{GP}$ property instead of using the
terminology of $t$-motives in [20] or dual $t$-motives in [3], since in this way it does
not let the reader be confused with the terminology of $t$-motives in [1] and [6].

2. $t$-motivic transcendence theory

$\mathbb{F}_q$ = the finite field of $q$ elements with characteristic $p$.
$t, \theta$ = independent variables.
$A = \mathbb{F}_q[\theta] = $ the polynomial ring in the variable $\theta$ over $\mathbb{F}_q$.
$A_+ = $ the set of all monic polynomials in $A$.
$k = \mathbb{F}_q(\theta) = $ the fraction field of $A$.
$k_{\infty} = \mathbb{F}_q((\frac{1}{\theta}))$, the completion of $k$ with respect to the place at infinity.
\( \overline{k_\infty} \) = a fixed algebraic closure of \( k_\infty \).
\( \bar{k} \) = the algebraic closure of \( k \) in \( \overline{k_\infty} \).
\( \mathbb{C}_\infty \) = the completion of \( \overline{k_\infty} \) with respect to the canonical extension of \( \infty \).
\( | \cdot |_\infty \) = a fixed absolute value for the completed field \( \mathbb{C}_\infty \) with \( |\theta|_\infty = q \).
\( T \) = \( \{ f \in \mathbb{C}_\infty[[t]] : f \) converges on \( |t|_\infty \leq 1 \} \) (the Tate algebra of \( \mathbb{C}_\infty \)).
\( \mathbb{L} \) = the fraction field of \( T \).
\( \mathbb{G}_a \) = the additive group.
\( \text{GL}_{r/F} \) = for a field \( F \), the \( F \)-group scheme of invertible \( r \times r \) matrices.
\( \mathbb{G}_m \) = \( \text{GL}_1 \) = the multiplicative group.

### 2.1. Notation and Frobenius twisting.
For \( n \in \mathbb{Z} \), given a Laurent series \( f = \sum a_i t^i \in \mathbb{C}_\infty((t)) \) we define the \( n \)-fold Frobenius twist of \( f \) by the rule \( f^{(n)} := \sum a_i q^n t^i \). For each \( n \), the Frobenius twisting operation is an automorphism of the Laurent series field \( \mathbb{C}_\infty((t)) \) stabilizing several subrings, e.g., \( k[[t]] \), \( k[t] \) and \( T \). More generally, for any matrix \( B \) with entries in \( \mathbb{C}_\infty((t)) \) we define \( B^{(n)} \) by the rule \( B^{(n)}_{ij} := B_{ij}^{(n)} \).

A power series \( f = \sum_{i=0}^\infty a_i t^i \in \mathbb{C}_\infty[[t]] \) that satisfies
\[
\lim_{i \to \infty} \sqrt[|a_i|_\infty]{i} = 0 \quad \text{and} \quad [k_\infty(a_0, a_1, a_2, \ldots) : k_\infty] < \infty
\]
is called an entire power series. As a function of \( t \), such a power series \( f \) converges on all of \( \mathbb{C}_\infty \) and, when restricted to \( \overline{k_\infty} \), \( f \) takes values in \( \overline{k_\infty} \). The ring of entire power series is denoted by \( \mathbb{E} \).

Let \( A_i \in \text{Mat}_{m_i}(\mathbb{L}) \) for \( i = 1, \ldots, n \), and \( m := m_1 + \cdots + m_n \). We define \( \bigoplus_{i=1}^n A_i \in \text{Mat}_m(\mathbb{L}) \) to be the block diagonal matrix, i.e., the matrix with \( A_1, \ldots, A_n \) down the diagonal and zeros elsewhere.

### 2.2. Tannakian formulation.
In this section we follow [20] for relative background and terminology. Let \( k(t)[\sigma, \sigma^{-1}] \) be the noncommutative ring of Laurent polynomials in \( \sigma \) with coefficients in \( k(t) \), subject to the relation
\[
\sigma f = f^{(-1)} \sigma, \quad \forall f \in \bar{k}(t).
\]
A pre-t-motive is a left \( \bar{k}(t)[\sigma, \sigma^{-1}] \)-module that is finite dimensional over \( \bar{k}(t) \). Let \( \mathcal{P} \) be the category of pre-t-motives. Morphisms in \( \mathcal{P} \) are left \( \bar{k}(t)[\sigma, \sigma^{-1}] \)-module homomorphisms.

Given a pre-t-motive \( M \) of dimension \( r \) over \( \bar{k}(t) \), let \( \mathbf{m} \in \text{Mat}_{r \times 1}(M) \) comprise a \( \bar{k}(t) \)-basis of \( M \). Multiplication by \( \sigma \) on \( M \) is given by
\[
\sigma \mathbf{m} = \Phi \mathbf{m}
\]
for some matrix \( \Phi \in \text{GL}_r(\bar{k}(t)) \).

The Laurent series field \( \mathbb{C}_\infty((t)) \) carries the natural structure of a left \( \bar{k}(t)[\sigma, \sigma^{-1}] \)-module by setting \( \sigma(f) = f^{(-1)} \). As such, the subfields \( \mathbb{L} \) and \( \bar{k}(t) \) are \( \bar{k}(t)[\sigma, \sigma^{-1}] \)-submodules. For any \( \sigma \)-invariant subfield \( F \) of \( \mathbb{C}_\infty((t)) \), we denote by \( F^\sigma \) the subfield consisting of all elements in \( F \) fixed by \( \sigma \). Then we have
\[
\mathbb{L}^\sigma = \bar{k}(t)^\sigma = \mathbb{F}_q(t).
\]
There are several important objects in the category $\mathcal{P}$:

(i) **Tensor products of pre-t-motives.** Given two pre-t-motives $M_1$ and $M_2$, we define $M_1 \otimes M_2$ to be the pre-t-motive whose underlying $k(t)$-vector space is $M_1 \otimes_{k(t)} M_2$, on which $\sigma$ acts diagonally.

(ii) **The Carlitz motive.** We define the Carlitz motive to be the pre-t-motive $C$ whose underlying $\bar{k}(t)$-space is $\bar{k}(t)$ itself, on which $\sigma$ acts by

$$\sigma f = (t - \theta)f^{(-1)}$$

for $f \in \bar{k}(t)$.

(iii) **Internal Hom.** Given two pre-t-motives $M_1$ and $M_2$, we set

$$\text{Hom}(M_1, M_2) := \text{Hom}_{k(t)}(M_1, M_2).$$

Then $\text{Hom}(M_1, M_2)$ is a $\bar{k}(t)$-vector space and we define a left $\bar{k}(t)$-$[\sigma, \sigma^{-1}]$-module structure on $\text{Hom}(M_1, M_2)$ by setting

$$\sigma \cdot \rho := \sigma \circ \rho \circ \sigma^{-1}$$

for $\rho \in \text{Hom}(M_1, M_2)$.

(iv) **Identity object.** We let $1 := \bar{k}(t)$ and give a $\sigma$-action on it by

$$\sigma f = f^{(-1)}$$

for $f \in \bar{k}(t)$.

It has the properties:

- For any $M \in \mathcal{P}$, the natural isomorphisms $M \otimes \bar{k}(t) 1 \cong 1 \otimes \bar{k}(t) M \cong M$ are isomorphisms of pre-t-motives;
- $\text{End}_\mathcal{P}(1) = \mathbb{F}_q(t)$.

(v) **Duals.** Given any $M \in \mathcal{P}$, we define

$$M^\vee := \text{Hom}(M, 1).$$

It has the property that $(M^\vee)^\vee \cong M$.

Finally, we define the notion of rigid analytic trivialization. Let $M$ be a pre-t-motive and let $\Phi \in \text{GL}_r(\bar{k}(t))$ be the matrix representing the multiplication by $\sigma$ on $M$ with respect to a $\bar{k}(t)$-basis $\mathbf{m}$ of $M$. We say that $M$ is rigid analytically trivial if there exists $\Psi \in \text{GL}_r(L)$ so that

$$\Psi^{(-1)} = \Phi \Psi.$$

The matrix $\Psi$ is called a rigid analytic trivialization for $\Phi$. It is unique up to right multiplication by a matrix in $\text{GL}_r(\mathbb{F}_q(t))$ (cf. [20, §4.1.6]).

An example of a rigid analytically trivial pre-t-motive is the Carlitz motive $C$.

Let $(-\theta)^{\frac{1}{q-1}}$ be a fixed choice of $(q-1)$st root of $-\theta$. Define

$$\Omega(t) := (-\theta)^{\frac{1}{q-1}} \prod_{i=1}^{\infty} (1 - \frac{t}{\theta^i}) \in \mathbb{E}.$$  \hspace{1cm} (1)
Then we have \( \Omega^{(−1)} = (t − \theta)\Omega \) and hence \( \Omega \) is a rigid analytic trivialization for \( (t − \theta) \). Note that \( \Omega(\theta) = \frac{1}{\pi} \), where \( \pi \) is a fundamental period of the Carlitz \( \mathbb{F}_q[t] \)-module, and it is fixed throughout this article. Note that \( \pi \) is transcendental over \( k \) by the work of Wade [30]. For more details and relative background, see [6].

Given a pre-t-motive \((M, \Phi, \mathbf{m})\) as above, we consider \( M^\dagger := L \otimes_{\bar{k}(t)} M \), where we give \( M^\dagger \) a left \( k(t)[\sigma, \sigma^{-1}] \)-module structure by letting \( \sigma \) act diagonally:

\[
\sigma(f \otimes m) := f^{(-1)} \otimes \sigma m, \quad \forall f \in \bar{k}(t), m \in M.
\]

Let

\[
M^B := (M^\dagger)^\sigma := \{ \mu \in M^\dagger : \sigma \mu = \mu \}.
\]

Then \( M^B \) is a vector space over \( \mathbb{F}_q(t) \). Note that the natural map \( L \otimes_{\mathbb{F}_q(t)} M^B \to M^\dagger \) is an isomorphism if and only if \( M \) is rigid analytically trivial (cf. [20, §3.3]). In this situation, the entries of \( \Psi^{-1}\mathbf{m} \) comprise an \( \mathbb{F}_q(t) \)-basis of \( M^B \), where \( \Psi \) is a rigid analytic trivialization for \( \Phi \).

**Theorem 2.1.** (Papanikolas, [20, Thm. 3.3.15]) The category of rigid analytically trivial pre-t-motives \( \mathcal{R} \) forms a neutral Tannakian category over \( \mathbb{F}_q(t) \) with fiber functor \( M \mapsto M^B \).

Given any \( M \in \mathcal{R} \), let \( \mathcal{R}_M \) be the strictly full Tannakian subcategory of \( \mathcal{R} \) generated by \( M \). That is, \( \mathcal{R}_M \) consists of all objects of \( \mathcal{R} \) isomorphic to subquotients of finite direct sums of \( M \otimes^\sigma (M^\dagger)^\sigma \) for various \( u, v \). By Tannakian duality there is an affine algebraic group scheme \( \Gamma_M \) over \( \mathbb{F}_q(t) \) so that \( \mathcal{R}_M \) is equivalent to the category of finite dimensional representations of \( \Gamma_M \) over \( \mathbb{F}_q(t) \). The algebraic group \( \Gamma_M \) is called the (motivic) Galois group of \( M \). In the next section, we will see that the Galois group \( \Gamma_M \) and the faithful representation

\[
\Gamma_M \hookrightarrow \text{GL}(M^B)
\]

coming from Tannakian duality can be described explicitly.

**2.3. Difference Galois groups.** From now on, we denote by \((M, \Phi, \Psi, \mathbf{m})\) the object \( M \in \mathcal{R} \) endowed with the difference equation \( \Psi^{(-1)} = \Phi \Psi \) for a given \( \bar{k}(t) \)-basis \( \mathbf{m} \) of \( M \) described above. Let \( r \) be the dimension of \( M \) over \( \bar{k}(t) \) and let \( X \) be an \( r \times r \) matrix with \( r^2 \) independent variables \( X_{ij} \). Define the \( \bar{k}(t) \)-algebra homomorphism

\[
\mu_{\Psi} : \bar{k}(t)[X, 1/\det(X)] \to L \\
X_{ij} \mapsto \Psi_{ij}.
\]

Put \( Z_{\Psi} := \text{Spec} \text{Im} \mu_{\Psi} \). Then \( Z_{\Psi} \) is a closed \( \bar{k}(t) \)-subscheme of \( \text{GL}_r/\bar{k}(t) \).

We define two matrices \( \Psi_1, \Psi_2 \in \text{GL}_r(\mathbb{L} \otimes_{\bar{k}(t)} \mathbb{L}) \) by

\[
(\Psi_1)_{ij} := \Psi_{ij} \otimes 1, \quad (\Psi_2)_{ij} := 1 \otimes \Psi_{ij}.
\]

Put \( \tilde{\Psi} := \Psi_1^{-1} \Psi_2 \in \text{GL}_r(\mathbb{L} \otimes_{\bar{k}(t)} \mathbb{L}) \) and define the \( \mathbb{F}_q(t) \)-algebra homomorphism

\[
\mu_{\tilde{\Psi}} : \mathbb{F}_q(t)[X, 1/\det(X)] \to \mathbb{L} \otimes_{\bar{k}(t)} \mathbb{L} \\
X_{ij} \mapsto \tilde{\Psi}_{ij}.
\]
Define $\Gamma_{\Psi} := \text{Spec } \mathcal{O}_{\Psi}$. Then $\Gamma_{\Psi}$ is a closed subscheme of $GL_{r/\mathbb{F}_q(t)}$. Finally, we denote by $\bar{k}(t)(\Psi)$ the field generated by all the entries of $\Psi$ over $\bar{k}(t)$.

**Theorem 2.2.** (Papanikolas, [20]) Given $(M, \Phi, \Psi, m) \in \mathcal{R}$, let $Z_{\Psi}, \Gamma_{\Psi}$ be defined as above. Then we have:

(a) $\Gamma_{\Psi}$ is an affine algebraic group scheme over $\mathbb{F}_q(t)$.

(b) $\Gamma_{\Psi}$ is smooth over $\mathbb{F}_q(t)$ and is geometrically connected.

(c) $Z_{\Psi}$ is a torsor for $\Gamma_{\Psi} \times \mathbb{F}_q(t)$ over $\bar{k}(t)$.

(d) $\dim \Gamma_{\Psi} = \text{tr.deg}_{\bar{k}(t)} \bar{k}(t)(\Psi)$.

(e) $\Gamma_{\Psi}$ is isomorphic to the Galois group $\Gamma_M$ of $M$.

Moreover, the faithful representation $\Gamma_M \hookrightarrow GL(M^B)$ is described as follows: for any $\mathbb{F}_q(t)$-algebra $R$,

$$\Gamma_M(R) \hookrightarrow GL(R \otimes_{\mathbb{F}_q(t)} M^B) \quad \gamma \mapsto (1 \otimes \Psi^{-1}m \mapsto (\gamma^{-1} \otimes 1)(1 \otimes \Psi^{-1}m)).$$

### 2.4. ABP criterion and connection with difference Galois groups.

We recall the linear independence criterion developed by Anderson-Brownawell-Papanikolas [3], the so-called ABP criterion.

**Theorem 2.3.** (Anderson-Brownawell-Papanikolas, [3, Thm. 3.1.1])

Let $\Phi \in \text{Mat}_r(\bar{k}[t])$ be given so that $\det \Phi = c(t - \theta)^s$, $c \in \bar{k}^\times$. Fix a column vector $\psi \in \text{Mat}_{r \times 1}(\mathbb{T})$ satisfying $\psi(-1) = \Phi \psi$. For every $\rho \in \text{Mat}_{1 \times r}(\bar{k})$ such that $\rho \psi(\theta) = 0$, there exists a vector $P \in \text{Mat}_{1 \times r}(\bar{k}[t])$ so that $P \psi = 0$ and $P(\theta) = \rho$.

In the situation of Theorem 2.3, we first note that by [3, Prop. 3.1.3] the condition of $\Phi$ implies $\psi \in \text{Mat}_{r \times 1}(\mathbb{T})$. We further mention that the spirit of the ABP criterion is that every $\bar{k}$-linear relation among the entries of $\psi(\theta)$ can be lifted to a $\bar{k}[t]$-linear relation among the entries of $\psi$.

Let $\Phi$ and $\psi$ be given in Theorem 2.3. For any $n \geq 1$, we consider the Kronecker tensor product $\psi^\otimes n$. Then the entries of $\psi^\otimes n$ comprise all monomials of total degree $n$ in the entries of $\psi$. Fix any $d \geq 1$ and take $\overline{\psi} \in \text{Mat}_{N \times 1}(\mathbb{T})$ to be the column vector whose entries are the concatenation of 1 and each of the columns of $\psi^\otimes n$ for $n \leq d$. (Here $N = (r^d+1)/(r-1)$). Then if $\overline{\mathbf{F}} \in \text{Mat}_N(\bar{k}[t]) \cap \text{GL}_N(\bar{k}(t))$ is the block diagonal matrix

$$\overline{\mathbf{F}} := [1] \oplus \Phi \oplus \Psi \otimes 2 \oplus \ldots \oplus \Psi \otimes d,$$

it follows that

$$\overline{\psi}(-1) = \overline{\mathbf{F}} \psi.$$
Note that $\Phi$ and $\psi$ satisfy the conditions of the ABP criterion. Thus, every $k$-polynomial relation among the entries of $\psi(\theta)$ can be lifted to a $\bar{k}[t]$-polynomial relation among the entries of $\psi$. By calculating the Hilbert series, Papanikolas showed that
$$\text{tr. deg}_{k(t)} \bar{k}(t)(\psi) = \text{tr. deg}_{\bar{k}} \bar{k}(\psi(\theta)),$$
where $\bar{k}(\psi(\theta))$ is the field generated by all entries of $\psi(\theta)$ over $\bar{k}$. Combining this with Theorem 2.2, one has the following important equality.

**Theorem 2.4.** (Papanikolas, [20]) Suppose we are given $(M, \Phi, \Psi, m) \in \mathcal{R}$ and suppose that $\Phi \in \text{Mat}_r(\bar{k}[t])$, $\det \Phi = c(t - \theta)^s$, $c \in \bar{k}^\times$, and that $\Psi \in \text{GL}_r(\mathcal{T})$. Then we have
$$\dim \Gamma_M = \text{tr. deg}_{\bar{k}} \bar{k}(\Psi(\theta)).$$

For the $\Psi$ in the theorem above, we shall call $\Psi^{-1}(\theta)$ the period matrix of $M$. (In §5.1, we will see an explicit connection between $\Psi^{-1}(\theta)$ and period matrices of Drinfeld modules). Therefore, Theorem 2.4 can be regarded as an analogue of Grothendieck’s periods conjecture for abelian varieties. For those $(M, \Phi, \Psi, m) \in \mathcal{R}$ having the two properties:

- each entry of $\Psi$ is analytic at $\theta$;
- $\text{tr. deg}_{\bar{k}(t)} \bar{k}(t)(\Psi) = \text{tr. deg}_{\bar{k}} \bar{k}(\Psi(\theta))$,

we say that $M$ has the GP property (Grothendieck’s periods property), since it follows that
$$\dim \Gamma_M = \text{tr. deg}_{\bar{k}} \bar{k}(\Psi(\theta)).$$

So using $t$-motivic transcendence theory displays in the following way. Suppose we are given a set $S$ of certain special values. If $S$ has $t$-motivic interpretation in the sense that there is an object $(M, \Phi, \Psi, m) \in \mathcal{R}$ which has the GP property and $\bar{k}(\Psi(\theta)) \supseteq S$, then we have a chance to figure out all the $\bar{k}$-algebraic relations among the entries of $S$, since we have the equality (2). However, computing the dimension of $\Gamma_M$ in terms of the known relations among the special values in question might be difficult in general.

3. Carlitz logarithms and special $\zeta$-values

3.1. Carlitz logarithms. The first application of Theorem 2.4 is the breakthrough on Carlitz logarithms due to Papanikolas. (For the background of Carlitz module, we refer the reader to [6]).

**Theorem 3.1.** (Papanikolas, [10, Thm. 1.2.6]) Let $\mathcal{C}$ be the Carlitz $\mathbb{F}_q[t]$-module and $\exp_C(z)$ be its exponential function. Let $\lambda_1, \ldots, \lambda_m \in \mathbb{C}_\infty$ satisfy $\exp_C(\lambda_i) \in \bar{k}$ for all $1 \leq i \leq m$. If $\lambda_1, \ldots, \lambda_m$ are linearly independent over $k$, then they are algebraically independent over $\bar{k}$.

The theorem above is an analogue of the classical conjecture on logarithms of algebraic numbers.
Conjecture 3.2. Let $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$ satisfy $e^{\lambda_i} \in \overline{\mathbb{Q}}$ for all $1 \leq i \leq m$. If $\lambda_1, \ldots, \lambda_m$ are linearly independent over $\mathbb{Q}$, then they are algebraically independent over $\overline{\mathbb{Q}}$.

Under the assumptions in the conjecture above, one only knows the $\mathbb{Q}$-linear independence of $1, \lambda_1, \ldots, \lambda_m$ by the celebrated work of Baker in the 1960s. The analogue of Baker’s work for Drinfeld modules was established by Yu [37].

Back to Theorem 3.1, in order to apply Theorem 2.4 suitably we have to give an $t$-motivic interpretation of the Carlitz logarithms as well as the Carlitz polylogarithms.

Given $n \in \mathbb{N}$ and any $\alpha \in \mathbb{k}^\times$ with $|\alpha|_\infty < |\theta|_\infty^{-\frac{m+n}{nq}}$, we consider the following power series

$$L_{\alpha,n}(t) := \alpha + \sum_{i=1}^{\infty} \frac{\alpha_i}{(t - \theta \alpha)(t - \theta \alpha^2) \cdots (t - \theta \alpha^n)}. \quad (3)$$

Substituting $\theta$ for $t$, one sees that $L_{\alpha,n}(\theta)$ is exactly the $n$-th Carlitz polylogarithm of $\alpha$, denoted by $\log^n_C(\alpha)$. (For $n = 1$, $\log^1_C(\alpha)$ is the Carlitz logarithm of $\alpha$).

Let $\Omega$ be given in (1). From the defining series $L_{\alpha,n}(t)$ one has that

$$(\Omega^n L_{\alpha,n})^{(-1)} = \alpha^{(-1)}(t - \theta)^n \Omega^n + \Omega^n L_{\alpha,n}. \quad (4)$$

More generally, given $m$ nonzero algebraic numbers $\alpha_1, \ldots, \alpha_m \in \mathbb{k}^\times$ with $|\alpha_i|_\infty < |\theta|_\infty^{-\frac{m+n}{nq}}$, we let $L_{\alpha_i,n}$ be the series as in (3) for $i = 1, \ldots, m$. We define

$$\Phi_n = \Phi(\alpha_1, \ldots, \alpha_m) := \begin{pmatrix} (t - \theta)^n & 0 & \cdots & 0 \\ \alpha_1^{(-1)}(t - \theta)^n & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_m^{(-1)}(t - \theta)^n & 0 & \cdots & 1 \end{pmatrix} \in \text{Mat}_{m+1}(\mathbb{F}[t]), \quad (5)$$

$$\Psi_n = \Psi(\alpha_1, \ldots, \alpha_m) := \begin{pmatrix} \Omega^n & 0 & \cdots & 0 \\ \Omega^n L_{\alpha_1,n} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \Omega^n L_{\alpha_m,n} & 0 & \cdots & 1 \end{pmatrix} \in \text{GL}_{m+1}(\mathbb{T}), \quad (6)$$

then by (4) we have

$$\Psi_n^{(-1)} = \Phi_n \Psi_n. \quad (7)$$

It follows that $\Phi_n$ defines a rigid analytically trivial $t$-motive $M_n$ that has the GP property because of Theorem 2.4. Note that $M_n$ fits into the short exact sequence of $t$-motives

$$0 \to C \to M_n \to 1^{\oplus m} \to 0.$$
From the definition of $\Gamma_{\Psi_n}$, which is identified with $\Gamma_{M_n}$ by Theorem 2.2, we see that

$$\Gamma_{M_n} \subseteq \left\{ \begin{pmatrix} * & 0 & \cdots & 0 \\ * & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & 0 & \cdots & 1 \end{pmatrix} \in \text{GL}_{m+1}/F_q(t) \right\}.$$  

Note that by Theorem 2.2 the Galois group of Carlitz motive is isomorphic to $G_m$, since $\Omega$ is transcendental over $\bar{k}(t)$. Since $C$ is a sub-pre-$t$-motive of $M_n$, we have a surjective map

$$\pi : \Gamma_{M_n} \twoheadrightarrow \Gamma_C \cong G_m.$$  

More precisely, for any $F_q(t)$-algebra $R$ the restriction of the action of any element $\gamma \in \Gamma_{M_n}(R)$ to $R \otimes_{F_q(t)} C^R$ is the same as the action of the upper left corner of $\gamma$. That is, $\pi$ is the projection on the upper left corner of any element of $\Gamma_{M_n}$.

As we have that $\bar{k}(\Psi_n(\theta)) = \bar{k}(\tilde{\pi}^n, \log_{C}\log^{[n]}(\alpha_1), \ldots, \log^{[n]}(\alpha_m))$ and that $M_n$ has the GP property, Theorem 3.1 is a consequence of the following (for $n = 1$).

**Theorem 3.3.** ([20, Thm. 6.3.2], [15, Thm. 3.1]) Let notation and assumptions be above. Set $N_n = k\text{-Span} \left\{ \tilde{\pi}^n, \log_{C}\log^{[n]}(\alpha_1), \ldots, \log^{[n]}(\alpha_m) \right\}$. Then we have

$$\dim \Gamma_{\Psi_n} = \text{tr.deg}_k \bar{k}(\tilde{\pi}^n, \log_{C}\log^{[n]}(\alpha_1), \ldots, \log^{[n]}(\alpha_m)) = \dim_k N_n.$$  

We sketch the proof of Theorem 3.3 due to Papanikolas as the following steps.

(I) The defining equations of $\Gamma_{\Psi_n}$ are given by degree one polynomials over $F_q(t)$.

(II) By (I) and Theorem 2.2, the defining equations of $Z_{\Psi_n}$ (cf. §2.3) are given by degree one polynomials over $\bar{k}(t)$.

(III) From the definition of $Z_{\Psi_n}$, we have $\bar{k}(t)$-linear relations among the functions $1, \Omega^n, \Omega^n L_{\alpha_1}, \ldots, \Omega^n L_{\alpha_m}$. By specializing at $t = \theta$ of these functions, we are able to obtain $k$-linear relations among $\tilde{\pi}^n, \log_{C}\log^{[n]}(\alpha_1), \ldots, \log^{[n]}(\alpha_m)$.

Finally, using (III) we can show that $\dim_k N_n \leq \dim \Gamma_{\Psi_n}$. As we have

$$\text{tr.deg}_k \bar{k}(\tilde{\pi}^n, \log_{C}\log^{[n]}(\alpha_1), \ldots, \log^{[n]}(\alpha_m)) \leq \dim_k N_n,$$

the result of Theorem 3.3 follows.

**3.2. Special $\zeta$-values.** The object of this section is to explain all the algebraic relations among the following $\zeta$-values of positive characteristic:

$$\zeta_C(n) := \sum_{a \in A_n} 1/a^n \in F_q((1/t^n)), \quad n = 1, 2, 3, \cdots.$$  

(8)

These $\zeta$-values were introduced in 1935 by L. Carlitz [8], where he obtained the Euler-Carlitz relations: $\zeta_C(n)/\tilde{\pi}^n$ falls in $k$ if $n$ is divisible by $q - 1$. We call the
positive integer \( n \) \textbf{even} provided it is a multiple of \( q - 1 \). The ratios \( \zeta_C(n)/\tilde{\pi}^n \) for \textbf{even} \( n \) involve what are now called Bernoulli-Carlitz “numbers”, just as the case of Riemann zeta function the ratios \( \zeta(n)/(2\pi\sqrt{-1})^n \) for even positive integer \( n \) can be expressed in terms of the classical Bernoulli numbers.

In late 1980s, Anderson-Thakur \cite{4} were able to relate the \( \zeta \)-values \( \zeta_C(n) \) to the \( n \)-th tensor powers of the Carlitz module for all positive integers \( n \). As a result of this big step forward, Yu \cite{36} was able to prove the transcendence of \( \zeta_C(n) \) for all positive integers \( n \), in particular for \textbf{odd} \( n \) (i.e., \( n \) is not divisible by \( q - 1 \)). Later in \cite{37}, Yu was also able to determine all \( \bar{k} \)-linear relations among these transcendental \( \zeta \)-values. Because it is in the characteristic \( p \) world, Frobenius \( p \)-th power relations certainly are also there for all positive integers \( n \) and \( m \):

\[
\zeta_C(p^m n) = \zeta_C(n)^{p^m}. \tag{9}
\]

Using Papanikolas’ theory, Chang-Yu \cite{15} demonstrated that these, i.e., Euler-Carlitz relations and \( p \)-th power relations, account for all the algebraic relations among the \( \zeta \)-values \( \zeta_C(1), \zeta_C(2), \zeta_C(3), \ldots \).

**Theorem 3.4.** (Chang-Yu, \cite[Cor. 4.6]{15}) For any positive integer \( s \), we have

\[
\text{tr. deg}_{\bar{k}} \bar{k}(\tilde{\pi}, \zeta_C(1), \ldots, \zeta_C(s)) = s - \lfloor s/p \rfloor - \lfloor s/(q - 1) \rfloor + \lfloor s/p(q - 1) \rfloor + 1.
\]

In the classical case for Riemann zeta function, the transcendence of \( \zeta(2n + 1) \) for each positive integer \( n \), is still an open problem. For the algebraic relations among the special \( \zeta \)-values \( \{\zeta(2), \zeta(3), \zeta(4), \ldots\} \), conjecturally one expects that the Euler-Bernoulli relations, i.e., \( \zeta(2n)/(2\pi\sqrt{-1})^{2n} \in \mathbb{Q} \) for \( n \in \mathbb{N} \), are the only algebraic relations.

**Conjecture 3.5.** Given an integer \( s > 2 \), we have

\[
\text{tr. deg}_{\bar{k}} \bar{k}(\zeta(2), \zeta(3), \ldots, \zeta(s)) = s - \lfloor s/2 \rfloor.
\]

### 3.2.1. Anderson-Thukar formula

Now, we review the \( t \)-motivic interpretation of Carlitz zeta values using an work of Anderson-Thakur. In \cite{4}, Anderson-Thakur successfully related the values \( \zeta_C(n) \) to the \( n \)-th tensor power of the Carlitz module, thereby obtained the following formula connecting \( \zeta_C(n) \) with the \( n \)-th Carlitz polylogarithms of \( 1, \theta, \cdots, \theta^n \) with \( l_n < \frac{mq}{q-1} \).

**Theorem 3.6.** (Anderson-Thakur, \cite[§3.9]{4}) Given any positive integer \( n \), one can find a finite sequence \( h_{n,0}, \ldots, h_{n,l_n} \in k \), \( l_n < \frac{mq}{q-1} \), such that the following identity holds

\[
\zeta_C(n) = \sum_{i=0}^{l_n} h_{n,i} L_{\theta^n, i}(\theta). \tag{10}
\]
3.2.2. The Galois group of $\zeta$-motive. In this section, we assume $q > 2$ since all positive integers are even for $q = 2$. Given a positive integer $n$ not divisible by $q - 1$ we set

$$N_n := k\text{-span}\{\tilde{\pi}^n, L_{1,n}(\theta), L_{\theta,n}(\theta), \ldots, L_{\theta^{n-1},n}(\theta)\}. \quad (11)$$

By (10) we have $\zeta_C(n) \in N_n$ and $m_n + 2 := \dim_k N_n \geq 2$ since $\zeta_C(n)$ and $\tilde{\pi}^n$ are linearly independent over $k$. (Note that $\tilde{\pi}^n \notin k_\infty$ for $(q - 1) \nmid n$). For each such $n$ we fix once for all a finite subset

$$\{\alpha_{n0}, \ldots, \alpha_{nmn}\} \subseteq \{1, \theta, \ldots, \theta^n\}$$

such that both

$$\{\tilde{\pi}^n, L_{n0}(\theta), \ldots, L_{nmn}(\theta)\}$$

and

$$\{\tilde{\pi}^n, \zeta_C(n), L_{n1}(\theta), \ldots, L_{nmn}(\theta)\}$$

are bases of $N_n$ over $k$, where $L_{nj}(t) := L_{\alpha_{nj},n}(t)$ for $j = 0, \ldots, m_n$. This can be done because of (10).

To each such odd integer $n$, take $M_n$ to be the pre-$t$-motive defined by the matrix

$$\Phi_n = \Phi(\alpha_{n0}, \ldots, \alpha_{nmn}).$$

The Galois group of $M_n$ has dimension $m_n + 2$ by Theorem 3.3. Since $M_n$ has the GP property, $m_n + 2$ also equals the transcendence degree over $k$ of

$$k(\tilde{\pi}^n, L_{n0}(\theta), \ldots, L_{nmn}(\theta)) = k(\tilde{\pi}^n, \zeta_C(n), L_{n1}(\theta), \ldots, L_{nmn}(\theta)).$$

In particular, the elements

$$\tilde{\pi}^n, \zeta_C(n), L_{n1}(\theta), \ldots, L_{nmn}(\theta)$$

are algebraically independent over $k$.

Given any positive integer $s$, we set $U(s) := \{1 \leq n \leq s \mid p \nmid n, q - 1 \nmid n\}$. Define the block diagonal matrices

$$\Phi(s) := \oplus_{n \in U(s)} \Phi_n,$$

$$\Psi(s) := \oplus_{n \in U(s)} \Psi_n.$$  

The matrix $\Phi(s)$ defines a pre-$t$-motive $M(s) := M_{\Phi(s)}$ which is the direct sum of the pre-$t$-motives $M_n$ with $n \in U(s)$. Clearly $\Psi(s)$ gives a rigid analytic trivialization for $\Phi(s)$ and $M(s)$ has the GP property by Theorem 2.4. We shall call $M(s)$ a $\zeta$-motive and in [15] the authors computed the Galois group $\Gamma(s)$ of this $\zeta$-motive explicitly.

**Theorem 3.7.** (Chang-Yu, [15, Thm. 4.5])

*Fix any $s \in \mathbb{N}$. Then we have an exact sequence of algebraic groups over $\mathbb{F}_q(t)$:

$$1 \to V(s) \to \Gamma(s) \to \mathbb{G}_m \to 1,$$
where $V(s)$ is isomorphic to the vector group $\prod_{n \in U(s)} \mathbb{G}_m^{m_n + 1}$. In particular, we have
\[ \dim \Gamma_1(s) = 1 + \sum_{n \in U(s)} (m_n + 1). \]

By the theorem above we find that $1 + \sum_{n \in U(s)} (m_n + 1)$ is exactly the transcendence degree over $\bar{k}$ of the following field:
\[ \bar{k}(\tilde{\pi}, \bigcup_{n \in U(s)} \{ L_{\xi C(n)}(\theta), \cdots, L_{nm_n}(\theta) \}). \]

It follows that the set
\[ \{ \tilde{\pi} \} \bigcup_{n \in U(s)} \{ \zeta_C(n), L_{n1}(\theta), \cdots, L_{nm_n}(\theta) \} \]
is algebraically independent over $\bar{k}$, hence also $\{ \zeta_C(n) \mid n \in U(s) \}$ is algebraically independent over $\bar{k}$. Counting cardinality of $U(s)$ the result of Theorem 3.4 follows.

4. Special values of geometric and arithmetic $\Gamma$-functions

4.1. Geometric $\Gamma$-function. Working in analogy with the classical Euler $\Gamma$-function, Thakur [25] studied the geometric $\Gamma$-function over $A$, which is a specialization of the two-variable $\Gamma$-function of Goss [18],
\[ \Gamma(z) := \frac{1}{z} \prod_{n \in A_+} \left(1 + \frac{z}{n} \right)^{-1}, \quad z \in \mathbb{C}_\infty. \]

It is a meromorphic function on $\mathbb{C}_\infty$ with poles at zero and $-n \in -A_+$ and satisfies several functional equations, which are analogous to the translation, reflection, and Gauss multiplication identities satisfied by the classical $\Gamma$-function.

Special geometric $\Gamma$-values are those values $\Gamma(r)$ with $r \in k \setminus A$. Since, when $a \in A$, $\Gamma(a)$ is either $\infty$ or in $k$, we can restrict to special geometric $\Gamma$-values for transcendence questions. Now the functional equations for $\Gamma(z)$ induce families of algebraic relations among the special values in question. Moreover, if for $x, y \in \mathbb{C}_\infty$ we set $x \sim y$ when $x/y \in \bar{k}^\times$, then for all $r \in k \setminus A, a \in A, g \in A_+$ with $\deg g = d$, we have the following relations:
- $\Gamma(r + a) \sim \Gamma(r)$;
- $\prod_{\xi \in \mathbb{P}_q^\times} \Gamma(\xi r) \sim \tilde{\pi}$;
- $\prod_{a \in A/g} \Gamma(\frac{r + a}{g}) \sim \tilde{\pi}^{d/r-1} \Gamma(r)$.

As observed by Thakur [25], for $q = 2$ all values of $\Gamma(r), r \in k \setminus A$, are $k$-multiples of $\tilde{\pi}$ and hence are transcendental over $k$. Thakur also related some special geometric
Γ-values to periods of Drinfeld modules and deduced their transcendence. Sinha [24] proved the first transcendence result for a general class of special geometric Γ-values: he showed that \( \Gamma(\frac{a}{f} + b) \) is transcendental over \( k \) whenever \( a, f \in A_+ \), \( \deg a < \deg f \), and \( b \in A \). Sinha’s result was obtained by representing the Γ-values in question as periods of certain \( t \)-modules over \( \overline{k} \) using the soliton functions of Anderson [2] and then invoking a transcendence criterion of Gelfond-Schneider type established by Yu [34]. Expanding on Sinha’s method, Brownawell and Papanikolas [7] represented all values \( \Gamma(r) \), \( r \in k \setminus A \), as periods of \( t \)-modules over \( k \) and thus proved transcendence for all special geometric Γ-values.

For algebraic relations among special geometric Γ-values, Thakur [25] adapted the Deligne-Koblitz-Ogus criterion to this setting and devised a diamond bracket criterion to determine which algebraic relations among special geometric Γ-values arise from the functional equation relations. More specifically, a geometric Γ-monomial is a monomial, with positive or negative exponents, in \( \tilde{\pi} \) and special geometric Γ-values, and Thakur’s criterion can decide whether a given geometric Γ-monomial is in \( k \). In [7], Brownawell and Papanikolas showed that the only \( k \)-linear relations among 1, \( \tilde{\pi} \), and special geometric Γ-values are those explained by the diamond bracket relations. This result was obtained by analyzing the sub-\( t \)-module structure of Sinha’s \( t \)-modules and then invoking Yu’s sub-\( t \)-module theorem [37].

In 2004, Anderson, Brownawell, and Papanikolas [3] adapted Sinha’s construction to create rigid analytically trivial pre-\( t \)-motives whose period matrices contain the special geometric Γ-values in question. Again a key component was the interpolation of these special values via Anderson’s soliton functions [2]. Anderson, Brownawell, and Papanikolas used the ABP-criterion to show that all algebraic relations over \( \kappa \) among special geometric Γ-values arise from diamond bracket relations among geometric Γ-monomials, and thus showed that all algebraic relations among special geometric Γ-values can be explained by the standard functional equations. As a consequence, the transcendence degree of the field generated by special geometric Γ-values in question can be obtained explicitly.

**Theorem 4.1.** (Anderson-Brownawell-Papanikolas, [3, Cor. 1.2.2]) For any \( f \in A_+ \) of positive degree, the transcendence degree of the field

\[
\kappa \left( \{ \tilde{\pi} \} \cup \left\{ \Gamma(r); r \in \frac{1}{f} A \setminus (\{0\} \cup -A \cup \ldots) \right\} \right)
\]

over \( \kappa \) is

\[
1 + \frac{q-2}{q-1} \cdot \#(A/f)^x.
\]

The theorem above is an analogue of Rohrlich-Lang conjecture on the special values of Euler Γ-function at proper fractions, which are called special Γ-values. The conjecture asserts that all \( \mathbb{Q} \)-algebraic relations among the special Γ-values and \( 2\pi \sqrt{-1} \) are explained by the translation, reflection and Gauss multiplication identities satisfied by the Γ-function. However, Rohrlich-Lang conjecture can be also formulated as the assertion that all \( \mathbb{Q} \)-linear relations among the monomials of special Γ-values and \( 2\pi \sqrt{-1} \) follow linearly from the two-term relations provided by the Deligne-Koblitz-Ogus criterion. Its transcendence degree formulation is the following.
Conjecture 4.2. For any integer \( n > 2 \) the transcendence degree of the field generated by the set
\[
\{2\pi\sqrt{-1}\} \cup \left\{\Gamma(r); r \in \frac{1}{n}\mathbb{Z} \setminus \mathbb{Z}_{\leq 0}\right\}
\]
over \( \overline{\mathbb{Q}} \) is \( 1 + \frac{1}{2} \cdot \#(\mathbb{Z}/n)^{\times} \).

4.1.1. \( t \)-motivic interpretation of special geometric \( \Gamma \)-values. Fix an \( f \in A_+ \) with positive degree. Let \( A_f \) be the free abelian group on symbols of the form \( [x] \), where \( x \in \frac{1}{f}A/A \). Every \( a \in A_f \) has a unique expression of the form
\[
a = \sum_{a \in A, \deg a < \deg f} m_a[a/f], \quad m_a \in \mathbb{Z}.
\]
If all of the coefficients \( m_a \) are non-negative, then we say that \( a \) is effective.

Let \( \text{wt} : A_f \to \mathbb{Z}/(q - 1) \) be the unique group homomorphism such that for \( x \in \frac{1}{f}A/A, \)
\[
\text{wt}[x] = \begin{cases} 0 & \text{if } x \in A, \\ \frac{1}{q-1} & \text{if } x \not\in A. \end{cases}
\]
For each \( a \in A \) relatively prime to \( f \), there exists a unique automorphism \( (a \mapsto a \ast a) : A_f \to A_f \) of abelian groups such that
\[
a \ast [x] = [ax], \quad x \in \frac{1}{f}A/A.
\]
Hence \( (A/f)^{\times} \) acts on \( A_f \) via \( \ast \). Finally we define
\[
\Pi(z) := z\Gamma(z) = \prod_{a \in A_+} \left(1 + \frac{z}{a}\right)^{-1},
\]
which is sometimes called the “geometric factorial” function, and for \( a \in A_f \) we define \( \Pi(a) \in \mathbb{C}_\infty^{\times} \) so that
\[
\Pi([x]) = \Pi(x), \quad x \in \frac{1}{f}A/A, \quad |x|_\infty < 1.
\]
The elements of the image of \( \Pi \) on \( A_f \) are called \( \Pi \)-monomials of level \( f \).

Let \( r \) be the cardinality of \( (A/f)^{\times} \). A crucial point of [3] is to give an explicit construction of a rigid analytically trivial \( t \)-motive that has the \( \text{GP} \) property and whose period matrix is given in terms of special geometric \( \Gamma \)-values.

Theorem 4.3 (Anderson-Brownawell-Papanikolas [3, Prop. 6.4.4]). Let \( a \in A_f \) be effective with \( \text{wt} a > 0 \). Then there exists (via explicit construction) an object \( (M_a, \Phi_a, \Psi_a, m) \in \mathcal{R} \) with \( \Phi_a \in \text{Mat}_r(k[t]), \) \( \det \Phi_a = c(t - \theta)^s \), \( c \in \bar{k}^{\times} \) and \( \Psi_a \in \text{GL}_r(T) \cap \text{Mat}_r(E) \). Moreover, the matrix \( \Psi_a(\theta) \) has the property that the sets
\[
\{\Psi_a(\theta)_{ij} \mid i, j = 1, \ldots, r\}, \quad \{\Pi(a \ast a)^{-1} \mid a \in A, \ (a, f) = 1\},
\]
Lemma 4.4. The Galois group \( \Gamma \) of \( \mathbb{F}_q(t) \) over \( \bar{\mathbb{F}}_q \) is the group algebra \( \mathbb{F}_q[t] \overline{\mathbb{F}}_q \setminus \mathbb{F}_q[t] \). In particular, a transcendence basis of \( k(\Psi_\alpha(\theta)_{i,j} \mid i,j = 1, \ldots, r) \) over \( k \) can be chosen among elements of the set of \( \Pi \)-monomials \( \{ \Pi(a \ast a)^{-1} \mid a \in A, \ (a,f) = 1 \} \) that are linearly independent over \( k \).

4.1.2. The Galois group of geometric \( \Gamma \)-motive. Let \( a \in A_f \) and \( M_a \) be given in the theorem above, we shall call \( M_a \) a geometric \( \Gamma \)-motive. According to the construction of \( M_a \) in [3], the endomorphism ring \( \text{End}_f(M_a) \) contains the \( \mathbb{F}_q(t) \)-extension of \( \mathbb{F}_q(t) \) by \( \mathbb{F}_q(t)^\times \). In other words, the geometric \( \Gamma \)-motive \( M_a \) has geometric complex multiplication by \( R_f \). Using this property and the fact that the faithful representation \( \pi_{M_a} \to \text{GL}(M_{a}^{\mathbb{F}_q}) \) is functorial in \( M_a \), the authors of [13] showed that the Galois group \( \Gamma_{M_a} \) is contained inside the restriction of scalars \( \text{Res}_{R_f/k}(\text{Gm}_{1/R_f}) \), whence a torus. For more details, see [13, Prop. 3.3.1].

Let \( E_f = \bar{k} \left( \{ \pi \} \cup \{ \pi \mid r \in \frac{1}{2} A \setminus \{ 0 \} \} \right) \). We choose a finite subset \( B_f \) of \( A_f \) whose elements are effective and of positive weight so that

\[
E_f = \left( \bigcup_{a \in B_f} \{ \Pi(a \ast a)^{-1} \mid a \in A, \ (a,f) = 1 \} \right).
\]

For each \( a \in B_f \), let \( \Phi_a \) and \( \Psi_a \) be given as in Theorem 4.3. Defining

\[
M_f := \oplus_{a \in B_f} M_a,
\]

we see that multiplication by \( \sigma \) on \( M_f \) is represented by \( \Phi_f := \oplus_{a \in B_f} \Phi_a \) and has rigid analytic trivialization \( \Psi_f := \oplus_{a \in B_f} \Psi_a \). Moreover, \( M_f \) has the \( \text{GP} \) property because of Theorems 4.3 and 2.4. Since each Galois group \( \Gamma_{M_a} \) is a torus, we see that \( \Gamma_{M_f} \subseteq \bigcap_{a \in B_f} \Gamma_{M_a} \) and hence is a torus over \( \mathbb{F}_q(t) \). On the other hand, Theorem 4.3 and (12) imply that \( k(\Psi_f(\theta)) = E_f \). As a summary, by Theorem 4.1 and Theorem 2.4 we have the following.

**Lemma 4.4.** The Galois group \( \Gamma_{M_f} \) is a torus over \( \mathbb{F}_q(t) \), which is split over \( R_f \). Moreover, its dimension is equal to

\[
1 + \frac{q - 2}{q - 1} \cdot \#(A/f)^\times.
\]

4.2. Arithmetic \( \Gamma \)-function. Let

\[
D_n := \prod_{i=0}^{n-1} (\theta^q^i - \theta^i), \quad \overline{D_n} := D_n / (\theta^\deg D_n).
\]

The Carlitz factorial of \( n \) is defined to be \( \prod D_i^{n_i} \in \mathbb{F}_q[\theta] \) for \( n = \sum n_i q^i \in \mathbb{N}, \ 0 \leq n_i < q \), and the interpolation of its unit part for \( n \in \mathbb{Z}_p \), due to Goss [17], is

\[
n! := \prod D_i^{n_i} \in k_\infty \text{ for } n = \sum n_i q^i, \ 0 \leq n_i < q.
\]

By special arithmetic \( \Gamma \)-values, we mean values at proper fractions of this function.
We are interested in the special values \( r! \in k_\infty \) for \( r \in \mathbb{Q} \cap \mathbb{Z}_p \). We see from the definition that \( r! \in k \) for a non-negative integer \( r \). For \( r \) a negative integer, \( r! \) is a \( k^x \)-multiple of \( \tilde{\pi} \) (cf. [25, p. 34]), and it is thus transcendental over \( k \). Moreover, for \( r \in \mathbb{Q} \cap (\mathbb{Z}_p \setminus \mathbb{Z}) \), \( r! \) depends up to multiplication by a factor in \( \bar{k} \) only on \( r \) modulo \( \mathbb{Z} \) (cf. [25]). Hence, without loss of generality we focus on those \( r! \) with \(-1 < r < 0\).

Given such an \( r \), we write \( r = \frac{a}{b} \), where \( a \) and \( b \) are integers and \( b \) is not divisible by \( p \). By Fermat’s little theorem we see that \( b \) divides \( q^\ell - 1 \) for some \( \ell \in \mathbb{N} \). Hence \( r! \) can be written in the form
\[
r! = \prod_{i=0}^{\ell-1} (1 - q^i)^{c_i}.
\]
(13)

Hence, to determine all the algebraic relations among \( \left\{ (1 - q^\ell)! , (\frac{2}{1 - q^\ell})! , \ldots , (\frac{q^\ell - 2}{1 - q^\ell})! \right\} \), we need only concentrate on these \( \ell \) values
\[
\left\{ (1 - q^\ell)! , (\frac{q}{1 - q^\ell})! , \ldots , (\frac{q^{\ell-1}}{1 - q^\ell})! \right\}.
\]

One of the main theorems in [12] is the following.

**Theorem 4.5.** (Chang-Papanikolas-Thakur-Yu, [12, Cor. 3.3.3]) Fix a positive integer \( \ell \). Then the \( \ell \) values
\[
(1 - q^\ell)! , (\frac{2}{1 - q^\ell})! , \ldots , (\frac{q^\ell - 2}{1 - q^\ell})!
\]
are algebraically independent over \( \bar{k} \). Particularly, we have
\[
\text{tr. deg}_k \bar{k} \left( \frac{1}{1 - q^\ell} , \frac{2}{1 - q^\ell} , \ldots , \frac{q^\ell - 2}{1 - q^\ell} \right) = \ell.
\]

**Remark 4.6.** A uniform framework for arithmetic, geometric and classical \( \Gamma \)-functions is described in [25, §7], [27, §4.12]. In particular, a ‘bracket criterion’ for the transcendence of arithmetic \( \Gamma \)-monomials at proper fractions is described. Our result implies that a set of arithmetic \( \Gamma \)-monomials is \( \bar{k} \)-linearly dependent exactly when the ratio of some pair of them satisfies the bracket criterion. (The exact parallel statement is proved for geometric \( \Gamma \)-monomials in [3], see [27, Thm. 10.5.3]). In fact, by the proof of [27, Thm. 4.6.4] a given arithmetic \( \Gamma \)-monomial satisfies the bracket criterion precisely when, by integral translations of arguments, it is expressible as a monomial in \((q^j/(1 - q^\ell))!\)'s (with fixed \( \ell \) and \( 0 \leq j < \ell \) and up to an element of \( k \)) and the latter monomial is trivial. Hence Theorem 4.5 implies that all algebraic relations over \( \bar{k} \) among special arithmetic \( \Gamma \)-values are generated by their bracket relations.
4.2.1. Drinfeld modules of Carlitz type. For a fixed positive integer \( \ell \), we recall the Carlitz \( \mathbb{F}_q[t] \)-module, denoted by \( C_\ell \), which is given by the \( \mathbb{F}_q \)-linear ring homomorphism
\[
C_\ell : \mathbb{F}_q[t] \rightarrow \text{End}_{\mathbb{F}_q}(\mathbb{G}_a),
\]
\[
t \mapsto (x \mapsto \theta x + x^q).
\]
One has the Carlitz exponential,
\[
\exp_{C_\ell}(z) = z \prod_{\theta \neq a \in \mathbb{F}_q} (1 - \frac{z}{a \tilde{\pi}_\ell}),
\]
where
\[
\tilde{\pi}_\ell := \theta (-\theta)^{\frac{1}{q^{\ell-1}}} \prod_{i=1}^{\infty} \left( 1 - \theta \theta^i \right)^{-1}
\]
is a fundamental period of \( C_\ell \) over \( \mathbb{F}_q[t] \). Throughout this article we fix a choice of \((-\theta)^{\frac{1}{q^{\ell-1}}} \) so that \( \tilde{\pi}_\ell \) is a well-defined element in \( k_{\infty} \). We also choose these roots in a compatible way so that \( \tilde{\pi}_1 = \tilde{\pi} \) fixed in §2 and when \( \ell | \ell' \) the number \((-\theta)^{\frac{1}{q^{\ell-1}}} \) is a power of \((-\theta)^{\frac{1}{q^{\ell'-1}}} \). Note that \( C_1 = C \).

We can regard \( C_\ell \) also as a Drinfeld \( \mathbb{F}_q[t] \)-module, and then it is of rank \( \ell \) with complex multiplication by \( \mathbb{F}_q[t] \) for \( \ell \geq 2 \) (see [6], [19], and [27]). There is a canonical pre-\( t \)-motive associated to this Drinfeld \( \mathbb{F}_q[t] \)-module \( C_\ell \), which we denote by \( M_\ell \). Its construction is given below (cf. [10, §2.4]).

Define \( \Phi_\ell := \left( \begin{array}{c} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ (t - \theta) & 0 & 0 & \cdots & 0 \end{array} \right) \in \text{GL}_\ell(\bar{k}(t)) \cap \text{Mat}_\ell(\bar{k}[t]). \]

Let \( \xi_\ell \) be a primitive element of \( \mathbb{F}_q \) and define \( \Psi_\ell := \Omega_\ell \) if \( \ell = 1 \), and otherwise let
\[
\Psi_\ell := \left( \begin{array}{cccc} \Omega_\ell & \xi_\ell \Omega_\ell & \cdots & \xi_\ell^{\ell-1} \Omega_\ell \\ \Omega_\ell^{(\ell-1)} & (\xi_\ell \Omega_\ell)^{(\ell-1)} & \cdots & (\xi_\ell^{\ell-1} \Omega_\ell)^{(\ell-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \Omega_\ell^{(\ell-1)} & (\xi_\ell \Omega_\ell)^{(\ell-1)} & \cdots & (\xi_\ell^{\ell-1} \Omega_\ell)^{(\ell-1)} \end{array} \right) \in \text{Mat}_\ell(T),
\]
where
\[
\Omega_\ell(t) := (-\theta)^{\frac{1}{q^{\ell-1}}} \prod_{i=1}^{\infty} \left( 1 - \frac{t}{\theta q^i} \right) \in k_{\infty}[t] \subseteq \mathbb{C}_{\infty}(t).
\]
Observe that \( \Omega_\ell \) is an entire function and that \( \Omega_\ell(\theta) = \frac{1}{\tilde{\pi}_\ell} \). Moreover, one has the following functional equation,
\[
\Omega_\ell^{(-\ell)}(t) = (t - \theta) \Omega_\ell,
\]
(15)
which implies $\Psi_{\ell}^{(-1)} = \Phi_{\ell}^t \Psi_{\ell}$. Since $\{1, \xi_{\ell}, \ldots, \xi_{\ell}^{\ell-1}\}$ is a basis of $F_q \ell$ over $F_q$, we have that $\Psi_{\ell} \in \text{GL}_\ell(L)$.

Let $M_{\ell}$ be an $\ell$-dimensional vector space over $\tilde{k}(t)$ with a basis $m \in \text{Mat}_{\ell \times 1}(M_{\ell})$. We give $M_{\ell}$ the structure of a left $\tilde{k}(t)[\sigma, \sigma^{-1}]$-module by defining $\sigma \cdot m := \Phi_{\ell}^t m$, thus making $M_{\ell}$ a rigid analytically trivial pre-$t$-motive that has the GP property. Working out its Galois group, we have the following lemma.

**Lemma 4.7.** ([12], Lem. 3.2.1) Let $M_{\ell}$ be the $t$-motive defined above. Then its Galois group $\Gamma_{M_{\ell}} \subseteq \text{GL}_\ell/F_q(t)$ is an $\ell$-dimensional torus over $F_q(t)$, which is split over $F_q^{\ell}(t)$.

### 4.2.2. $t$-motivic interpretation of special arithmetic $\Gamma$-values

The following first theorem is an analogue (see [27, §4.12]) of the Chowla-Selberg formula and the following second theorem is its quasi-periods counterpart

**Theorem 4.8.** (Thakur, [25]) For each positive integer $\ell$, we have

$$
\frac{(\frac{1}{1-q})!}{(\frac{q-1}{1-q})!} \sim \Omega_{\ell}(\theta).
$$

(17)

**Theorem 4.9.** ([12, Thm. 3.3.2]) Fix an integer $\ell \geq 2$. For each $j$, $1 \leq j \leq \ell - 1$, we have

$$
\frac{(\frac{q^j}{1-q^j})!}{(\frac{q^{j-1}}{1-q^{j-1}})!} \sim \Omega_{\ell}^{(-j)}(\theta).
$$

(18)

Proofs for both follow in exactly the same fashion by straightforward manipulation. Use $D_t/D_t^{\ell-1} = (\theta^t - \theta)$ and take unit parts to verify that the left side in each formula is the one-unit part of the corresponding right side.

To prove Theorem 4.5, we first suppose $q > 2$ or $\ell > 1$. Set

$$
L = \tilde{k}\left(\frac{1}{1-q^j}!, \frac{q}{1-q^j}!, \ldots, \frac{q^{\ell-1}}{1-q^j}!\right).
$$

From (17) and (18) we observe that

$$
\tilde{k}\left(\Omega_{\ell}(\theta), \Omega_{\ell}^{(-1)}(\theta), \ldots, \Omega_{\ell}^{(-(\ell-1))}(\theta)\right) \subseteq L.
$$

Hence Lemma 4.7 gives $\text{tr. deg}_{\tilde{k}} L \geq \ell$, whence the result of Theorem 4.5.

For $q = 2$ and $\ell = 1$, we interpret $(\frac{1}{1-q^j})$ as $\{(n)!\}$ and thus Theorem 4.5 holds in that case too.

**Remark 4.10.** As we have seen that the period matrix of $M_{\ell}$ is given in terms of special arithmetic $\Gamma$-values, we shall call $M_{\ell}$ an arithmetic $\Gamma$-motive. By [10, §2.4] the endomorphism ring $\text{End}_{R}(M_{\ell})$ is isomorphic to $F_q(\ell)$, that is, $M_{\ell}$ has arithmetic complex multiplication by $F_q(\ell)$. 

4.3. Γ-values and ζ-values. Concerning the special values of the Euler Γ-function at proper fractions and the special values of the Riemann ζ-function at positive integers bigger than 1, it is obvious that $2\pi\sqrt{-1}$ is closely related to them. In a conference at Hanoi (2006), Yu provided an open question concerning the algebraic relations among the special Γ-values and the special ζ-values put together. Naturally this question raises the analogous question in the function field setting. In Lemmas 4.4, 4.7 and Theorem 3.7 we have seen that the Galois groups associated to special Γ-values are tori, and the Galois groups associated to ζ-values are extensions of $G_m$ by vector groups. (Here $G_m$ is coming from the Carlitz motive, whence related to $\tilde{\pi}$). Since there are no relations between tori and vector groups, there are expectedly no nontrivial algebraic relations among the Γ-values and ζ-values. (Note that the trivial relations are those coming from the connection with $\tilde{\pi}$). This is indeed the case when taking Γ-motives and ζ-motives at the same stage.

**Theorem 4.11.** (Chang-Papanikolas-Yu, [13, Thm. 5.12]) Given $f \in A_+^*$ with positive degree and $s$ a positive integer, the transcendence degree of the field

$$\bar{k}\left(\{\tilde{\pi}\} \cup \{\Gamma(r) \mid r \in \frac{1}{f}A \setminus (\{0\} \cup -A_+)\} \cup \{\zeta_C(1), \ldots, \zeta_C(s)\}\right)$$

over $\bar{k}$ is

$$1 + \frac{q-2}{q-1} \cdot \#(A/f)^\times + s - [s/p] - [s/(q-1)] + [s/(p(q-1))].$$

**Theorem 4.12.** (Chang-Papanikolas-Thakur-Yu, [12, Thm. 4.2.2]) Given any two positive integers $s$ and $\ell$, let $E$ be the field over $\bar{k}$ generated by the set

$$\{\tilde{\pi}, \zeta_C(1), \ldots, \zeta_C(s)\} \cup \left\{\left(\frac{c}{1-q}\right)!; 1 \leq c \leq q^\ell - 2\right\}.$$ 

Then the transcendence degree of $E$ over $\bar{k}$ is

$$s - [s/p] - [s/(q-1)] + [s/p(q-1)] + \ell.$$ 

The further question is to determine all the algebraic relations among the special values of arithmetic and geometric Γ-functions and the Carlitz ζ-values put together. As expected there are no nontrivial relations among them.

**Theorem 4.13.** (Tien-Yu, [28]) Given $f \in A_+^*$ with positive degree and $\ell, s$ positive integers, let $E$ be the field over $\bar{k}$ generated by the Γ-values

$$\{\tilde{\pi}\} \cup \{\Gamma(r) \mid r \in \frac{1}{f}A \setminus (\{0\} \cup -A_+)\} \cup \left\{\left(\frac{c}{1-q}\right)!; 1 \leq c \leq q^\ell - 2\right\}$$

and the ζ-values

$$\{\zeta_C(1), \ldots, \zeta_C(s)\}.$$ 

Then the transcendence degree of $E$ over $\bar{k}$ is

$$\frac{q-2}{q-1} \cdot \#(A/f)^\times + \ell + s - [s/p] - [s/(q-1)] + [s/p(q-1)].$$
5. Periods and logarithms of Drinfeld modules

One of the main themes of transcendence theory for Drinfeld modules is the study of the periods and quasi-periods. In this section, our objects are Drinfeld $\mathbb{F}_q[t]$-modules $\rho$ defined over $\bar{k}$. Following the point of view of Grothendieck, we consider the period matrix $P_\rho$ of $\rho$, which is related to the isomorphism between the de Rham and Betti cohomologies of $\rho$ (cf. [38]). Nonzero periods and quasi-periods occur as entries of $P_\rho$, and they are transcendental over $k$ by the work of Yu [33, 35]. Determination of the algebraic relations among the entries of $P_\rho$ is the central problem. In the following contexts, we will review the recent progress on this problem as well as the determination of the algebraic relations among the Drinfeld logarithms of algebraic points.

5.1. Drinfeld modules of rank 2. In this section, we will focus on rank 2 Drinfeld $\mathbb{F}_q[t]$-modules $\rho$ defined over $\bar{k}$ and see its analogy with classical transcendence theory for elliptic curves. Let $\tau$ be the Frobenius $q$th power operator on $C_\infty$. For simplicity we assume that $\rho : \mathbb{F}_q[t] \to \bar{k}[\tau]$ satisfies $\rho(t) = \theta + \kappa \tau + \tau^2$, $\kappa \in \bar{k}$.

For applications to general setting (of rank 2 Drinfeld modules) we do not lose any generality (see [10, Remark 3.4.2]). Following Anderson, there is a natural fully faithful functor from the category of Drinfeld $\mathbb{F}_q[t]$-modules up to isogeny to $R$ (cf. [10, §2.4]). The image of $\rho$ under this functor is the pre-$t$-motive $M_\rho$ which is of dimension 2 over $\bar{k}(t)$, and on which multiplication on $\sigma$ is given by the matrix

$$\Phi_\rho := \begin{pmatrix} 0 & 1 \\ (t - \theta) & -\kappa \end{pmatrix} \in \text{Mat}_2(\bar{k}[t]).$$

We will use Anderson generating functions to create a rigid analytic trivialization $\Psi_\rho \in \text{GL}_r(T) \cap \text{Mat}_2(\mathbb{E})$ for $\Phi_\rho$ and connect it with the period matrix of $\rho$ (for more details, see [22, §4.2] and [10, §2.5]).

Let $\exp_\rho(z) := z + \sum_{i=1}^{\infty} \alpha_i z^q$ be the exponential function of $\rho$. Given $u \in C_\infty$ we consider the Anderson generating function

$$f_u(t) := \sum_{i=0}^{\infty} \exp_\rho \left( \frac{u}{\theta^{i+1}} \right) t^i = \sum_{i=0}^{\infty} \alpha_i u^q t^{i} \in T$$

and note that $f_u(t)$ is a meromorphic function on $C_\infty$. It has simple poles at $\theta^q, \ldots$ with residues $-u, -\alpha_1 u^q, \ldots$ respectively. Using the functional equation $\rho_t(\exp_\rho(u/\theta^{i+1})) = \exp_\rho(u/\theta^i)$, we have

$$\kappa f_u^{(1)}(t) + f_u^{(2)} = (t - \theta) f_u(t) + \exp_\rho(u).$$

Since $f_u^{(m)}(t)$ converges away from $\{\theta^m, \theta^{m+1}, \ldots\}$ and $\text{Res}_{t=\theta} f_u(t) = -u$, we have

$$\kappa f_u^{(1)}(\theta) + f_u^{(2)}(\theta) = -u + \exp_\rho(u)$$

by specializing (20) at $t = \theta$. 

Fixing an \( A \)-basis \( \{ \omega_1, \omega_2 \} \) of the period lattice \( A_\rho := \ker \exp_\rho \), we set \( f_i := f_{\omega_i}(t) \) for \( i = 1, 2 \). Let \( F_\rho(z) \) be the quasi-periodic function of \( \rho \) associated to the biderivation given by \( (t \mapsto \tau) : \mathbb{F}_q[t] \to \mathbb{C}_\infty[\tau] \tau \) (cf. [38]).

Recall the analogue of the Legendre relation proved by Anderson,

\[
\omega_1 F_\tau(\omega_2) - \omega_2 F_\tau(\omega_1) = \tilde{\pi}/\xi \quad \text{(cf. [27, Thm. 6.4.6])},
\]

where \( \xi \in \mathbb{F}_q^\times \) satisfies \( \xi(-1) = -\xi \). We fix such a \( \xi \) throughout this section. Put \( \Psi_\rho := \xi \Omega \left( \begin{array}{cc} -f(1)/2 & f(1)/2 \\ \kappa f(2)/2 + f(2)/2 & -\kappa f(1)/2 - f(2)/2 \end{array} \right) \).

By (20) we have \( \Psi_\rho^{-1} = \Phi_\rho^{-1} \Psi_\rho \), and thus \( \det(\Psi_\rho) \) is an \( \mathbb{F}_q \)-multiple of \( \xi \Omega \). By specializing at \( t = \theta \) and using the Legendre relation, we see that \( \det(\Psi_\rho) = \xi \Omega \), whence \( \Psi_\rho \in \text{GL}_2(T) \). Therefore, \( M_\rho \) is rigid analytically trivial and has the GP property by Theorem 2.4.

Since \( F_\tau(\omega_i) = \sum_{j=0}^{\infty} \exp_\rho \left( \frac{\omega_i}{\theta+j+1} \right)^{q \theta j}, \quad \text{for } i = 1, 2, \) (cf. [27, §6.4]), by evaluating \( \Psi_\rho^{-1} \) at \( t = \theta \) we obtain

\[
\Psi_\rho^{-1}(\theta) = \left( \begin{array}{cc} \omega_1 & F_\tau(\omega_1) \\ \omega_2 & F_\tau(\omega_2) \end{array} \right)
\]

and hence \( \tilde{k}(\Psi_\rho(\theta)) = \tilde{k}(\omega_1, \omega_2, F_\tau(\omega_1), F_\tau(\omega_2)) \). When \( \rho \) has complex multiplication, Thiery [29] proved that \( \text{tr. deg}_k \tilde{k}(\omega_1, \omega_2, F_\tau(\omega_1), F_\tau(\omega_2)) = 2 \). In [10], the authors dealt with the case without complex multiplication.

**Theorem 5.1.** (Chang-Papanikolas, [10, Thm. 3.4.1]) Let \( \rho \) be a rank 2 Drinfeld \( \mathbb{F}_q[t] \)-module defined by \( \rho_t = \theta + \kappa \tau + \tau^2 \) with \( \kappa \in \mathbb{k} \). Let \( \omega_1, \omega_2 \) generate the period lattice of \( \rho \). If \( p \neq 2 \) and \( \rho \) is without complex multiplication, then we have \( \Gamma_{M_\rho} = \text{GL}_2 \) and hence

\[
\text{tr. deg}_k \tilde{k}(\omega_1, \omega_2, F_\tau(\omega_1), F_\tau(\omega_2)) = 4.
\]

Let notation and assumptions be in Theorem 5.1, we list the ingredients of the proof as follows.

(I) Establish an analogue of \( t \)-motivic version of Tate’s conjecture in this setting:

\[
\text{End}(M_\rho) \cong \text{Cent}_{\text{Mat}_2(F_q(t))}(\Gamma_{M_\rho}(F_q(t))).
\]

(II) Show that the motivic Galois representation \( \Gamma_{M_\rho} \hookrightarrow \text{GL}(M_\rho^B) \) is absolutely irreducible.
(III) Show that the determinant map \( \det : \Gamma_M \to \mathbb{G}_m \) is surjective.

Note that the property (II) relies on (I). Having these properties at hand, we are able to prove Theorem 5.1. Suppose that \( \Gamma_M, \subseteq \text{GL}_2 \). Then we have \( \dim \Gamma_M \leq 3 \), since \( \Gamma_M \) is connected. We claim that \( \Gamma_M \) is solvable, which contradicts (II) and hence \( \Gamma_M = \text{GL}_2 \).

To prove the claim, we let \( G \) be the kernel of the determinant map \( \det : \Gamma_M \to \mathbb{G}_m \) from (III). Let \( G_0 \) be the identity component of \( G \). Since \( G \) is normal in \( \Gamma_M \), for any \( \gamma \in \Gamma_M \) we have \( \gamma^{-1}G\gamma = G \) and hence \( \gamma^{-1}G^0\gamma = G^0 \). We note that \( G_0 \) is solvable since \( \dim G_0 \leq 2 \), and \( \Gamma_M/G_0 \) is abelian since it is a one-dimensional connected algebraic group. It follows that \( \Gamma_M \) is solvable.

Concerning the \( \rho \)-logarithms of algebraic points, in \( \S 3.1 \) we have seen that the \( t \)-motivic interpretation of Carlitz logarithms is related to the extensions of the direct sum of identity object \( 1 \) by the Carlitz motive. Here the \( t \)-motivic interpretation of \( \rho \)-logarithms is related the extensions of direct sum of \( 1 \) by \( M \). Following this direction, Chang and Papanikolas proved the following result.

**Theorem 5.2.** (Chang-Papanikolas [10, Thm. 1.2.4]) Let \( \rho \) be a rank 2 Drinfeld \( \mathbb{F}_q[t] \)-module defined over \( \bar{k} \) without complex multiplication. Let \( \lambda_1, \ldots, \lambda_m \in \mathbb{C}_\infty \) satisfy \( \exp_\rho(\lambda_i) \in \bar{k} \) for each \( 1 \leq i \leq m \). Suppose that \( p \) is odd. If \( \lambda_1, \ldots, \lambda_m \) are linearly independent over \( k \), then the \( 2m \) quantities

\[
\lambda_1, \ldots, \lambda_m, F_\tau(\lambda_1), \ldots, F_\tau(\lambda_m)
\]

are algebraically independent over \( \bar{k} \).

One can compare these results to classical conjectures about elliptic curves. Specifically, let \( E \) be an elliptic curve over \( \mathbb{Q} \). Let \( \Lambda := Z\omega_1 + Z\omega_2 \) be its period lattice in \( \mathbb{C} \), and let \( \wp \) be the Weierstrass \( \wp \)-function associated to \( \Lambda \). One has the Weierstrass \( \zeta \)-function satisfying \( \zeta'(z) = -\wp(z) \). Then each \( \eta_i := 2\zeta(\frac{1}{2}\omega_i) \), for \( i = 1, 2 \), is called a quasi-period of \( E \). The matrix

\[
P_E := \begin{pmatrix}
\omega_1 & \eta_1 \\
\omega_2 & \eta_2
\end{pmatrix}
\]

is called the period matrix of \( E \) and the Legendre relation says that \( \det P_E = \pm 2\pi \sqrt{-1} \). Conjecturally, one expects

\[
\text{tr. deg}_{\mathbb{Q}}(P_E) = \begin{cases}
4 & \text{if } \text{End}(E) = \mathbb{Z}, \\
2 & \text{if } \text{End}(E) \neq \mathbb{Z}.
\end{cases}
\]

This conjecture is known, by work of Chudnovsky, if \( E \) has complex multiplication, but in the non-CM case, one only knows, by work of Masser, that the periods and quasi-periods are linearly independent over \( \mathbb{Q} \).

Furthermore, one can conjecture results on elliptic logarithms. That is, suppose \( \lambda_1, \ldots, \lambda_m \in \mathbb{C} \) satisfy \( \wp(\lambda_i) \in \mathbb{Q} \). If \( \lambda_1, \ldots, \lambda_m \) are linearly independent over the multiplication ring of \( E \), then one expects that the \( 2m \) quantities,

\[
\lambda_1, \ldots, \lambda_m, \zeta(\lambda_1), \ldots, \zeta(\lambda_m),
\]

are algebraically independent over \( \mathbb{Q} \).
are algebraically independent over $\overline{\mathbb{Q}}$. However, the best known results involve only linear independence over $\overline{\mathbb{Q}}$, due to Masser (elliptic logarithms in the CM case), Bertrand-Masser (elliptic logarithms in the non-CM case), and Wüstholz (elliptic integrals of both the first and second kind). See [31, §4] for more details.

5.2. Algebraic independence for Drinfeld modules of arbitrary rank. In §5.1, we have sketched an approach to calculate the Galois group of the pre-$t$-motive associated to a given Drinfeld module of rank 2. Now suppose we are given a Drinfeld $\mathbb{F}_q[t]$-module $\rho$ of arbitrary rank defined over $\overline{k}$. In [11], the authors have established that the image of the Galois representation on $t$-adic Tate’s module of $\rho$ is naturally contained inside the $t$-adic valued points of the Galois group of the pre-$t$-motive associated to $\rho$. Hence using the fundamental theorem of Pink [23] on the openness of the image of the Galois representation on the $t$-adic Tate module of $\rho$, they are able to obtain an explicit description of the Galois group in question and so obtain the following result.

**Theorem 5.3.** (Chang-Papanikolas, [11]) Let $\rho$ be a Drinfeld $\mathbb{F}_q[t]$-module of rank $r$ defined over $k$ and let $s$ be the rank of $\text{End}(\rho)$ over $A$. Let $\omega_1, \ldots, \omega_r$ be an $A$-basis of the period lattice $\Lambda_\rho := \text{Ker} \exp_\rho$, and $F_{\tau j}$ be the quasi-periodic function of $\rho$ associated to the biderivation given by $t \mapsto \tau^j$, $1 \leq j \leq r - 1$. Then we have

$$\text{tr. deg}_k (\omega_i, F_{\tau j}(\omega_i); 1 \leq i \leq r, 1 \leq j \leq r - 1) = r^2/s.$$

In other words, the theorem above asserts that all the $k$-algebraic relations among the entries of the period matrix of $\rho$ are those linear relations induced by the endomorphisms of $\rho$ (cf. [38]). Based on Theorem 5.3, Chang and Papanikolas further prove the following algebraic independence result concerning the Drinfeld logarithms.

**Theorem 5.4.** (Chang-Papanikolas, [11]) Let $\rho$ be a Drinfeld $\mathbb{F}_q[t]$-module of rank $r$ defined over $k$. Let $\lambda_1, \ldots, \lambda_m \in \mathbb{C}_\infty$ satisfy $\exp_\rho(\lambda_i) \in k$ for all $1 \leq i \leq m$. If $\lambda_1, \ldots, \lambda_m$ are linearly independent over $\text{End}(\rho)$, then they are algebraically independent over $k$.

6. Transcendence problems with varying constant fields

In this section, we will investigate the transcendence problem concerning Carlitz $\zeta$-values with varying finite constant fields.

6.1. A refined version of the ABP criterion. Given a left $k(t)[\sigma, \sigma^{-1}]$-module $M$ that is finite dimensional over $k(t)$, we observe that the iteration of the $\sigma$-action on $M$ makes $M$ be a left $k(t)[\sigma^r, \sigma^{-r}]$-module for any $r \in \mathbb{N}$. This motivates the following definition, which specifies the corresponding operators as powers of $\sigma$.

**Definition 6.1.** Let $r$ and $s$ be positive integers.
(I) A pre-	-motive of level \( r \) is a left \( \bar{k}(t)[\sigma^r, \sigma^{-r}] \)-module \( M \) that is finite dimensional over \( \bar{k}(t) \).

(II) For a pre-	-motive \( M \) of level \( r \), we define its \( s \)-th derived pre-	-motive \( M^{(s)} \) that is a pre-	-motive of level \( rs \); the underlying space is the same as \( M \), but it is regarded now as a left \( \bar{k}(t)[\sigma^{rs}, \sigma^{-rs}] \)-module.

Note that pre-	-motives of level \( 1 \) here are the pre-	-motives we used in the previous sections. We shall give more precise description of the \( M \) in part (II) above. Let \( m \in \text{Mat}_{n \times 1}(M) \) comprise a \( \bar{k}(t) \)-basis of \( M \) and suppose that the matrix representing multiplication by \( \sigma^r \) on \( M \) is given by \( \sigma^r m = \Phi m \) for some \( \Phi \in \text{GL}_n(\bar{k}(t)) \). Then the matrix representing multiplication by \( \sigma^{rs} \) on \( M \) is given by

\[
\sigma^{rs} m = \Phi^{(-r(s-1))} \cdots \Phi^{(-r)} \Phi m.
\]

To be compatible with the definition of rigid analytic triviality of pre-	-motives before, here we say that the given pre-	-motive \( M \) of level \( r \) is rigid analytically trivial (with respect to the operator \( \sigma^r \)) if the there exists \( \Psi \in \text{GL}_n(\bar{L}) \) so that \( \Psi^{(-r)} = \Phi \Psi \). Notice that the category of rigid analytically trivial pre-	-motives of level \( r \) is a neutral Tannakian category over \( F_{q^r}(t) \) and that the Galois group \( \Gamma_M \) of \( M \) is defined over \( F_{q^r}(t) \). Again, we say that \( M \) has the \( \text{GP} \) property if there exists a \( \bar{k}(t) \)-basis \( m \in \text{Mat}_{n \times 1}(M) \) of \( M \) so that there exists a rigid analytic trivialization \( \Psi \in \text{GL}_n(\bar{L}) \) of \( M \) with respect to \( m \) for which

- all the entries of \( \Psi \) are analytic at \( t = \theta \);
- \( \text{tr. deg} \bar{k}(t) \bar{k}(\Psi(\theta)) = \text{tr. deg} \bar{k} \bar{k}(\Psi(\theta)) \).

Note that the second property above implies \( \dim \Gamma_M = \text{tr. deg} \bar{k} \bar{k}(\Psi(\theta)) \). Moreover, the \( \text{GP} \) property is independent of the choices of \( \Psi \) for a fixed \( m \).

Following the definition of \( \text{GP} \) property, one has the following property.

**Proposition 6.2.** Let \( M \) be a rigid analytically trivial pre-	-motive of level \( r \) which has the \( \text{GP} \) property. For any positive integer \( s \), the \( s \)-th derived pre-	-motive \( M^{(s)} \) of \( M \) is also rigid analytically trivial and has the \( \text{GP} \) property.

Following the proof of the ABP criterion in [3], the author of the present article obtained a refined version of the ABP criterion.

**Theorem 6.3.** (Chang, [9, Thm. 1.2]) Fix a positive integer \( r \). Fix a matrix \( \Phi = \Phi(t) \in \text{Mat}_r(\bar{k}[t]) \) such that \( \det \Phi \) is a polynomial in \( t \) satisfying \( \det \Phi(0) \neq 0 \). Fix a vector \( \psi = [\psi_1(t), \cdots, \psi_l(t)]^T \in \text{Mat}_{l \times 1}(\bar{T}) \) satisfying the functional equation \( \psi^{(-r)} = \Phi \psi \). Let \( \xi \in \bar{k}^\times \) satisfy \( |\xi|_\infty \neq 1 \) and

\[
\det \Phi(\xi^{(i)} \sigma^r) \neq 0 \text{ for all } i = 1, 2, 3, \cdots.
\]

Then we have:

1. For every vector \( \rho \in \text{Mat}_{1 \times l}(\bar{k}) \) such that \( \rho \psi(\xi) = 0 \) there exists a vector \( P = P(t) \in \text{Mat}_{1 \times l}(\bar{k}[t]) \) such that \( P(\xi) = \rho \), and \( P \psi = 0 \).
(2) \[ \text{tr. deg}_{\bar{k}(t)} \tilde{k}(t)(\psi_1(t), \cdots, \psi_r(t)) = \text{tr. deg}_{\bar{k}} \tilde{k}(\psi_1(\xi), \cdots, \psi_r(\xi)). \]

Note that by Proposition 3.1.3 of [3] the condition \( \det \Phi(0) \neq 0 \) of the theorem above implies \( \psi \in \text{Mat}_\ell(\bar{E}) \).

Theorem 6.3 can be also thought of as a function field analogue of the Siegel-Shidlovskii theorem concerning the \( E \)-functions satisfying the linear differential equations (cf. [5]).

A direct consequence of Theorem 6.3 is an extension of Theorem 2.4.

**Corollary 6.4.** Suppose that \( \Phi \in \text{Mat}_n(\bar{k}[t]) \) defines a rigid analytically trivial pre-\( t \)-motive \( M \) of level \( r \) with a rigid analytic trivialization \( \Psi \in \text{Mat}_n(T) \cap \text{GL}_n(L) \). If \( \det \Phi(0) \neq 0 \) and \( \det \Phi(\theta^ri) \neq 0 \) for all \( i = 1, 2, 3, \ldots \), then \( M \) has the GP property.

Moreover, we have the following result which takes the pre-\( t \)-motives having the GP property with respect to different constant fields at the same stage for a larger constant field.

**Corollary 6.5.** Given an integer \( d \geq 2 \), we let \( \ell := \text{lcm}(1, \ldots, d) \). For each \( 1 \leq r \leq d \), let \( \ell_r := \ell \) and let \( \Phi_r \in \text{Mat}_n(\bar{k}[t]) \cap \text{GL}_n(\bar{k}(t)) \) define a pre-\( t \)-motive \( M_r \) of level \( r \) with a rigid analytic trivialization \( \Psi_r \in \text{Mat}_n(T) \cap \text{GL}_n(L) \). Suppose that each \( \Phi_r \) satisfies the hypotheses of Theorem 6.3 for \( r = 1, \ldots, d \). Then the direct sum

\[ M := \bigoplus_{r=1}^d M_r^{(\ell_r)} \]

is a rigid analytically trivial pre-\( t \)-motive of level \( \ell \) that has the GP property.

### 6.2. Application to \( \zeta \)-values with varying constant fields.

Our aim in this section is to determine all the algebraic relations among the Carlitz \( \zeta \)-values:

\[ \zeta_r(n) := \sum_{\substack{a \in \mathbb{F}_{q^r}[\theta] \\text{monic} \\text{monic}}} a^n \in \mathbb{F}_{q^r}((1/\theta)) \subseteq \mathbb{F}_q((1/\theta)), \]

where \( r \) and \( n \) vary over all positive integers.

In 1998, Denis [16] proved the algebraic independence of all fundamental periods \( \{ \tilde{\pi}_1, \tilde{\pi}_2, \tilde{\pi}_3, \ldots \} \) as the constant field varies. Thus, in view of Theorem 3.4, one expects for the bigger set of zeta values,

\[ \cup_{r=1}^{\infty} \{ \zeta_r(1), \zeta_r(2), \zeta_r(3), \ldots \}, \]

the Euler-Carlitz relations and the Frobenius \( p \)-th power relations in individual family \( \{ \zeta_r(1), \zeta_r(2), \zeta_r(3), \ldots \} \) still account for all the algebraic relations. This is indeed the case we find from the following theorem.

**Theorem 6.6.** (Chang-Papanikolas-Yu, [14]) Given any positive integers \( s \) and \( d \), the transcendence degree of the field

\[ \mathbb{F}_p(\theta)(\cup_{r=1}^d \{ \tilde{\pi}_r, \zeta_r(1), \ldots, \zeta_r(s) \}) \]
over $\overline{F}_p(\theta)$ is

$$\sum_{r=1}^{d} \left( s - \left\lfloor \frac{s}{p} \right\rfloor - \left\lfloor \frac{s}{q^r - 1} \right\rfloor + \left\lfloor \frac{s}{\prod_{q^r - 1} q^r - 1} \right\rfloor + 1 \right).$$

Now, we sketch the ideas of the proof as the following.

In §3.2 we have already constructed suitable pre-$t$-motives that fit into our consideration of $\zeta$-values for each fixed polynomial ring over a finite field. As concerning the situation with varying constant fields in equal characteristic, we shall go back to the constructions in §3.2. So we fix a positive integer $r$ and consider the operator $\sigma^r$.

Recall that in (15) we have defined the entire function $\Omega^r$ that satisfies the functional equation $\Omega^r(-r) = \Omega^r$ and $\Omega^r(\theta) = -\frac{\tilde{\pi}^r}{\tilde{\pi}^r}$. Recall further that given $n \in \mathbb{N}$ and $\alpha \in \overline{k}^\times$ with $|\alpha|_\infty < |\theta|_\infty^{\frac{1}{qr^r}}$, we have defined the power series

$$L_{\alpha,rn}(t) := \alpha + \sum_{i=1}^{\infty} \frac{\alpha_{q^r}}{(t - \theta^{q^r})^n \cdots (t - \theta^{q^r})^n},$$

which as a function on $\mathbb{C}_\infty$ converges on $|t|_\infty < |\theta|_\infty^{\frac{1}{qr^r}}$, and $L_{\alpha,rn}(\theta)$ is exactly the $n$-th polylogarithm of $\alpha$ associated to the Carlitz $\mathbb{F}_q[t]$-module $C_r$, i.e., $L_{\alpha,rn}(\theta) = \log_{[n]}^\alpha(\alpha)$. Given a collection of such numbers $\alpha$, say $\alpha_1, \ldots, \alpha_m$, we have defined the following matrices and objects constructed before, but we add one more index $r$ to specify the operator $\sigma^r$:

$$\Phi_{rn} := \Phi_{rn}(\alpha_1, \ldots, \alpha_m) \rightarrow \Phi_n := \Phi_n(\alpha_1, \ldots, \alpha_m) \quad (\text{cf. (5)})$$

$$\Psi_{rn} := \Psi_{rn}(\alpha_1, \ldots, \alpha_m) \rightarrow \Psi_n := \Psi_n(\alpha_1, \ldots, \alpha_m) \quad (\text{cf. (6)})$$

$$M_{rn} \rightarrow M_n \quad (\text{cf. §3.1}).$$

Note that the object $M_{rn}$ defined by $\Phi_{rn}$ is a rigid analytically trivial pre-$t$-motive of level $r$ that has the GP property, because of $\Psi_{rn}(-r) = \Phi_{rn} \Psi_{rn}$.

Now, Anderson-Thakur formula is restated as follows: given any positive integer $n$, one can find a sequence $h_{rn,0}, \ldots, h_{rn,l_n} \in \mathbb{F}_q[t]$, $l_n < \frac{qn}{q-1}$, such that the following identity holds

$$\zeta_r(n) = \sum_{i=0}^{l_n} h_{rn,i} \log_{[n]}^\alpha(\theta^i). \quad (23)$$

For each $n \in \mathbb{N}$, $(q^r - 1) \nmid n$, with $l_n$ given above, we fix a finite subset

$$\{\alpha_0, r_n, \ldots, \alpha_{m_n, r_n}\} \subseteq \{1, \theta, \ldots, \theta^{l_n}\}$$

such that

$$\{\tilde{\pi}^n, L_{0, r_n}(\theta), \ldots, L_{m_n, r_n}(\theta)\}$$

and

$$\{\tilde{\pi}^n, \zeta_r(n), L_{1, r_n}(\theta), \ldots, L_{m_n, r_n}(\theta)\}$$
are $\mathbb{F}_{q^r}(\theta)$-bases for the vector space
$$N_{rn} := \mathbb{F}_{q^r}(\theta)\text{-Span}\left\{\pi^n_r, \log^{[n]}_{\omega^r}(1), \log^{[n]}_{\omega^r}(\theta), \ldots, \log^{[n]}_{\omega^r}(\theta^{rn})\right\},$$
where $L_{j, rn}(t) := L_{j, rn}(t)$ for $j = 0, \ldots, m_{rn}$. This can be done because of (23) and note that $m_{rn} + 2$ is the dimension of $N_{rn}$ over $\mathbb{F}_{q^r}(\theta)$. In the case of $q = 2$ and $r = 1$, the $\mathbb{F}_q(\theta)$-dimension of $N_{11}$ is 1 and we set $m_{11} := -1$.

**Definition 6.7.** Given any positive integers $s$ and $d$ with $d \geq 2$. For each $1 \leq r \leq d$ we define
\[
U_r(s) = \begin{cases} 
\{1\}, & \text{if } q = 2 \text{ and } r = 1; \\
\{1 \leq n \leq s; p \nmid n, (q^r - 1) \nmid n\}, & \text{otherwise.}
\end{cases}
\]
For each $n \in U_r(s)$, we define that if $q = 2$ and $r = 1$,
\[
\Phi_{rn} := (t - \theta) \in \text{GL}_1(k(t)),
\Psi_{rn} := \Omega_1 \in \text{GL}_1(L),
\]
otherwise
\[
\Phi_{rn} := \Phi_{rn}(\alpha_{0, rn}, \ldots, \alpha_{m_{rn}, rn}) \in \text{GL}_{(m_{rn} + 2)}(k(t)),
\Psi_{rn} := \Psi_{rn}(\alpha_{0, rn}, \ldots, \alpha_{m_{rn}, rn}) \in \text{GL}_{(m_{rn} + 2)}(L).
\]
Note that we have
\[
\dim \Gamma_{\psi_{rn}} = \text{tr. deg}_{k(t)}(\hat{k}(t)(\psi_{rn})) = m_{rn} + 2.
\]
Put $\Phi_{(rs)} := \oplus_{n \in U_r(s)}\Phi_{rn}$, then $\Phi_{(rs)}$ defines a rigid analytically trivial pre-$t$-motive $M_{(rs)}$ of level $r$ with rigid analytical trivialization $\Psi_{(rs)} := \oplus_{n \in U_r(s)}\Psi_{rn}$. Moreover, $M_{(rs)}$ has the GP property. For clarity, we summarize the notations that fit in §3.2.2:
\[
N_{rn} \rightarrow N_n \text{ (cf. (11))}
\]
\[
\Phi_{rn} \rightarrow \Phi_n
\]
\[
\Psi_{rn} \rightarrow \Psi_n
\]
\[
U_r(s) \rightarrow U(s)
\]
\[
\Phi_{(rs)} \rightarrow \Phi_{(s)}
\]
\[
\Psi_{(rs)} \rightarrow \Psi_{(s)}
\]
\[
M_{(rs)} \rightarrow M_{(s)}.
\]
Now, we put $\ell := \text{lcm}(1, \ldots, d)$ and $\ell_r := \frac{\ell}{r}$ for $r = 1, \ldots, d$. For each $1 \leq r \leq d$ let $M_r := M_{(rs)}(\ell_r) \otimes_{\mathbb{Q}} \mathbb{Q}$ be the $\ell_r$-th derived pre-$t$-motive of $M_{(rs)}$ defined as above. Note that $M_r$ is a rigid analytically trivial pre-$t$-motive of level $\ell$. By Proposition 6.2 each $M_r$ has the GP property and by Theorem 3.7 its Galois group $\Gamma_{M_r}$ has dimension $1 + \sum_{n \in U_r(s)}(1 + m_{rn})$.

Now, put $M := \oplus_{r=1}^{d} M_r$, and the natural rigid analytic triviality of $M$ is given by $\Psi := \oplus_{r=1}^{d} \Psi_{(rs)}$. By Corollary 6.5 we see that $M$ has the GP property. Notice that
\[
\hat{k}(\Psi(\theta)) = \hat{k}(\cup_{r=1}^{d} \cup_{n \in U_r(s)}(\pi^n_r, \zeta_{r}(n), \mathcal{L}_{1, rn}(\theta), \ldots, \mathcal{L}_{m_{rn}, rn}(\theta)))
\]
Theorem 6.6 follows from the following explicit description of $\Gamma_M$. 


Theorem 6.8. Given any positive integers \( s \) and \( d \) with \( d \geq 2 \), let \( M \) be defined as above. Then the Galois group \( \Gamma_M \) is an extension of a \( d \)-dimensional split torus by an vector group and its dimension is given by

\[
\dim \Gamma_M = d + \sum_{r=1}^{d} (m_{rn} + 1).
\]

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